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A NOTE ON WALSH-FOURIER COEFFICIENTS

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Abstract. In this note we have estimate the order of magnitude of Walsh-Fourier coefficients for functions of the class $\Lambda BV(p(n) \uparrow \infty, \varphi)$.

1. Introduction

In 1949 N. J. Fine [1], using second mean value theorem, proved that if f is of bounded variation over [0,1] then its Walsh-Fourier coefficients $\hat{f}(n) = O(\frac{1}{n})$. U. Goginava [2] has studied uniform convergence of Walsh-Fourier series of a function of $BV(p(n), \varphi)$. In 2008 [3], the order of magnitude of Walsh-Fourier coefficients of functions of $\Lambda BV^{(p)}$ and $\phi\Lambda BV$ are estimated. Here we have estimate the order of magnitude of Walsh-Fourier coefficients for a function of $\Lambda BV(p(n) \uparrow \infty, \varphi)$.

Let f be a function defined on $(-\infty, \infty)$ with period 1. $\mathbf{P}$ is said to be a partition with period 1 if

$$\mathbf{P} : ... < x_{-1} < x_0 < x_1 < ... < x_m < ...$$

satisfies $x_{k+m} = x_k + 1$ for $k = 0, \pm 1, \pm 2, ...$, where m is a positive integer.

Definition 1.1. *Let $\varphi(n)$ be a real sequence such that $\varphi(1) \geq 2$ and $\lim_{n\to\infty} \varphi(n) = \infty$. For a given sequence $\Lambda = \{\lambda_m\}$ $(m = 1, 2, \ldots)$ of non-decreasing positive real numbers λ_m such that $\sum_{m=1}^{\infty} \frac{1}{\lambda_m}$ diverges and $1 \leq p(n) \uparrow p$ as $n \to \infty$, where $1 \leq p \leq \infty$, we say that $f \in \Lambda BV(p(n) \uparrow p, \varphi)$ (that is, f is a function of $p(n)$-Λ-bounded variation over [0,1]) if*

$$V_\Lambda(f, p(n), \varphi) = \sup_{n \geq 1} \sup_{\mathbf{P}} \{ V_\Lambda(\mathbf{P}, f, p(n), \varphi) \ : \rho\{\mathbf{P}\} \geq \frac{1}{\varphi(n)}\} < \infty,$$

where

$$V_\Lambda(\mathbf{P}, f, p(n), \varphi) = (\sum_{k=1}^{m} \frac{| f(x_k) - f(x_{k-1}) |^{p(n)}}{\lambda_k})^{1/p(n)}$$

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and

$$\rho\{\mathbf{P}\} = \inf_{k} \mid x_k - x_{k-1} \mid .$$

For $p = \infty$, we denote the class $\Lambda BV(p(n) \uparrow \infty, \varphi)$ by simply $\Lambda BV(p(n), \varphi)$.

Note that, if $\varphi(n) = 2^n,\ \forall n$, and $p = \infty$ then one gets the class $\Lambda BV(p(n) \uparrow \infty)$; if $\lambda_m = 1,\ \forall m$, then one gets the class $BV(p(n) \uparrow p, \varphi)$; if $p(n) = p,\ \forall n$, one gets the class $\Lambda BV^{(p)}$.

Let $\{\phi_n\}$ ($n \in \mathbb{N}_0 = \{0, 1, 2, ...\}$) denotes the complete orthonormal Walsh system defined on the interval [0,1] in the Paley enumeration, where the subscript denote the number of zeros (that is, sign-changes) in the interior of the interval [0,1].

Any $x \in [0, 1)$ can be written as

$$x = \sum_{k=0}^{\infty} x_k\ 2^{-(k+1)},\ each\ x_k = 0\ or\ 1.$$

For any $x \in [0, 1)\backslash Q$, there is only one expression of this form, where Q is the class of dyadic rationals in $[0, 1)$. When $x \in Q$ there are two expression of this form, one which terminates in 0's and one which terminates in 1's. For any $x, y \in [0, 1)$ their dyadic sum is defined as

$$x \dotplus y = \sum_{k=0}^{\infty} |x_k - y_k| 2^{-(k+1)}.$$

Observed that, for each $n \in \mathbb{N}_0,\ \phi_n(x \dotplus y) = \phi_n(x)\phi_n(y),\ x \dotplus y \notin Q$.

For a 1-periodic function $f \in L^1[0, 1]$, its Walsh-Fourier series is defined by

$$f(x) \sim \sum_{n \in \mathbb{N}_0} \hat{f}(n)\phi_n(x), \tag{1.1}$$

where $\hat{f}(n) = \int_0^1 f(x)\ \phi_n(x)\ dx,\ \forall\ n \in \mathbb{N}_0$, are the Walsh-Fourier coefficients of f.

2. Statement of the result

Here, we prove the following theorem.

Theorem 2.1. *If 1-periodic $f \in \Lambda BV(p(n) \uparrow \infty, \varphi, [0, 1]), 1 \leq p(n) \uparrow \infty$ as $n \to \infty$, then*

$$\hat{f}(m) = O\Big(\frac{1}{(\sum_{j=1}^{m} \frac{1}{\lambda_j})^{1/p(\tau(m))}}\Big),$$

where

$$\tau(m) = min\{k : k \in \mathbb{N}, \varphi(k) \geq m\},\ m \geq 1. \tag{2.1}$$

Remark 1. *Here $\lambda_n = 1,\ for\ all\ n$, reduces the class $\Lambda BV(p(n), \varphi)$ to the class $BV(p(n), \varphi)$, and $O(1/(\sum_{i=1}^{m} \frac{1}{\lambda_i})^{1/p(\tau(m))})$ reduces to $O(1/m)^{1/p(\tau(m))}$.*

We need the following lemma to prove the result.

Lemma 2.1. *([5, Lemma 3.1]). The class $\Lambda BV(p(n) \uparrow p, \varphi, [0, 1])$ $(1 \leq p \leq \infty)$ $\subseteq$ $B[0, 1]$.*

3. Proof of result

Proof of Theorem 2.1 . In view of Lemma 2.1, $f \in \Lambda BV(p(n),\varphi)$ over [0,1] implies f is bounded and hence $f \in L^1[0,1]$.

Fix $k \in \mathbb{N}_0$ and $h = \frac{1}{2^{k+1}}$. If we put

$$g(x) = f(x \dotplus \frac{1}{2^k} \dotplus \frac{1}{2^{k+1}}) - f(x),\ for\ all\ x.$$

Then $g \in L^1[0,1]$. For $m = 2^k$, $\phi_m(h) = -1$ and $\phi_m(\frac{1}{2^k}) = 1$ implies

$$\hat{g}(m) = \hat{f}(m)\phi_m(\frac{1}{2^k})\phi_m(h) - \hat{f}(m) = -2\hat{f}(m)$$

and

$$2|\hat{f}(m)| \leq \int_0^1 |f(x \dotplus \frac{1}{2^k} \dotplus \frac{1}{2^{k+1}}) - f(x)|dx$$

$$= \int_0^1 |f((x \dotplus \frac{1}{2^{k+1}}) \dotplus (\frac{1}{2^k} \dotplus \frac{1}{2^{k+1}})) - f(x \dotplus \frac{1}{2^{k+1}})|dx$$

$$= \int_0^1 |f(x \dotplus \frac{1}{2^k}) - f(x \dotplus \frac{1}{2^{k+1}})|dx.$$

Similarly, we get

$$2|\hat{f}(m)| \leq \int_0^1 |f(x \dotplus \frac{4}{2^{k+1}}) - f(x \dotplus \frac{3}{2^{k+1}})|dx$$

and in general we have

$$2|\hat{f}(m)| \leq \int_0^1 |f(x \dotplus \frac{2j}{2^{k+1}}) - f(x \dotplus \frac{(2j-1)}{2^{k+1}})|dx,\ for\ all\ j = 1\ to\ 2^k - 1.$$

Dividing both the sides of the above inequality by λ_j and summing over $j = 1$ to $2^k - 1$, we get

$$2|\hat{f}(2^k)|(\sum_{j=1}^{2^k-1} \frac{1}{\lambda_j}) \leq (\int_0^1 \sum_{j=1}^{2^k-1} \frac{|f_j(x)|}{\lambda_j^{(\frac{1}{p(\tau(2^k))}+\frac{1}{q(\tau(2^k))})}} dx),$$

where $f_j(x) = f(x \dotplus \frac{2j}{2^{k+1}}) - f(x \dotplus \frac{(2j-1)}{2^{k+1}})$ and $q(\tau(2^k))$ is the index conjugate of $p(\tau(2^k))$. Then by applying Holder's inequality on the right side we have

$2|\hat{f}(2^k)|(\sum_{j=1}^{2^k-1} \frac{1}{\lambda_j})$

$$\leq \int_0^1 (\sum_{j=1}^{2^k-1} \frac{|f_j(x)|^{p(\tau(2^k))}}{\lambda_j})^{1/p(\tau(2^k))} (\sum_{j=1}^{2^k-1} \frac{1}{\lambda_j})^{1/q(\tau(2^k))}\ dx.$$

For any $x \in \mathbb{R}$, all these points $x \dotplus 2jh,\ x \dotplus (2j-1)h,\ for\ j = 1, 2, ..., 2^k - 1$ lie in the interval of length 1. Thus, $f \in \Lambda BV(p(n),\varphi)$ over [0,1] implies

$(\sum_{j=1}^{2^k-1} \frac{|f_j(x)|^{p(\tau(2^k))}}{\lambda_j})^{1/p(\tau(2^k))} = O(1)$. This together with $\sum_{j=1}^{2^k} \frac{1}{\lambda_j} \approx \sum_{j=1}^{2^k-1} \frac{1}{\lambda_j}$ and the above inequality implies

$$|\hat{f}(2^k)| = O((\sum_{j=1}^{2^k} \frac{1}{\lambda_j})^{-1/p(\tau(2^k))}).$$

This proves the theorem.

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References

[1] N.J. Fine, *On the Walsh functions*, Trans. Amer. Math. Soc., **65** (1949), 372-414.
[2] U. Goginava, *On the uniform convergence of Walsh-Fourier series*, Acta Math. Hungar, **93** (2001), No.1-2, 59-70.
[3] B. L. Ghodadra and J. R. Patadia, *A note on the magnitude of Walsh Fourier coefficients*, J. Inequal. Pure Appl. Math., **9** (2008), No.2, Art.44, 7 pp.
[4] Ferenc Moricz, *Absolute convergence of Walsh Fourier series and related results*, Analysis Mathematica, **36** (2010), 275-286.
[5] R. G. Vyas, *A note on functions of $p(n) - \Lambda$-bounded variation*, J. Indian Math. Soc., V.78, **1-4** (2011), 215-220.

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