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COEFFICIENT ESTIMATES FOR CERTAIN NEW SUBCLASSES OF STARLIKE FUNCTIONS OF COMPLEX ORDER

(COMMUNICATED BY HARI SRIVASTAVA)

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Abstract. In the present paper, we consider the coefficient estimates for functions in certain new subclasses of starlike and convex functions of complex order γ, which are introduced by means of a generalized differential operator and non-homogeneous Cauchy-Euler type differential equation. Several corollaries and consequences of the main results are also obtained.

1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

that are analytic in the open unit disc $U = \{z \in C : |z| < 1\}$.

For two functions $f(z)$ and $g(z)$, analytic in U, we say that $f(z)$ is subordinate to $g(z)$ in U, and we note $f(z) \prec g(z),\ (z \in U)$, if there exists a Schwarz function $\omega(z)$ analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1$ $(z \in U)$, such that $f(z) = g(\omega(z)),\ (z \in U)$. In particular, if the function $g(z)$ is univalent in U, then the subordination is equivalent to $f(0) = g(0)$ and $f(U) = g(U)$.

A function $f(z) \in \mathcal{A}$ is said to be in the $S^*(\gamma)$ of starlike functions of complex order γ if it satisfies the following inequality:

$$Re\left\{1 + \frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0\ (z \in U;\ \gamma \in C^* = C \setminus \{0\}). \tag{2}$$

Furthermore, a function $f(z) \in \mathcal{A}$ is said to be in the $C(\gamma)$ of convex functions of complex order γ if it satisfies the following inequality:

$$Re\left\{1 + \frac{1}{\gamma}\left(\frac{zf''(z)}{f'(z)}\right)\right\} > 0\ (z \in U;\ \gamma \in C^*). \tag{3}$$

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The function classes $S^*(\gamma)$ and $C(\gamma)$ were considered earlier by Nasr and Aouf [1] and Wiatrowshi [2], respectively, and (very recently) by Altintas et al.[3-9], Deng [10], Murugusundaramoorthy and Srivastava [11], Xu et al.[12], and Srivastava et al.[13-15].

For a function $f(z) \in \mathcal{A}$, Raducanu and Orhan [16] introduced a generalized differential operator $D^n_{\alpha,\delta}$ as follows:

$$D^0_{\alpha,\delta} f(z) = f(z)$$

$$D^1_{\alpha,\delta} f(z) = D_{\alpha,\delta} f(z) = \alpha\delta z^2 (f(z))'' + (\alpha-\delta) z (f(z))' + (1-\alpha+\delta) f(z)$$

$$\vdots$$

$$D^n_{\alpha,\delta} f(z) = D_{\alpha,\delta}(D^{n-1}_{\alpha,\delta} f(z)),\ (\alpha \ge \delta \ge 0,\ n \in N_0 = N \cup \{0\}). \tag{4}$$

If f is given by (1), then from the definition of operator $D^n_{\alpha,\delta}$ it is easy to see that

$$D^n_{\alpha,\delta} f(z) = z + \sum_{k=2}^{\infty} \Phi_k^n a_k z^k, \tag{5}$$

where $\Phi_k = [1 + (\alpha\delta k + \alpha - \delta)(k-1)],\ (\Phi^n_k = [\Phi_k]^n);\ \alpha \ge \delta \ge 0$ and $n \in N_0$.

When $\alpha = 1$ and $\delta = 0$, we get the Salagean differential operator $D^n f(z)$ (see [18]), and when $\delta = 0$, we obtain the Al-Oboudi differential operator $D^n_\alpha f(z)$ (see [17]).

Next, by using the differential operator $D^n_{\alpha,\delta}$, we define new subclasses of functions belonging to the class $\mathcal{A}$.

Definition 1. Let $\gamma \ne 0$ be any complex number, $\alpha \ge \delta \ge 0; 0 \le \lambda \le 1, n \in N_0$ and for the parameters A and B such that $-1 \le B < A \le 1$, we say that a function $f(z) \in \mathcal{A}$ is in the class $H^n_{\gamma,\lambda,\alpha,\delta}(A, B)$ if it satisfies the following subordination condition:

$$1 + \frac{1}{\gamma}\left(\frac{z(F^n_{\lambda,\alpha,\delta}(z))'}{F^n_{\lambda,\alpha,\delta}(z)} - 1\right) \prec \frac{1+Az}{1+Bz},\ z \in U, \tag{6}$$

where $F^n_{\lambda,\alpha,\delta}(z) = (1-\lambda) D^n_{\alpha,\delta} f(z) + \lambda D^{n+1}_{\alpha,\delta} f(z)$.

The special classes $H^0_{1,\lambda,1,0}(1-2\alpha,-1)$ and $H^0_{\gamma,\lambda,1,0}(A,B)$ were introduced and studied by Altintas et al.[4] and Srivastava et al.[14], respectively.

Definition 2. A function $f(z) \in \mathcal{A}$ is said to be in the class $K^{m;n}_{\gamma,\lambda,\alpha,\delta}(A, B; \mu)$ if it satisfies the following non-homogeneous Cauchy-Euler type differential equation of order m:

$$z^m \frac{d^m w}{dz^m} + \begin{pmatrix} m \\ 1 \end{pmatrix} (\mu + m - 1) z^{m-1} \frac{d^{m-1} w}{dz^{m-1}} + \cdots + \begin{pmatrix} m \\ m \end{pmatrix} w \prod_{i=0}^{m-1} (\mu + i)$$

$$= g(z) \prod_{i=0}^{m-1} (\mu + i + 1), \tag{7}$$

where $w = f(z) \in \mathcal{A}$, $g(z) \in H^n_{\gamma,\lambda,\alpha,\delta}(A, B)$, $\mu \in R \setminus (-\infty, -1]$ and $m \in N^* = N \setminus \{1\} = \{2, 3, \cdots\}$.

The special cases of the class $K^{2,0}_{1,\lambda,1,0}(A, B; \mu)$ and $K^{3,0}_{1,\lambda,1,0}(A, B; \mu)$ were also introduced and studied by Altintas et al.[4]. The object of the present paper is to derive the coefficient estimates for functions in the classes $H^n_{\gamma,\lambda,\alpha,\delta}(A, B)$ and $K^{m,n}_{\gamma,\lambda,\alpha,\delta}(A, B; \mu)$ employing the techniques used earlier by Srivastava et al.[14].

2. **MAIN RESULTS**

The first property for $f(z) \in H^n_{\gamma,\lambda,\alpha,\delta}(A,B)$ is contained in

Theorem 1. Let the function $f(z)$ given by (1) be in the class $H^n_{\gamma,\lambda,\alpha,\delta}(A,B)$. Then

$$|a_k| \leq \frac{\prod_{j=0}^{k-2}\left(j+2|\gamma|\frac{A-B}{1-B}\right)}{(k-1)!\Phi_k^n[1+\lambda(\Phi_k-1)]}, \tag{8}$$

where $\Phi_k = [1+(\alpha\delta k+\alpha-\delta)(k-1)]$, $k \in N^*$ and $n \in N_0$.

Proof. By the definitions of $D^n_{\alpha,\delta}f(z)$ and $F^n_{\lambda,\alpha,\delta}(z)$, we can write

$$F^n_{\lambda,\alpha,\delta}(z) = z + \sum_{k=2}^{\infty} A_k z^k \ (z \in U), \tag{9}$$

in which

$$A_k = \Phi_k^n[1+\lambda(\Phi_k-1)]a_k \ (k \in N^*). \tag{10}$$

Then, clearly, $F^n_{\lambda,\alpha,\delta}(z)$ is analytic in U with

$$F^n_{\lambda,\alpha,\delta}(0) = (F^n_{\lambda,\alpha,\delta})'(0) - 1 = 0. \tag{11}$$

Thus, by virtue of the subordination condition in equation (6) of Definition 1, we have

$$1 + \frac{1}{\gamma}\left(\frac{z(F^n_{\lambda,\alpha,\delta}(z))'}{F^n_{\lambda,\alpha,\delta}(z)} - 1\right) \subset g(U), \tag{12}$$

where the function $g(z)$ is given by

$$g(z) = \frac{1+Az}{1+Bz} \ (z \in U,\ -1 \leq B < A \leq 1). \tag{13}$$

By setting

$$h(z) = 1 + \frac{1}{\gamma}\left(\frac{z(F^n_{\lambda,\alpha,\delta}(z))'}{F^n_{\lambda,\alpha,\delta}(z)} - 1\right), \tag{14}$$

we deduce also that $h(0) = g(0) = 1$ and $h(U) \subset g(U)$ $(z \in U)$ for the the function $g(z)$ given by (13). Therefore, we have

$$h(z) = \frac{1+A\omega(z)}{1+B\omega(z)} \ (\omega(0) = 0,\ |\omega(z)| < 1) \tag{15}$$

and

$$|\omega(z)| = \left|\frac{h(z)-1}{A-Bh(z)}\right| < 1,\ h(z) = u + iv. \tag{16}$$

Now, by using of (16), we obtain that

$$2u(1-AB) > 1 - A^2 + (1-B^2)(u^2+v^2).$$

Also, since $(Re(h(z)))^2 \leq |h(z)|^2$, we have $(1-B^2)u^2 - 2u(1-AB) + 1 - A^2 < 0$, which implies that

$$\frac{1-A}{1-B} < u = Re(h(z)) < \frac{1+A}{1+B}. \tag{17}$$

If

$$Re(h(z)) > \frac{1-A}{1-B},\ h(z) = 1 + p_1 z + p_2 z^2 + \cdots \in P, \tag{18}$$

then we have that

$$|p_k| \leq 2\left(\frac{A-B}{1-B}\right). \tag{19}$$

By (14), we have

$$z(F^n_{\lambda,\alpha,\delta}(z))' - F^n_{\lambda,\alpha,\delta}(z) = \gamma(h(z)-1)F^n_{\lambda,\alpha,\delta}(z). \tag{20}$$

Then, from (9) and (18), equating the coefficient of z^k in (20), we obtain that

$$(k-1)A_k = \gamma(p_{k-1} + p_{k-2}A_2 + \cdots + p_1 A_{k-1}). \tag{21}$$

In particular, when $n = 2, 3, 4$, (21) yields

$$|A_2| \leq 2|\gamma|\frac{A-B}{1-B},\ |A_3| \leq \frac{2|\gamma|\frac{A-B}{1-B}\left(1+2|\gamma|\frac{A-B}{1-B}\right)}{2!},$$

and

$$|A_4| \leq \frac{2|\gamma|\frac{A-B}{1-B}\left(1+2|\gamma|\frac{A-B}{1-B}\right)\left(2+2|\gamma|\frac{A-B}{1-B}\right)}{3!},$$

respectively. Thus, by using the principle of mathematical induction, we have

$$|A_k| \leq \frac{\prod_{j=0}^{k-2}\left(j+2|\gamma|\frac{A-B}{1-B}\right)}{(k-1)!}. \tag{22}$$

Also, since $A_k = \Phi_k^n[1+\lambda(\Phi_k - 1)]a_k$ $(k \in N^*)$. Then, by (22), we have that inequality (8). This completes the proof of Theorem 1.

Corollary 1. Let the function $f(z) \in \mathcal{A}$ be given by (1). If $f(z) \in H^n_{\gamma,\lambda,1,0}(A,B)$, then

$$|a_k| \leq \frac{\prod_{j=0}^{k-2}\left(j+2|\gamma|\frac{A-B}{1-B}\right)}{(k-1)!k^n[1+\lambda(k-1)]}\ (k \in N^*).$$

Corollary 2 ([14]). Let the function $f(z) \in \mathcal{A}$ be given by (1). If $f(z) \in H^0_{\gamma,\lambda,1,0}(A,B) \equiv S(\lambda,\gamma,A,B)$, then

$$|a_k| \leq \frac{\prod_{j=0}^{k-2}\left(j+2|\gamma|\frac{A-B}{1-B}\right)}{(k-1)![1+\lambda(k-1)]}\ (k \in N^*).$$

Corollary 3. Let the function $f(z) \in \mathcal{A}$ be given by (1). If $f(z) \in H^n_{\gamma,\lambda,1,0}(1-2\alpha,-1)$, then

$$|a_k| \leq \frac{\prod_{j=0}^{k-2}(j+2|\gamma|(1-\alpha))}{(k-1)!k^n[1+\lambda(k-1)]}\ (k \in N^*).$$

Corollary 4 ([10]). Let the function $f(z) \in \mathcal{A}$ be given by (1). If $f(z) \in H^0_{\gamma,\lambda,1,0}(1-2\alpha,-1) \equiv B(0,\lambda,\alpha,b)$, then

$$|a_k| \leq \frac{\prod_{j=0}^{k-2}(j+2|b|(1-\alpha))}{(k-1)![1+\lambda(k-1)]}\ (k \in N^*).$$

Corollary 5. Let the function $f(z) \in \mathcal{A}$ be given by (1). If $f(z) \in H^n_{\gamma,\lambda,\alpha,0}(A,B)$, then

$$|a_k| \leq \frac{\prod_{j=0}^{k-2}\left(j+2|\gamma|\frac{A-B}{1-B}\right)}{(k-1)![1+\alpha(k-1)]^n[1+\lambda\alpha(k-1)]}\ (k \in N^*).$$

Corollary 6. Let the function $f(z) \in \mathcal{A}$ be given by (1). If $f(z) \in H^n_{\gamma,\lambda,\alpha,0}(1-2\alpha,-1)$, then

$$|a_k| \leq \frac{\prod_{j=0}^{k-2}(j+2|\gamma|(1-\alpha))}{(k-1)![1+\alpha(k-1)]^n[1+\lambda\alpha(k-1)]}\ (k \in N^*).$$

Theorem 2. Let the function $f(z) \in \mathcal{A}$ be given by (1). If $f(z) \in K^{m,n}_{\gamma,\lambda,\alpha,\delta}(A,B;\mu)$, then

$$|a_k| \leq \frac{\prod_{j=0}^{k-2}\left(j+2|\gamma|\frac{A-B}{1-B}\right)\prod_{i=0}^{m-1}(\mu+i+1)}{(k-1)!\Phi_k^n[1+\lambda(\Phi_k-1)]\prod_{i=0}^{m-1}(\mu+i+k)} \ (k,m \in N^*;\ n \in N_0), \quad (23)$$

$$(0 \leq \lambda \leq 1; \gamma \in C^*; -1 \leq B < A \leq 1; \mu \in R \setminus (-\infty,-1]).$$

Proof. Suppose that the function $f(z) \in \mathcal{A}$ be given by (1). Let

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in H^n_{\gamma,\lambda,\alpha,\delta}(A,B).$$

By Theorem 1, we have

$$|b_k| \leq \frac{\prod_{j=0}^{k-2}\left(j+2|\gamma|\frac{A-B}{1-B}\right)}{(k-1)!\Phi_k^n[1+\lambda(\Phi_k-1)]} \ (k \in N^*,\ n \in N_0). \quad (24)$$

Then we deduce from (7) that

$$a_k = \left(\frac{\prod_{i=0}^{m-1}(\mu+i+1)}{\prod_{i=0}^{m-1}(\mu+i+k)}\right) b_k \ (k,m \in N^*;\ \mu \in R \setminus (-\infty,-1]). \quad (25)$$

Using (24) and (25), we have the assertion (23) of Theorem 2. This completes the proof.

Corollary 7. Let the function $f(z) \in \mathcal{A}$ be given by (1). If $f(z) \in K^{m,n}_{\gamma,\lambda,\alpha,0}(A,B;\mu)$, then

$$|a_k| \leq \frac{\prod_{j=0}^{k-2}\left(j+2|\gamma|\frac{A-B}{1-B}\right)\prod_{i=0}^{m-1}(\mu+i+1)}{(k-1)![1+\alpha(k-1)]^n[1+\lambda\alpha(k-1)]\prod_{i=0}^{m-1}(\mu+i+k)} \ (k,m \in N^*).$$

Corollary 8 ([14]). Let the function $f(z) \in \mathcal{A}$ be given by (1). If $f(z) \in K^{m,0}_{\gamma,\lambda,1,0}(A,B;\mu) \equiv K(\lambda,\gamma,A,B,m;\mu)$, then

$$|a_k| \leq \frac{\prod_{j=0}^{k-2}\left(j+2|\gamma|\frac{A-B}{1-B}\right)\prod_{i=0}^{m-1}(\mu+i+1)}{(k-1)![1+\lambda(k-1)]\prod_{i=0}^{m-1}(\mu+i+k)} \ (k,m \in N^*).$$

Corollary 9. Let the function $f(z) \in \mathcal{A}$ be given by (1). If $f(z) \in K^{m,n}_{\gamma,\lambda,\alpha,0}(1-2\alpha,-1;\mu)$, then

$$|a_k| \leq \frac{\prod_{j=0}^{k-2}(j+2|\gamma|(1-\alpha))\prod_{i=0}^{m-1}(\mu+i+1)}{(k-1)![1+\alpha(k-1)]^n[1+\lambda\alpha(k-1)]\prod_{i=0}^{m-1}(\mu+i+k)} \ (k,m \in N^*).$$

Corollary 10 ([13]). Let the function $f(z) \in \mathcal{A}$ be given by (1). If $f(z) \in K^{2,0}_{\gamma,\lambda,1,0}(1-2\alpha,-1;\mu) \equiv T(0,\lambda,\alpha,b;\mu)$, then

$$|a_k| \leq \frac{(1+\mu)(2+\mu)\prod_{j=0}^{k-2}(j+2|b|(1-\alpha))}{(k-1)!(k+\mu)(k+\mu+1)[1+\lambda(k-1)]} \ (k \in N^*).$$

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