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OSCILLATION CRITERIA FOR A CLASS OF SECOND ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DAMPING

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Abstract. In this paper, some oscillation criteria for solutions of a general second order non-linear differential equations with damping of the form

$$\left(a(t)\Psi\left(x\left(t\right)\right)k\left(x'(t)\right)\right)' + p\left(t\right)k\left(x'(t)\right) + q\left(t\right)f\left(x\left(t\right)\right) = 0,$$

are given. The results obtained extend some existing results in the literature by using the refined integral averaging technique introduced by Rogovchenko and Tuncay ([1], [2]).

1. Introduction

In this paper, we are concerned with the oscillation of solutions of the second-order nonlinear differential equations with damping terms of the following form

$$(a(t)\Psi\left(x\left(t\right)\right)k\left(x'(t)\right))' + p\left(t\right)k\left(x'(t)\right) + q\left(t\right)f\left(x\left(t\right)\right) = 0, \tag{1.1}$$

where $t \geq t_0 \geq 0, a(t), p(t), q(t) \in C\left([t_0, \infty); \mathbb{R}\right)$ and $\Psi, k, f \in C(\mathbb{R}, \mathbb{R})$. It is also assumed that there are positive constants c, c_1, μ and γ such that the following conditions are satisfied:

(C1) $a(t) > 0$ and $xf(x) > 0$ for all $x \neq 0$;
(C2) $0 < c \leq \Psi\left(x\right) \leq c_1$ for all x;
(C3) $\gamma > 0$ and $k^2(y) \leq \gamma y k(y)$ for all $y \in \mathbb{R}$;
(C4) $q(t) \geq 0$, $\frac{f(x)}{x} \geq \mu > 0$ for $x \neq 0$.

We recall that a function $x : [t_0, t_1) \to \mathbb{R}, t_1 > t_0$ is called a solution of Eq. (1.1) if $x(t)$ satisfies Eq. (1.1) for all $t \in [t_0, t_1)$. In what follows, it will be always assumed that solutions of Eq. (1.1) exist for any $t_0 \geq 0$. Furthermore, a solution $x(t)$ of Eq. (1.1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. Finally, we say that Eq. (1.1) is oscillatory if all its solutions are oscillatory.

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In the last decades, there has been an increasing interest in obtaining sufficient conditions for the oscillation of solutions for different classes of second order differential equations. Especially, by using the integral averaging technique and the generalized Riccati technique, the oscillation problem for Eq. (1.1) and its special cases such as the nonlinear equations with damping term

$$(r(t)x'(t))' + p(t)\,x'(t) + q(t)\,f(x(t)) = 0 \tag{1.2}$$

and

$$(r(t)\psi(x(t))x'(t))' + p(t)\,x'(t) + q(t)\,f(x(t)) = 0. \tag{1.3}$$

has been studied extensively in recent years (see, for example, [1]-[13] and the references cited therein).

Following Philos [10], we define a family of functions $\mathcal{P}$ which will be used in the rest of the article. For this purpose, let

$$D = \{(t,s) : t \geq s \geq t_0\}.$$

A function $H \in C(D, \mathbb{R})$ is said to belong to the class $\mathcal{P}$ if

(i) $H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ for $t > s \geq t_0$;

(ii) $H(t,s)$ has a continuous and nonpositive partial derivative on D with respect to the second variable, and there is a function $h \in C(D; [0, +\infty))$ such that

$$-\frac{\partial H}{\partial s}(t,s) = h(t,s)\sqrt{H(t,s)} \text{ for all } (t,s) \in D.$$

In this connection, in 2004, Wang [5] established oscillation criteria for Eq. (1.1). We now state one of his main results for easier reference.

Theorem 1.1. *([5], Theorem 3.3). Let assumptions (C1)-(C4) be fulfilled. Let the function $H \in \mathcal{P}$, and suppose also that*

$$0 < \inf_{s \geq t_0} \left\{ \liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} \right\} \leq \infty. \tag{1.4}$$

If there exist functions $R, \phi \in C([t_0,\infty);\mathbb{R})$ and $\Phi \in C^1([t_0,\infty);(0,\infty))$ such that $(aR) \in C^1([t_0,\infty);\mathbb{R})$ and

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \Phi(s)a(s)h_2^2(t,s)ds < \infty, \tag{1.5}$$

$$\int_{t_0}^{\infty} \frac{\phi_+^2(s)}{\Phi(s)a(s)} ds = \infty,$$

and for every $T \geq t_0$

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left[H(t,s)Q_2(s) - \frac{c_1\gamma}{4}\Phi(s)a(s)h_2^2(t,s) \right] ds \geq \phi(T),$$

where

$$Q_2(t) = \Phi(t)\left\{ \mu q(t) - \frac{\gamma}{4}\left(\frac{1}{c} - \frac{1}{c_1}\right)\frac{p^2(t)}{a(t)} - \frac{1}{c_1}p(t)R(t) + \frac{1}{c_1\gamma}a(t)R^2(t) - (a(t)R(t))' \right\},$$

$$h_2(t,s) = h(t,s) - \sqrt{H(t,s)}\left(\frac{\Phi'(s)}{\Phi(s)} + \frac{2R(s)}{c_1\gamma} - \frac{p(s)}{c_1 a(s)} \right)$$

and

$$\phi_+(s) = \max\{\phi(s), 0\},$$

then Eq. (1.1) is oscillatory.

We have two aims in this paper. The first aim is to remove the condition (1.5) in Theorem 1.1 and to demonstrate this with an example. The second goal is to extend the technique developed by Rogovchenko and Tuncay ([1], [2]) for (1.2) and (1.3) to Eq. (1.1).

2. Main Results

Theorem 2.1. *Suppose that (C1)-(C4) are satisfied. Suppose also that there exist functions* $H \in \mathcal{P}, g \in C^1([t_0,\infty);\mathbb{R})$ *and* $\chi \in C([t_0,\infty);\mathbb{R})$ *such that (1.4) holds and for all* $t > t_0$*, all* $T \geq t_0$*, and for some* $\beta > 1$*,*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \frac{\chi_+^2(s)}{a(s)v(s)} ds = \infty, \tag{2.1}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} \left(H(t,s)\phi(s) - \frac{\beta\gamma c_1}{4} a(s)v(s)h^2(t,s) \right) ds \geq \chi(T), \tag{2.2}$$

where

$$\phi(t) = v(t)\left(\mu q(t) + \frac{g^2(t)}{\gamma c_1 a(t)} - \frac{p(t)g(t)}{c_1 a(t)} - g'(t) + \left(\frac{1}{c_1} - \frac{1}{c}\right)\frac{\gamma p^2(t)}{4a(t)}\right), \tag{2.3}$$

$$v(t) = \exp\left(-\frac{2}{c_1}\int^{t}\left(\frac{g(s)}{\gamma a(s)} - \frac{p(s)}{2a(s)}\right)ds\right), \tag{2.4}$$

and

$$\chi_+(s) = \max(\chi(s), 0).$$

Then Eq. (1.1) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of Eq. (1.1). Then there exists a $T_0 \geq t_0$ such that $x(t) \neq 0$ for all $t \geq T_0$. Without loss of generality, we may assume that $x(t) > 0$ for all $t \geq T_0$, for some $T_0 \geq t_0$. A similar argument holds for the case when $x(t)$ is eventually negative. As in [3], define a generalized Riccati transformation by

$$u(t) = v(t)\left[\frac{a(t)\Psi(x(t))\,k(x'(t))}{x(t)} + g(t)\right] \quad \text{for all } t \geq T_0. \tag{2.5}$$

Then differentiating (2.5) and using Eq. (1.1), we obtain

$$u'(t) = \frac{v'(t)}{v(t)}u(t) + v(t)\left[\frac{-p(t)k(x'(t))}{x(t)} - \frac{q(t)f(x(t))}{x(t)} - \frac{a(t)\Psi(x(t))\,k(x'(t))\,x'(t)}{x^2(t)} + g'(t)\right].$$

In view of (C1)-(C4), we conclude that for all $t \geq T_0$,

$$\begin{aligned}
u'(t) \quad &\leq \left[-\tfrac{2g(t)}{\gamma c_1 a(t)} + \tfrac{p(t)}{c_1 a(t)}\right] u(t) \\
&+v(t)\left[-p(t)\tfrac{k\left(x'(t)\right)}{x(t)} - \mu q(t) - \tfrac{a(t)\Psi(x(t))k^2\left(x'(t)\right)}{\gamma x^2(t)} + g'(t)\right] \\
&= \left[-\tfrac{2g(t)}{\gamma c_1 a(t)} + \tfrac{p(t)}{c_1 a(t)}\right] u(t) + v(t)\left[-p(t)\left(\tfrac{1}{a(t)\Psi(x(t))}\left(\tfrac{u(t)}{v(t)} - g(t)\right)\right) - \mu q(t)\right] \\
&+v(t)\left[-\tfrac{a(t)\Psi(x(t))}{\gamma}\left(\tfrac{1}{a(t)\Psi(x(t))}\left(\tfrac{u(t)}{v(t)} - g(t)\right)\right)^2 + g'(t)\right] \\
&= -\mu q(t)v(t) - \tfrac{\left[u(t)+\frac{\gamma}{2}v(t)p(t)-v(t)g(t)\right]^2}{\gamma a(t)\Psi(x(t))v(t)} + \tfrac{\gamma v(t)p^2(t)}{4a(t)\Psi(x(t))} \\
&+v(t)g'(t) + \left[-\tfrac{2g(t)}{\gamma c_1 a(t)} + \tfrac{p(t)}{c_1 a(t)}\right] u(t) \\
&\leq -\mu q(t)v(t) - \tfrac{u^2(t)}{\gamma c_1 a(t)v(t)} + \tfrac{p(t)g(t)v(t)}{c_1 a(t)} - \tfrac{v(t)g^2(t)}{\gamma c_1 a(t)} \\
&+v(t)g'(t) + \left(\tfrac{1}{c} - \tfrac{1}{c_1}\right)\tfrac{\gamma v(t)p^2(t)}{4a(t)}.
\end{aligned}$$

Using (2.3) in the latter inequality, we have, for all $t \geq T_0$,

$$u'(t) \leq -\phi(t) - \frac{u^2(t)}{\gamma c_1 a(t)v(t)} \tag{2.6}$$

Multiplying both sides of (2.6) by $H(t,s)$, integrating it with respect to s from T to t, and using the properties of the function $H(t,s)$, we get, for all $t \geq T \geq T_0$,

$$\begin{aligned}
\int_T^t H(t,s)\phi(s)ds \quad &\leq -\int_T^t H(t,s)u'(s)ds - \int_T^t H(t,s)\tfrac{u^2(s)}{\gamma c_1 a(s)v(s)}ds \\
&= -H(t,s)u(s)\,|_T^t - \int_T^t \left[-\tfrac{\partial H(t,s)}{\partial s}u(s) + H(t,s)\tfrac{u^2(s)}{\gamma c_1 a(s)v(s)}\right]ds \\
&= H(t,T)u(T) - \int_T^t \left[h(t,s)\sqrt{H(t,s)}u(s) + H(t,s)\tfrac{u^2(s)}{\gamma c_1 a(s)v(s)}\right]ds.
\end{aligned} \tag{2.7}$$

Then, for any $\beta > 1$, (2.7) gives

$$\begin{aligned}
\int_T^t H(t,s)\phi(s)ds \quad &\leq H(t,T)u(T) - \int_T^t \left(\sqrt{\tfrac{H(t,s)}{\beta\gamma c_1 a(s)v(s)}}u(s) + \tfrac{1}{2}\sqrt{\beta\gamma c_1 a(s)v(s)}h(t,s)\right)^2 ds \\
&+\tfrac{\beta\gamma c_1}{4}\int_T^t a(s)v(s)h^2(t,s)ds - \int_T^t \tfrac{(\beta-1)H(t,s)}{\beta\gamma c_1 a(s)v(s)}u^2(s)ds
\end{aligned} \tag{2.8}$$

and, for all $t \geq T \geq T_0$,

$$\int_T^t \left(H(t,s)\phi(s) - \tfrac{\beta\gamma c_1}{4} a(s)v(s)h^2(t,s)\right) ds \leq H(t,T)u(T)$$

$$- \int_T^t \left(\sqrt{\tfrac{H(t,s)}{\beta\gamma c_1 a(s)v(s)}}u(s) + \tfrac{1}{2}\sqrt{\beta\gamma c_1 a(s)v(s)}h(t,s)\right)^2 ds - \int_T^t \tfrac{(\beta-1)H(t,s)}{\beta\gamma c_1 a(s)v(s)} u^2(s)ds. \tag{2.9}$$

From (2.9),

$$\tfrac{1}{H(t,T)} \int_T^t \left(H(t,s)\phi(s) - \tfrac{\beta\gamma c_1}{4} a(s)v(s)h^2(t,s)\right) ds \leq u(T) - \tfrac{1}{H(t,T)} \int_T^t \tfrac{(\beta-1)H(t,s)}{\beta\gamma c_1 a(s)v(s)} u^2(s)ds$$

$$- \tfrac{1}{H(t,T)} \int_T^t \left(\sqrt{\tfrac{H(t,s)}{\beta\gamma c_1 a(s)v(s)}}u(s) + \tfrac{1}{2}\sqrt{\beta\gamma c_1 a(s)v(s)}h(t,s)\right)^2 ds$$

$$\leq u(T) - \tfrac{1}{H(t,T)} \int_T^t \tfrac{(\beta-1)H(t,s)}{\beta\gamma c_1 a(s)v(s)} u^2(s)ds.$$

Therefore, for all $t > T \geq T_0$,

$$\limsup_{t\to\infty} \tfrac{1}{H(t,T)} \int_T^t \left(H(t,s)\phi(s) - \tfrac{\beta\gamma c_1}{4} a(s)v(s)h^2(t,s)\right) ds$$

$$\leq u(T) - \liminf_{t\to\infty} \tfrac{1}{H(t,T)} \int_T^t \tfrac{(\beta-1)H(t,s)}{\beta\gamma c_1 a(s)v(s)} u^2(s)ds. \tag{2.10}$$

It follows from (2.2) that

$$u(T) \geq \chi(T) + \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \frac{(\beta-1)H(t,s)}{\beta\gamma c_1 a(s)v(s)} u^2(s)ds,$$

for all $T \geq T_0$ and for any $\beta > 1$. This shows that

$$u(T) \geq \chi(T), \qquad \text{for all } T \geq T_0 \tag{2.11}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,T_0)} \int_{T_0}^t \frac{H(t,s)}{a(s)v(s)} u^2(s)ds \leq \frac{\beta\gamma c_1}{(\beta-1)} \left(u(T_0) - \chi(T_0)\right) < \infty. \tag{2.12}$$

We want to prove that

$$\int_{T_0}^\infty \frac{u^2(s)}{a(s)v(s)} ds < \infty. \tag{2.13}$$

Suppose to the contrary that

$$\int_{T_0}^\infty \frac{u^2(s)}{a(s)v(s)} ds = \infty. \tag{2.14}$$

By (1.4), there exists a positive constant ρ such that

$$\inf_{s\geq t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right] > \rho. \tag{2.15}$$

From (2.15),

$$\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)} > \rho > 0,$$

and there exists a $T_2 \geq T_1$ such that $H(t,T_1)/H(t,t_0) \geq \rho$, for all $t \geq T_2$. On the other hand, by (2.14) for any positive number δ, there exists a $T_1 > T_0$, such that, for all $t \geq T_1$,

$$\int_{T_0}^{t}\frac{u^2(s)}{a(s)v(s)}ds \geq \frac{\delta}{\rho}.$$

Using integration by parts, we obtain, for all $t \geq T_1$,

$$\begin{aligned}\frac{1}{H(t,T_0)}\int_{T_0}^{t}\frac{H(t,s)}{a(s)v(s)}u^2(s)ds &= \frac{1}{H(t,T_0)}\int_{T_0}^{t}\left[-\frac{\partial H(t,s)}{\partial s}\right]\left[\int_{T_0}^{s}\frac{u^2(\tau)}{a(\tau)v(\tau)}d\tau\right]ds\\ &\geq \frac{\delta}{\rho}\frac{1}{H(t,T_0)}\int_{T_1}^{t}\left[-\frac{\partial H(t,s)}{\partial s}\right]ds\\ &= \frac{\delta}{\rho}\frac{H(t,T_1)}{H(t,T_0)}.\end{aligned}$$

This implies that

$$\frac{1}{H(t,T_0)}\int_{T_0}^{t}\frac{H(t,s)}{a(s)v(s)}u^2(s)ds \geq \delta \quad \text{for all } t \geq T_2.$$

Since δ is an arbitrary positive constant,

$$\liminf_{t\to\infty}\frac{1}{H(t,T_0)}\int_{T_0}^{t}\frac{H(t,s)}{a(s)v(s)}u^2(s)ds = +\infty,$$

which conradicts (2.12). Because of that, (2.13) holds, and from (2.11)

$$\int_{T_0}^{\infty}\frac{\chi_+^2(s)}{a(s)v(s)}ds \leq \int_{T_0}^{\infty}\frac{u^2(s)}{a(s)v(s)}ds < +\infty,$$

which contradicts (2.1). Therefore, Eq. (1.1) is oscillatory.

Following the classical ideas of Kamenev [4], we define $H(t,s)$ as

$$H(t,s) = (t-s)^{n-1}, \qquad (t,s) \in D$$

where n is an integer and $n > 2$. Evidently, $H \in \mathcal{P}$ and

$$h(t,s) = (n-1)\,(t-s)^{(n-3)/2}, \quad (t,s) \in D.$$

Thus, by Theorem 2.1 we have the following oscillation result.

Corollary 2.2. *Let (C1)-(C4) hold. Suppose that there exist functions* $g \in C^1([t_0,\infty);\mathbb{R})$ *and* $\chi \in C([t_0,\infty);\mathbb{R})$ *such that, for all* $T \geq t_0$*, for some integer* $n > 2$*, and for some* $\beta > 1$*,*

$$\limsup_{t\to\infty} t^{1-n} \int_T^t \left((t-s)^{n-1}\phi(s) - \frac{\beta\gamma c_1 (n-1)^2}{4} a(s)v(s)(t-s)^{n-3} \right) ds \geq \chi(T)$$

and (2.1) holds, where $\phi(t)$ *and* $v(t)$ *are as in Theorem 2.1. Then Eq. (1.1) is oscillatory.*

Example 2.3. *Consider the differential equation of the form*

$$\left[t^2\left(\tfrac{1}{2}+\tfrac{e^{-|x(t)|}}{2}\right)\tfrac{x'(t)}{1+x'^2(t)}\right]' + 2t^3\tfrac{x'(t)}{1+x'^2(t)}$$
$$+\left(2+2t^4+6t^2-6t^2\sin^2 t\right)x(t)\left(1+x^4(t)\right)=0, \qquad (2.16)$$

where $x \in (-\infty,\infty)$ *and* $t \geq 1$*. Since* $\frac{f(x)}{x} = 1 + x^4 \geq 1 = \mu$, $c = 1/2, c_1 = 1$ *the assumptions (C1)-(C4) hold for* $\gamma = 1$*. Let us apply Corollary 2.2 with* $\beta = 2$ *and* $g(t) = t^3$*, then* $v(t) = 1$ *and* $\phi(t) = 2 + 3t^2 - 6t^2\sin^2 t$*. A direct computation yields with* $n = 3$

$$\limsup_{t\to\infty} t^{1-n} \int_T^t \left((t-s)^{n-1}\phi(s) - \tfrac{\beta\gamma c_1 (n-1)^2}{4} a(s)v(s)(t-s)^{n-3} \right) ds$$

$$= \limsup_{t\to\infty}\tfrac{1}{t^2}\int_T^t \left((t-s)^2\left(2+3s^2-6s^2\sin^2 s\right) - \tfrac{2\cdot 1\cdot 1\cdot 4}{4}s^2\right) ds$$

$$= \tfrac{3}{4} - 2T - 3T^2\sin T\cos T - \tfrac{3}{2}T\cos^2 T + \tfrac{3}{2}T\sin^2 T + \tfrac{3}{2}\sin T\cos T = \chi(T).$$

The relation

$$\frac{\chi_+^2(t)}{a(t)v(t)} = O(t^2) \qquad \text{as} \quad t\to\infty$$

implies that the condition (2.1) is satisfied. Therefore, Eq. (2.16) is oscillatory by Corollary 2.2. Note that in this example

$$\limsup_{t\to\infty}\frac{1}{t^2}\int_1^t \frac{\beta\gamma c_1}{4} a(s)v(s)h^2(t,s)ds = \limsup_{t\to\infty}\frac{1}{t^2}\int_1^t \frac{2\cdot 1\cdot 1}{4}s^2\cdot 1\cdot 4ds = \infty. \quad (2.17)$$

(2.17) shows that we do not need to impose any condition similar to the condition (1.5) in Theorem 1.1.

Theorem 2.4. *Suppose that (C1)-(C4) and (2.1) are satisfied. Suppose also that there exist functions* $H \in \mathcal{P}, g \in C^1([t_0,\infty);\mathbb{R})$ *and* $\chi \in C([t_0,\infty);\mathbb{R})$ *such that (1.4) holds and, for all* $T \geq t_0$*, and for some* $\beta > 1$*,*

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \left(H(t,s)\phi(s) - \frac{\beta\gamma c_1}{4}a(s)v(s)h^2(t,s)\right) ds \geq \chi(T)$$

where $\phi(t), v(t)$ *and* $\chi_+(t)$ *are the same as in Theorem 2.1. Then Eq. (1.1) is oscillatory.*

Proof. Since

$$\begin{aligned}\chi(T) &\le \liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \left(H(t,s)\phi(s)-\frac{\beta\gamma c_1}{4}a(s)v(s)h^2(t,s)\right)ds\\ &\le \limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \left(H(t,s)\phi(s)-\frac{\beta\gamma c_1}{4}a(s)v(s)h^2(t,s)\right)ds,\end{aligned}$$

that Eq. (1.1) is oscillatory follows readily from Theorem 2.1.

From now on, we present a new set of oscillation theorems. We want to point out that these theorems differ from Theorem 2.1 and 2.4. That is, they are neither a special case nor a generalized form of Theorem 2.1 and 2.4.

Theorem 2.5. *Let (C1)-(C4) hold. Suppose that there exists a function* $g \in C^1([t_0,\infty);\mathbb{R})$ *such that, for some* $\beta \ge 1$ *and for some* $H \in \mathcal{P}$

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t \left(H(t,s)\phi(s)-H(t,s)\frac{\gamma p^2(s)v(s)}{2c_1a(s)}-\frac{\beta\gamma c_1}{2}a(s)v(s)h^2(t,s)\right)ds=\infty, \tag{2.18}$$

where $\phi(s)$ *is defined by (2.3) and*

$$v(t)=\exp\left(-\frac{2}{c_1}\int^t \frac{g(s)}{\gamma a(s)}ds\right). \tag{2.19}$$

Then Eq. (1.1) is oscillatory.

Proof. Let $x(t)$ be a non-oscillatory solution of the differential equation (1.1). Then there exists a $T_0 \ge t_0$ such that $x(t) \ne 0$ for all $t \ge T_0$. Without loss of generality, we may assume that $x(t) > 0$ for all $t \ge T_0$. Define the function $u(t)$ as in (2.5), where $v(t)$ is given by (2.19). Then differentiating (2.5) and using (1.1) , we have

$$\begin{aligned}u'(t) &= \tfrac{v'(t)}{v(t)}u(t)\\ &+v(t)\left[\tfrac{-p(t)k\left(x'(t)\right)}{x(t)}-\tfrac{q(t)f(x(t))}{x(t)}-\tfrac{a(t)\Psi(x(t))k\left(x'(t)\right)x'(t)}{x^2(t)}+g'(t)\right].\end{aligned} \tag{2.20}$$

Using (C1)-(C4) in (2.20), we easily get

$$u'(t)\le -\phi(t)-\frac{p(t)u(t)}{c_1a(t)}-\frac{u^2(t)}{\gamma c_1a(t)v(t)}, \tag{2.21}$$

where $\phi(t)$ is defined by (2.3). On the other hand, since the inequality

$$mz-nz^2\le\frac{m^2}{2n}-\frac{n}{2}z^2, n>0, m,z\in R$$

which holds for all $n > 0$ and all $m, z \in \mathbb{R}$, we see from (2.21) that

$$\phi(t)-\frac{p^2(t)\gamma v(t)}{2c_1a(t)}\le -u'(t)-\frac{u^2(t)}{2\gamma c_1a(t)v(t)} \tag{2.22}$$

for all $t \geq T_0$. Multiplying (2.22) by $H(t,s)$ and integrating from T to t, we have for some $\beta \geq 1$ and for all $t \geq T \geq T_0$,

$$\int_T^t H(t,s)\left(\phi(s) - \frac{\gamma p^2(s)v(s)}{2c_1 a(s)}\right) ds$$

$$\leq H(t,T)u(T) - \int_T^t \left(\sqrt{\frac{H(t,s)}{2\beta\gamma c_1 a(s)v(s)}}u(s) + \frac{1}{2}\sqrt{2\beta\gamma c_1 a(s)v(s)}h(t,s)\right)^2 ds$$

$$+\frac{\beta\gamma c_1}{2}\int_T^t a(s)v(s)h^2(t,s)ds - \int_T^t \frac{(\beta-1)H(t,s)}{2\beta\gamma c_1 a(s)v(s)}u^2(s)ds.$$

This implies that, for all $t \geq T \geq T_0$,

$$\int_T^t \left(H(t,s)\phi(s) - H(t,s)\frac{\gamma p^2(s)v(s)}{2c_1 a(s)} - \frac{\beta\gamma c_1}{2}a(s)v(s)h^2(t,s)\right) ds \leq H(t,T)u(T)$$

$$-\int_T^t \frac{(\beta-1)H(t,s)}{2\beta\gamma c_1 a(s)v(s)}u^2(s)ds - \int_T^t \left(\sqrt{\frac{H(t,s)}{2\beta\gamma c_1 a(s)v(s)}}u(s) + \frac{1}{2}\sqrt{2\beta\gamma c_1 a(s)v(s)}h(t,s)\right)^2 ds.$$

Using the properties of $H(t,s)$, we see that for every $t \geq T_0$

$$\int_{T_0}^t \left(H(t,s)\phi(s) - H(t,s)\frac{\gamma p^2(s)v(s)}{2c_1 a(s)} - \frac{\beta\gamma c_1}{2}a(s)v(s)h^2(t,s)\right) ds$$
$$\leq H(t,T_0)u(T_0) \leq H(t,T_0)\left|u(T_0)\right| \leq H(t,t_0)\left|u(T_0)\right|.$$

Therefore,

$$\int_{t_0}^t \left(H(t,s)\phi(s) - H(t,s)\frac{\gamma p^2(s)v(s)}{2c_1 a(s)} - \frac{\beta\gamma c_1}{2}a(s)v(s)h^2(t,s)\right) ds$$

$$= \int_{t_0}^{T_0} \left(H(t,s)\phi(s) - H(t,s)\frac{\gamma p^2(s)v(s)}{2c_1 a(s)} - \frac{\beta\gamma c_1}{2}a(s)v(s)h^2(t,s)\right) ds$$

$$+ \int_{T_0}^t \left(H(t,s)\phi(s) - H(t,s)\frac{\gamma p^2(s)v(s)}{2c_1 a(s)} - \frac{\beta\gamma c_1}{2}a(s)v(s)h^2(t,s)\right) ds$$

$$\leq H(t,t_0)\left[\int_{t_0}^{T_0}\left|\phi(s)\right| ds + \left|u(T_0)\right|\right]$$

for all $t \geq T_0$. This gives

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t \left(H(t,s)\phi(s) - H(t,s)\frac{\gamma p^2(s)v(s)}{2c_1 a(s)} - \frac{\beta\gamma c_1}{2}a(s)v(s)h^2(t,s)\right) ds$$

$$\leq \int_{t_0}^{T_0}\left|\phi(s)\right| ds + \left|u(T_0)\right| < +\infty,$$

which contradicts with the assumption (2.18) of the theorem. This completes the proof of Theorem 2.5.

Therefore, by Theorem 2.5 we have the following oscillation result.

Corollary 2.6. *Let (C1)-(C4) hold. Suppose that there exists a function* $g \in C^1([t_0,\infty);\mathbb{R})$ *such that, for some integer* $n > 2$ *and some* $\beta \geq 1$

$$\limsup_{t\to\infty} t^{1-n} \int_{t_0}^{t} \left[(t-s)^{n-1} \left(\phi(s) - \frac{\gamma p^2(s) v(s)}{2c_1 a(s)} \right) - \frac{\beta\gamma c_1 (n-1)^2}{2} a(s) v(s) (t-s)^{n-3} \right] ds = \infty, \tag{2.23}$$

where $\phi(t)$ *and* $v(t)$ *are as in Theorem 2.5. Then Eq. (1.1) is oscillatory.*

Example 2.7. *For* $t \geq 1$, *consider the nonlinear differential equation*

$$\left[\left(1+\sin^2 t\right) \tfrac{2+x^2(t)}{1+x^2(t)} \tfrac{x'(t)}{1+x'^2(t)} \right] + t\sqrt{1+\sin^2 t}\, \tfrac{x'(t)}{1+x'^2(t)}$$

$$+ \left(2+\tfrac{3}{8}t^2\right) x(t) \left(1+\tfrac{1}{2+x^2(t)}\right) = 0, \tag{2.24}$$

Obviously, for all $x \in (-\infty,\infty)$ *one has* $1 \leq \Psi(x) \leq 2$ *and* $f(x)/x \geq 1 = \mu$. *Let* $g(t) = 0$ *and* $\gamma = 1$, *then* $v(t) = 1$, *and* $\phi(t) = 2 + \frac{1}{4}t^2$. *Let us take* $n = 3$, *and for any* $\beta \geq 1$,

$$\limsup_{t\to\infty} t^{1-n} \int_1^t \left[(t-s)^{n-1} \left(\phi(s) - \tfrac{\gamma p^2(s) v(s)}{2c_1 a(s)} \right) - \tfrac{\beta\gamma c_1 (n-1)^2}{2} a(s) v(s) (t-s)^{n-3} \right] ds$$

$$= \limsup_{t\to\infty} \tfrac{1}{t^2} \int_1^t \left[(t-s)^2 \times 2 - 4\beta\left(1+\sin^2 s\right) \right] ds = \infty$$

Therefore, Eq. (2.24) is oscillatory by Corollary 2.6.

Theorem 2.8. *Suppose that (C1)-(C4) are satisfied. Suppose also that there exist functions* $H \in \mathcal{P}, g \in C^1([t_0,\infty);\mathbb{R})$ *and* $\chi \in C([t_0,\infty);\mathbb{R})$ *such that (1.4) holds, and for all* $t > t_0$, *any* $T \geq t_0$, *and for some* $\beta > 1$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left(H(t,s)\phi(s) - H(t,s)\frac{\gamma p^2(s) v(s)}{2c_1 a(s)} - \frac{\beta\gamma c_1}{2} a(s) v(s) h^2(t,s) \right) ds \geq \chi(T), \tag{2.25}$$

where $\phi(t)$ *and* $v(t)$ *are the same as in Theorem 2.5. If (2.1) is satisfied, Eq. (1.1) is oscillatory.*

Proof. The proof of this theorem is similar to that of the Theorem 2.1 and hence it is omitted.

Theorem 2.9. *Let all assumptions of Theorem 2.8 satisfied except that condition (2.25) be replaced with*

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left(H(t,s)\phi(s) - H(t,s)\frac{\gamma p^2(s) v(s)}{2c_1 a(s)} - \frac{\beta\gamma c_1}{2} a(s) v(s) h^2(t,s) \right) ds \geq \chi(T).$$

Then Eq. (1.1) is oscillatory.

Proof. By a similar argument to that in the proof of Theorem 2.4, one can complete the proof of this theorem. Therefore, we omit the detailed proof for the theorem.

Remark 2.10. *If* $f(x) = x$, *then* $q(t) \geq 0$ *is not necessary in the above Theorems.*

Remark 2.11. *If (2.5) is replaced by*

$$u(t) = v(t)\left[\frac{a(t)\Psi\left(x\left(t\right)\right)k\left(x'(t)\right)}{f(x\left(t\right))} + g(t)\right],$$

then, without putting any sign condition on $q(t)$, we can obtain similar oscillation results that are derived in the main results section of this paper for Eq. (1.1). But in this case the assumption $f'(x) \geq \sigma > 0$ is necessary.

Remark 2.12. *When $k(x') = x'$, it is easy to see that Theorems 2.8 and 2.9 reduce to Theorems 9 and 10 of* [1] *with $\gamma = 1$, respectively.*

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