Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 4 Issue 3 (2012), Pages 99-114.

INVARIANT SUBMANIFOLDS OF KENMOTSU MANIFOLDS ADMITTING QUARTER SYMMETRIC METRIC CONNECTION

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Abstract. The object of this paper is to study invariant submanifolds M of Kenmotsu manifolds $\widetilde{M}$ admitting a quarter symmetric metric connection and to show that M admits quarter symmetric metric connection. Further it is proved that the second fundamental forms σ and $\overline{\sigma}$ with respect to Levi-Civita connection and quarter symmetric metric connection coincide. Also it is shown that if the second fundamental form σ is recurrent, 2-recurrent, generalized 2-recurrent, semiparallel, pseudoparallel, Ricci-generalized pseudoparallel and M has parallel third fundamental form with respect to quarter symmetric metric connection, then M is totally geodesic with respect to Levi-Civita connection.

1. Quarter symmetric metric connection

The study of the geometry of invariant submanifolds of Kenmotsu manifolds is carried out by C.S. Bagewadi and V.S. Prasad [4], S. Sular and C. Ozgur [13] and M. Kobayashi [10]. The author [10] has shown that the submanifold M of a Kenmotsu manifold $\widetilde{M}$ has parallel second fundamental form if and only if M is totally geodesic. The authors [4, 11, 13] have shown the equivalence of totally geodesicity of M with parallelism and semiparallelism of σ. Also they have shown that invariant submanifold of Kenmotsu manifold carries Kenmotsu structure and if $K \leq \widetilde{K}$, then M is totally geodesic. Further the author [13] have shown the equivalence of totally geodesicity of M, if σ is recurrent, M has parallel third fundamental form and σ is generalized 2-recurrent. Further the study has been carried out by B.S. Anitha and C.S. Bagewadi [2]. In this paper we extend the results to invariant submanifolds M of Kenmotsu manifolds admitting quarter symmetric metric connection.

We know that a connection ∇ on a manifold M is called a metric connection if there is a Riemannian metric g on M if $\nabla g = 0$ otherwise it is non-metric. In 1924, Friedman and J.A. Schouten [7] introduced the notion of a semi-symmetric linear

2000 *Mathematics Subject Classification.* 53D15, 53C21, 53C25, 53C40.
Key words and phrases. Invariant submanifolds; Sasakian manifold; Quarter symmetric metric connection.

Submitted July 23, 2012. Published date September 10, 2012.

connection on a differentiable manifold. In 1932, H.A. Hayden [9] introduced the idea of metric connection with torsion on a Riemannian manifold. In 1970, K. Yano [14] studied some curvature tensors and conditions for semi-symmetric connections in Riemannian manifolds. In 1975's S. Golab [8] defined and studied quarter symmetric linear connection on a differentiable manifold. A linear connection $\widetilde{\nabla}$ in an n-dimensional Riemannian manifold is said to be a quarter symmetric connection [8] if its torsion tensor T is of the form

$$T(X,Y) = \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] = A(Y)KX - A(X)KY, \tag{1.1}$$

where A is a 1-form and K is a tensor field of type $(1,1)$. If a quarter symmetric linear connection $\overline{\nabla}$ satisfies the condition

$$(\overline{\nabla}_X g)(Y,Z) = 0,$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields on the manifold M, then $\overline{\nabla}$ is said to be a quarter symmetric metric connection. For a contact metric manifold admitting quarter symmetric connection, we can take $A = \eta$ and $K = \phi$ to write (1.1) in the form:

$$T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y. \tag{1.2}$$

Now we obtain the relation between Levi-civita connection ∇ and quarter symmetric metric connection $\overline{\nabla}$ of a contact metric manifold as follows:

The relation between linear connection $\overline{\nabla}$ and a Riemannian connection ∇ of an almost contact metric manifold symmetric [8] is given as follows.

Let $\overline{\nabla}$ be a linear connection and ∇ be a Riemannian connection of an almost contact metric manifold as given below

$$\overline{\nabla}_X Y = \nabla_X Y + H(X,Y), \tag{1.3}$$

where H is a tensor of type $(1,1)$. For $\overline{\nabla}$ to be a quarter symmetric metric connection in M, we have

$$H(X,Y) = \frac{1}{2}[T(X,Y) + T'(X,Y) + T'(Y,X)] \quad \text{and} \tag{1.4}$$

$$g(T'(X,Y),Z) = g(T(Z,X),Y). \tag{1.5}$$

From (1.2) and (1.5), we get

$$T'(X,Y) = g(X,\phi Y)\xi - \eta(X)\phi Y. \tag{1.6}$$

Using (1.2) and (1.6) in (1.4), we get

$$H(X,Y) = -\eta(X)\phi Y. \tag{1.7}$$

Hence a quarter symmetric metric connection $\overline{\nabla}$ of an almost contact metric manifold is given by

$$\overline{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{1.8}$$

The covariant differential of the p^{th} order, $p \geq 1$, of a $(0,k)$-tensor field T, $k \geq 1$, defined on a Riemannian manifold (M,g) with the Levi-Civita connection ∇, is denoted by $\nabla^p T$. The tensor T is said to be *recurrent* and *2-recurrent* [12], if the following conditions hold on M, respectively,

$$(\nabla T)(X_1,...,X_k;X)T(Y_1,...,Y_k) = (\nabla T)(Y_1,...,Y_k;X)T(X_1,...,X_k), \tag{1.9}$$
$$(\nabla^2 T)(X_1,...,X_k;X,Y)T(Y_1,...,Y_k) = (\nabla^2 T)(Y_1,...,Y_k;X,Y)T(X_1,...,X_k),$$

where $X,Y,X_1,Y_1,...,X_k,Y_k \in TM$. From (1.9) it follows that at a point $x \in M$, if the tensor T is non-zero, then there exists a unique 1-form ϕ and a $(0,2)$-tensor ψ, defined on a neighborhood U of x such that

$$\nabla T = T \otimes \phi, \quad \phi = d(\log \|T\|) \tag{1.10}$$

and

$$\nabla^2 T = T \otimes \psi, \tag{1.11}$$

hold on U, where $\|T\|$ denotes the norm of T and $\|T\|^2 = g(T,T)$. The tensor T is said to be *generalized 2-recurrent* if

$$((\nabla^2 T)(X_1,...,X_k;X,Y) - (\nabla T \otimes \phi)(X_1,...,X_k;X,Y))T(Y_1,...,Y_k)$$
$$= ((\nabla^2 T)(Y_1,...,Y_k;X,Y) - (\nabla T \otimes \phi)(Y_1,...,Y_k;X,Y))T(X_1,...,X_k),$$

holds on M, where ϕ is a 1-form on M. From this it follows that at a point $x \in M$ if the tensor T is non-zero, then there exists a unique $(0,2)$-tensor ψ, defined on a neighborhood U of x, such that

$$\nabla^2 T = \nabla T \otimes \phi + T \otimes \psi, \tag{1.12}$$

holds on U.

2. Isometric Immersion

Let $f : (M,g) \to (\widetilde{M},\widetilde{g})$ be an isometric immersion from an n-dimensional Riemannian manifold (M,g) into $(n+d)$-dimensional Riemannian manifold $(\widetilde{M},\widetilde{g})$, $n \geq 2$, $d \geq 1$. We denote by ∇ and $\widetilde{\nabla}$ as Levi-Civita connection of M^n and $\widetilde{M}^{n+d}$

respectively. Then the formulas of Gauss and Weingarten are given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{2.1}$$
$$\widetilde{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \tag{2.2}$$

for any tangent vector fields X, Y and the normal vector field N on M, where σ, A and $\nabla^{\perp}$ are the second fundamental form, the shape operator and the normal connection respectively. If the second fundamental form σ is identically zero, then the manifold is said to be *totally geodesic*. The second fundamental form σ and A_N are related by

$$\widetilde{g}(\sigma(X,Y),N) = g(A_N X, Y),$$

for tangent vector fields X, Y. The first and second covariant derivatives of the second fundamental form σ are given by

$$(\widetilde{\overline{\nabla}}_X \sigma)(Y,Z) = \nabla^{\perp}_X(\sigma(Y,Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z), \tag{2.3}$$
$$\begin{aligned}(\widetilde{\overline{\nabla}}^2\sigma)(Z,W,X,Y) &= (\widetilde{\overline{\nabla}}_X \widetilde{\overline{\nabla}}_Y \sigma)(Z,W), \\ &= \nabla^{\perp}_X((\widetilde{\overline{\nabla}}_Y\sigma)(Z,W)) - (\widetilde{\overline{\nabla}}_Y\sigma)(\nabla_X Z, W) \\ &\quad -(\widetilde{\overline{\nabla}}_X\sigma)(Z,\nabla_Y W) - (\widetilde{\overline{\nabla}}_{\nabla_X Y}\sigma)(Z,W)\end{aligned} \tag{2.4}$$

respectively, where $\widetilde{\overline{\nabla}}$ is called the *vander Waerden-Bortolotti connection* of M [6]. If $\widetilde{\overline{\nabla}}\sigma = 0$, then M is said to have *parallel second fundamental form* [6]. We next define endomorphisms $R(X,Y)$ and $X \wedge_B Y$ of $\chi(M)$ by

$$\begin{aligned}R(X,Y)Z &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \\ (X \wedge_B Y)Z &= B(Y,Z)X - B(X,Z)Y\end{aligned} \tag{2.5}$$

respectively, where $X, Y, Z \in \chi(M)$ and B is a symmetric $(0,2)$-tensor.

Now, for a $(0,k)$-tensor field T, $k \geq 1$ and a $(0,2)$-tensor field B on (M,g), we define the tensor $Q(B,T)$ by

$$\begin{aligned}Q(B,T)(X_1,...,X_k;X,Y) &= -(T(X \wedge_B Y)X_1,...,X_k) \\ &\quad - \cdots -T(X_1,...,X_{k-1}(X \wedge_B Y)X_k).\end{aligned} \tag{2.6}$$

Putting into the above formula $T = \sigma$ and $B = g$, $B = S$, we obtain the tensors $Q(g,\sigma)$ and $Q(S,\sigma)$.

3. Kenmotsu Manifolds

Let $\widetilde{M}$ be a n-dimensional almost contact metric manifold with structure (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form and g is the

Riemannian metric satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \tag{3.1}$$
$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \tag{3.2}$$

for all vector fields X, Y on M. If

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{3.3}$$
$$\nabla_X \xi = X - \eta(X)\xi, \tag{3.4}$$

where ∇ denotes the Riemannian connection of g, then (M, ϕ, ξ, η, g) is called an almost Kenmotsu manifold [3].

Example of Kenmotsu manifold: Consider the 3-dimensional manifold $M = \{(x, y, z) \in R^3 : z \neq 0\}$, where (x, y, z) are the standard coordinates in R^3. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame field on M given by

$$E_1 = z\frac{\partial}{\partial x}, \quad E_2 = \frac{z}{y}\frac{\partial}{\partial y}, \quad E_3 = -z\frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$g(E_1, E_2) = g(E_2, E_3) = g(E_1, E_3) = 0,$$
$$g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1.$$

The (ϕ, ξ, η) is given by

$$\eta = -\frac{1}{z}dz, \quad \xi = E_3 = \frac{\partial}{\partial z},$$
$$\phi E_1 = E_2, \quad \phi E_2 = -E_1, \quad \phi E_3 = 0.$$

The linearity property of ϕ and g yields that

$$\eta(E_3) = 1, \quad \phi^2 U = -U + \eta(U)E_3,$$
$$g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W),$$

for any vector fields U, W on M. By definition of Lie bracket, we have

$$[E_1, E_3] = E_1, \quad [E_2, E_3] = E_2.$$

The Levi-Civita connection with respect to above metric g be given by Koszula formula

$$2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Then we have,

$$\begin{array}{lll} \nabla_{E_1}E_1 = -E_3, & \nabla_{E_1}E_2 = 0, & \nabla_{E_1}E_3 = E_1, \\ \nabla_{E_2}E_1 = 0, & \nabla_{E_2}E_2 = -E_3, & \nabla_{E_2}E_3 = E_2, \\ \nabla_{E_3}E_1 = 0, & \nabla_{E_3}E_2 = 0, & \nabla_{E_3}E_3 = 0. \end{array}$$

The tangent vectors X and Y to M are expressed as linear combination of E_1, E_2, E_3, i.e., $X = a_1E_1 + a_2E_2 + a_3E_3$ and $Y = b_1E_1 + b_2E_2 + b_3E_3$, where a_i and b_j are scalars. Clearly (ϕ, ξ, η, g) and X, Y satisfy equations $(3.1), (3.2), (3.3)$ and (3.4). Thus M is a Kenmotsu manifold.

In Kenmotsu manifolds the following relations hold [3]:

$$R(X,Y)Z = \{g(X,Z)Y - g(Y,Z)X\}, \tag{3.5}$$

$$R(X,Y)\xi = \{\eta(X)Y - \eta(Y)X\}, \tag{3.6}$$

$$R(\xi,X)Y = \{\eta(Y)X - g(X,Y)\xi\}, \tag{3.7}$$

$$R(\xi,X)\xi = \{X - \eta(X)\xi\}, \tag{3.8}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{3.9}$$

$$Q\xi = -(n-1)\xi. \tag{3.10}$$

4. Invariant submanifolds of Kenmotsu manifolds admitting Quarter symmetric metric connection

A submanifold M of a Kenmotsu manifold $\widetilde{M}$ with a quarter symmetric metric connection is called an invariant submanifold of $\widetilde{M}$ with a quarter symmetric metric connection, if for each $x \in M$, $\phi(T_xM) \subset T_xM$. As a consequence, ξ becomes tangent to M. For an invariant submanifold of a Kenmotsu manifold with a quarter symmetric metric connection, we have

$$\sigma(X,\xi) = 0, \tag{4.1}$$

for any vector X tangent to M.

Let $\widetilde{M}$ be a Kenmotsu manifold admitting a quarter symmetric metric connection $\widetilde{\nabla}$.

Lemma 4.1. *Let M be an invariant submanifold of contact metric manifold $\widetilde{M}$ which admits quarter symmetric metric connection $\overline{\widetilde{\nabla}}$ and let σ and $\overline{\sigma}$ be the second fundamental forms with respect to Levi-Civita connection and quarter symmetric metric connection, then (1) M admits quarter symmetric metric connection, (2) the second fundamental forms with respect to $\widetilde{\nabla}$ and $\overline{\widetilde{\nabla}}$ are equal.*

Proof. We know that the contact metric structure $(\widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$ on $\widetilde{M}$ induces (ϕ, ξ, η, g) on invariant submanifold. By virtue of (1.8), we get

$$\overline{\widetilde{\nabla}}_X Y = \widetilde{\nabla}_X Y - \eta(X)\phi Y. \tag{4.2}$$

By using (2.1) in (4.2), we get

$$\overline{\overline{\widetilde{\nabla}}}_X Y = \nabla_X Y + \sigma(X,Y) - \eta(X)\phi Y. \tag{4.3}$$

Now Gauss formula (2.1) with respect to quarter symmetric metric connection is given by

$$\overline{\overline{\widetilde{\nabla}}}_X Y = \overline{\nabla}_X Y + \overline{\sigma}(X,Y). \tag{4.4}$$

Equating (4.3) and (4.4), we get (1.8) and

$$\overline{\sigma}(X,Y) = \sigma(X,Y). \tag{4.5}$$

□

Now we introduce the definitions of semiparallel, pseudoparallel and Ricci-generalized pseudoparallel with respect to quarter symmetric metric connection.

definition 4.2. *An immersion is said to be semiparallel, pseudoparallel and Ricci-generalized pseudoparallel with respect to quarter symmetric metric connection, respectively, if the following conditions hold for all vector fields X, Y tangent to M*

$$\begin{aligned} \overline{\overline{\widetilde{R}}}\cdot\sigma &= 0, &(4.6)\\ \overline{\overline{\widetilde{R}}}\cdot\sigma &= L_1 Q(g,\sigma) \quad \text{and} &(4.7)\\ \overline{\overline{\widetilde{R}}}\cdot\sigma &= L_2 Q(S,\sigma), &(4.8)\end{aligned}$$

where $\overline{\overline{\widetilde{R}}}$ denotes the curvature tensor with respect to connection $\overline{\overline{\widetilde{\nabla}}}$. Here L_1 and L_2 are functions depending on σ.

Lemma 4.3. *Let M be an invariant submanifold of Contact manifold $\widetilde{M}$ which admits quarter symmetric metric connection. Then Gauss and Weingarten formulae with respect to quarter symmetric metric connection are given by*

$$\begin{aligned} &tan(\overline{\overline{\widetilde{R}}}(X,Y)Z) = R(X,Y)Z - \eta(X)\phi\nabla_Y Z - \eta(Y)\nabla_X\phi Z \\ &+\eta(Y)\phi\nabla_X Z + \eta(X)\nabla_Y\phi Z + \eta([X,Y])\phi Z + tan\Big\{\overline{\overline{\widetilde{\nabla}}}_X\{\sigma(Y,Z)\} \\ &-\overline{\overline{\widetilde{\nabla}}}_Y\{\sigma(X,Z)\} + \overline{\overline{\widetilde{\nabla}}}_Y\eta(X)\phi Z - \overline{\overline{\widetilde{\nabla}}}_X\eta(Y)\phi Z\Big\}, \end{aligned} \tag{4.9}$$

$$\begin{aligned} &nor(\overline{\overline{\widetilde{R}}}(X,Y)Z) = \sigma(X,\nabla_Y Z) - \eta(Y)\sigma(X,\phi Z) - \sigma(Y,\nabla_X Z) \\ &+\eta(X)\sigma(Y,\phi Z) - \sigma([X,Y],Z) + nor\Big\{\overline{\overline{\widetilde{\nabla}}}_X\{\sigma(Y,Z)\} - \overline{\overline{\widetilde{\nabla}}}_Y\{\sigma(X,Z)\} \\ &+\overline{\overline{\widetilde{\nabla}}}_Y\eta(X)\phi Z - \overline{\overline{\widetilde{\nabla}}}_X\eta(Y)\phi Z\Big\}. \end{aligned} \tag{4.10}$$

Proof. The Riemannian curvature tensor $\widetilde{R}$ on $\widetilde{M}$ with respect to quarter symmetric metric connection is given by

$$\overline{\overline{\widetilde{R}}}(X,Y)Z = \overline{\overline{\widetilde{\nabla}}}_X \overline{\overline{\widetilde{\nabla}}}_Y Z - \overline{\overline{\widetilde{\nabla}}}_Y \overline{\overline{\widetilde{\nabla}}}_X Z - \overline{\overline{\widetilde{\nabla}}}_{[X,Y]} Z. \tag{4.11}$$

Using (1.8) and (2.1) in (4.11), we get

$$\begin{aligned}
\overline{\overline{\widetilde{R}}}(X,Y)Z &= R(X,Y)Z + \sigma(X, \nabla_Y Z) - \eta(X)\phi\nabla_Y Z + \overline{\overline{\widetilde{\nabla}}}_X \{\sigma(Y,Z)\} \\
&- \overline{\overline{\widetilde{\nabla}}}_X \eta(Y)\phi Z - \eta(Y)\nabla_X \phi Z - \eta(Y)\sigma(X,\phi Z) - \sigma(Y, \nabla_X Z) + \eta(Y)\phi\nabla_X Z \\
&- \overline{\overline{\widetilde{\nabla}}}_Y \{\sigma(X,Z)\} + \overline{\overline{\widetilde{\nabla}}}_Y \eta(X)\phi Z + \eta(X)\nabla_Y \phi Z + \eta(X)\sigma(Y,\phi Z) \\
&- \sigma([X,Y],Z) + \eta([X,Y])\phi Z.
\end{aligned} \tag{4.12}$$

Comparing tangential and normal part of (4.12), we obtain Gauss and Weingarten formulae (4.9) and (4.10). □

We obtain the condition in the following lemma for semi, pseudo and Ricci-generalized pseudoparallelism for invariant submanifold M of Sasakian manifold $\widetilde{M}$.

Lemma 4.4. *Let M be an invariant submanifold of Contact manifold $\widetilde{M}$ which admits quarter symmetric metric connection. Then*

$$\begin{aligned}
(\overline{\overline{\widetilde{R}}}(X,Y)\cdot\sigma)(U,V) &= R^{\perp}(X,Y)\sigma(U,V) - \sigma(R(X,Y)U,V) \\
&- \sigma(R(X,Y)U,V) - \nabla_X A_{\sigma(U,V)}Y - \sigma(X, A_{\sigma(U,V)}Y) \\
&+ \eta(X)\phi A_{\sigma(U,V)}Y - A_{\nabla^{\perp}_Y \sigma(U,V)}X - \eta(X)\phi\nabla^{\perp}_Y \sigma(U,V) \\
&- \overline{\overline{\widetilde{\nabla}}}_X \eta(Y)\phi\sigma(U,V) + \nabla_Y A_{\sigma(U,V)}X + \sigma(Y, A_{\sigma(U,V)}X) \\
&- \eta(Y)\phi A_{\sigma(U,V)}X + A_{\nabla^{\perp}_X \sigma(U,V)}Y + \eta(Y)\phi\nabla^{\perp}_X \sigma(U,V) \\
&+ \overline{\overline{\widetilde{\nabla}}}_Y \eta(X)\phi\sigma(U,V) + A_{\sigma(U,V)}[X,Y] + \eta([X,Y])\phi\sigma(U,V) \\
&- \sigma(\sigma(X,\nabla_Y U),V) + \eta(X)\sigma(\phi\nabla_Y U, V) - \sigma(\overline{\overline{\widetilde{\nabla}}}_X \{\sigma(Y,U)\}, V) \\
&+ \sigma(\overline{\overline{\widetilde{\nabla}}}_X \eta(Y)\phi U, V) + \eta(Y)\sigma(\nabla_X \phi U, V) + \eta(Y)\sigma(\sigma(X,\phi U),V) \\
&+ \sigma(\sigma(Y,\nabla_X U),V) - \eta(Y)\sigma(\phi\nabla_X U, V) + \sigma(\overline{\overline{\widetilde{\nabla}}}_Y \{\sigma(X,U)\}, V) \\
&- \sigma(\overline{\overline{\widetilde{\nabla}}}_Y \eta(X)\phi U, V) - \eta(X)\sigma(\nabla_Y \phi U, V) - \eta(X)\sigma(\sigma(Y,\phi U),V) \\
&+ \sigma(\sigma([X,Y],U),V) - \eta([X,Y])\sigma(\phi U, V) - \sigma(U, \sigma(X, \nabla_Y V)) \\
&+ \eta(X)\sigma(U, \phi\nabla_Y V) - \sigma(U, \overline{\overline{\widetilde{\nabla}}}_X \{\sigma(Y,V)\}) + \sigma(U, \overline{\overline{\widetilde{\nabla}}}_X \eta(Y)\phi V) \\
&+ \eta(Y)\sigma(U, \nabla_X \phi V) + \eta(Y)\sigma(U, \sigma(X,\phi V)) + \sigma(U, \sigma(Y, \nabla_X V)) \\
&- \eta(Y)\sigma(U, \phi\nabla_X V) + \sigma(U, \overline{\overline{\widetilde{\nabla}}}_Y \{\sigma(X,V)\}) - \sigma(U, \overline{\overline{\widetilde{\nabla}}}_Y \eta(X)\phi V) \\
&- \eta(X)\sigma(U, \nabla_Y \phi V) - \eta(X)\sigma(U, \sigma(Y,\phi V)) + \sigma(U, \sigma([X,Y],V)) \\
&- \eta([X,Y])\sigma(U, \phi V),
\end{aligned} \tag{4.13}$$

for all vector fields X, Y, U and V tangent to M, where

$$R^{\perp}(X,Y) = [\nabla^{\perp}_X, \nabla^{\perp}_Y] - \nabla^{\perp}_{[X,Y]}.$$

Proof. We know, from tensor algebra, that

$$(\overline{\overline{R}}(X,Y)\sigma)(U,V) = \overline{\overline{R}}(X,Y)\sigma(U,V) - \sigma(\overline{\overline{R}}(X,Y)U,V) - \sigma(U,\overline{\overline{R}}(X,Y)V). \quad (4.14)$$

Replace Z by $\sigma(U,V)$ in (4.11) to get

$$\overline{\overline{R}}(X,Y)\sigma(U,V) = \overline{\overline{\nabla}}_X\overline{\overline{\nabla}}_Y\sigma(U,V) - \overline{\overline{\nabla}}_Y\overline{\overline{\nabla}}_X\sigma(U,V) - \overline{\overline{\nabla}}_{[X,Y]}\sigma(U,V). \quad (4.15)$$

In view of (1.8), (2.1) and (2.2) we have the following equalities:

$$\begin{aligned} \overline{\overline{\nabla}}_X\overline{\overline{\nabla}}_Y\sigma(U,V) &= \overline{\overline{\nabla}}_X(-A_{\sigma(U,V)}Y + \nabla^{\perp}_Y\sigma(U,V) - \eta(Y)\phi\sigma(U,V)), \quad (4.16)\\ &= -\nabla_X A_{\sigma(U,V)}Y - \sigma(X, A_{\sigma(U,V)}Y) + \eta(X)\phi A_{\sigma(U,V)}Y \\ &\quad -A_{\nabla^{\perp}_Y\sigma(U,V)}X + \nabla^{\perp}_X\nabla^{\perp}_Y\sigma(U,V) - \eta(X)\phi\nabla^{\perp}_Y\sigma(U,V)\\ &\quad -\overline{\overline{\nabla}}_X\eta(Y)\phi\sigma(U,V), \end{aligned}$$

$$\begin{aligned} \overline{\overline{\nabla}}_Y\overline{\overline{\nabla}}_X\sigma(U,V) &= -\nabla_Y A_{\sigma(U,V)}X - \sigma(Y, A_{\sigma(U,V)}X) \quad (4.17)\\ &\quad +\eta(Y)\phi A_{\sigma(U,V)}X - A_{\nabla^{\perp}_X\sigma(U,V)}Y + \nabla^{\perp}_Y\nabla^{\perp}_X\sigma(U,V)\\ &\quad -\eta(Y)\phi\nabla^{\perp}_X\sigma(U,V) - \overline{\overline{\nabla}}_Y\eta(X)\phi\sigma(U,V) \end{aligned}$$

and

$$\overline{\overline{\nabla}}_{[X,Y]}\sigma(U,V) = -A_{\sigma(U,V)}[X,Y] + \nabla^{\perp}_{[X,Y]}\sigma(U,V) - \eta([X,Y])\phi\sigma(U,V). \quad (4.18)$$

Substituting (4.16) – (4.18) into (4.15), we get

$$\begin{aligned} &\overline{\overline{R}}(X,Y)\sigma(U,V) = R^{\perp}(X,Y)\sigma(U,V) - \nabla_X A_{\sigma(U,V)}Y - \sigma(X, A_{\sigma(U,V)}Y) \quad (4.19)\\ &+\eta(X)\phi A_{\sigma(U,V)}Y - A_{\nabla^{\perp}_Y\sigma(U,V)}X - \eta(X)\phi\nabla^{\perp}_Y\sigma(U,V) - \overline{\overline{\nabla}}_X\eta(Y)\phi\sigma(U,V)\\ &+\nabla_Y A_{\sigma(U,V)}X + \sigma(Y, A_{\sigma(U,V)}X) - \eta(Y)\phi A_{\sigma(U,V)}X + A_{\nabla^{\perp}_X\sigma(U,V)}Y\\ &+\eta(Y)\phi\nabla^{\perp}_X\sigma(U,V) + \overline{\overline{\nabla}}_Y\eta(X)\phi\sigma(U,V) + A_{\sigma(U,V)}[X,Y] + \eta([X,Y])\phi\sigma(U,V). \end{aligned}$$

By using (4.12) in $\sigma(\overline{\overline{R}}(X,Y)U,V)$ and $\sigma(U,\overline{\overline{R}}(X,Y)V)$, we get

$$\sigma(\overline{\overline{\widetilde{R}}}(X,Y)U,V) = \sigma(R(X,Y)U,V) + \sigma(\sigma(X,\nabla_Y U),V) \tag{4.20}$$
$$-\eta(X)\sigma(\phi\nabla_Y U,V) + \sigma(\overline{\overline{\widetilde{\nabla}}}_X\{\sigma(Y,U)\},V) - \sigma(\overline{\overline{\widetilde{\nabla}}}_X\eta(Y)\phi U,V)$$
$$-\eta(Y)\sigma(\nabla_X\phi U,V) - \eta(Y)\sigma(\sigma(X,\phi U),V) - \sigma(\sigma(Y,\nabla_X U),V)$$
$$+\eta(Y)\sigma(\phi\nabla_X U,V) - \sigma(\overline{\overline{\widetilde{\nabla}}}_Y\{\sigma(X,U)\},V) + \sigma(\overline{\overline{\widetilde{\nabla}}}_Y\eta(X)\phi U,V)$$
$$+\eta(X)\sigma(\nabla_Y\phi U,V) + \eta(X)\sigma(\sigma(Y,\phi U),V) - \sigma(\sigma([X,Y],U),V)$$
$$+\eta([X,Y])\sigma(\phi U,V) \tag{4.21}$$

and

$$\sigma(U,\overline{\overline{\widetilde{R}}}(X,Y)V) = \sigma(U,R(X,Y)V) + \sigma(U,\sigma(X,\nabla_Y V)) \tag{4.22}$$
$$-\eta(X)\sigma(U,\phi\nabla_Y V) + \sigma(U,\overline{\overline{\widetilde{\nabla}}}_X\{\sigma(Y,V)\}) - \sigma(U,\overline{\overline{\widetilde{\nabla}}}_X\eta(Y)\phi V)$$
$$-\eta(Y)\sigma(U,\nabla_X\phi V) - \eta(Y)\sigma(U,\sigma(X,\phi V)) - \sigma(U,\sigma(Y,\nabla_X V))$$
$$+\eta(Y)\sigma(U,\phi\nabla_X V) - \sigma(U,\overline{\overline{\widetilde{\nabla}}}_Y\{\sigma(X,V)\}) + \sigma(U,\overline{\overline{\widetilde{\nabla}}}_Y\eta(X)\phi V)$$
$$+\eta(X)\sigma(U,\nabla_Y\phi V) + \eta(X)\sigma(U,\sigma(Y,\phi V)) - \sigma(U,\sigma([X,Y],V))$$
$$+\eta([X,Y])\sigma(U,\phi V).$$

Substituting (4.19) – (4.22) into (4.14), we get (4.13). □

5. Recurrent Invariant submanifolds of Kenmotsu manifolds admitting Quarter symmetric metric connection

We consider invariant submanifold of a Kenmotsu manifold when σ is recurrent, 2-recurrent, generalized 2-recurrent and M has parallel third fundamental form with respect to quarter symmetric metric connection. We write the equations (2.3) and (2.4) with respect to quarter symmetric metric connection in the form

$$(\overline{\overline{\widetilde{\nabla}}}_X\sigma)(Y,Z) = \overline{\nabla}^{\perp}_X(\sigma(Y,Z)) - \sigma(\overline{\nabla}_X Y,Z) - \sigma(Y,\overline{\nabla}_X Z), \tag{5.1}$$
$$(\overline{\overline{\widetilde{\nabla}}}^2\sigma)(Z,W,X,Y) = (\overline{\overline{\widetilde{\nabla}}}_X\overline{\overline{\widetilde{\nabla}}}_Y\sigma)(Z,W), \tag{5.2}$$
$$= \overline{\nabla}^{\perp}_X((\overline{\overline{\widetilde{\nabla}}}_Y\sigma)(Z,W)) - (\overline{\overline{\widetilde{\nabla}}}_Y\sigma)(\overline{\nabla}_X Z,W) - (\overline{\overline{\widetilde{\nabla}}}_X\sigma)(Z,\overline{\nabla}_Y W) - (\overline{\overline{\widetilde{\nabla}}}_{\overline{\nabla}_X Y}\sigma)(Z,W).$$

and prove the following theorems

Theorem 5.1. *Let M be an invariant submanifold of a Kenmotsu manifold $\widetilde{M}$ admitting quarter symmetric metric connection. Then σ is recurrent with respect to quarter symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Let σ be recurrent with respect to quarter symmetric metric connection. Then from (1.10) we get

$$(\overline{\overline{\widetilde{\nabla}}}_X\sigma)(Y,Z) = \phi(X)\sigma(Y,Z),$$

where ϕ is a 1-form on M. By using (5.1) and $Z = \xi$ in the above equation, we have

$$\overline{\nabla}^{\perp}_X \sigma(Y,\xi) - \sigma(\overline{\nabla}_X Y,\xi) - \sigma(Y,\overline{\nabla}_X \xi) = \phi(X)\sigma(Y,\xi), \tag{5.3}$$

which by virtue of (4.1) reduces to

$$-\sigma(\overline{\nabla}_X Y,\xi) - \sigma(Y,\overline{\nabla}_X \xi) = 0. \tag{5.4}$$

Using (1.8), (3.4) and (4.1) in (5.4), we obtain $\sigma(X,Y) = 0$, i.e., M is totally geodesic. The converse statement is trivial. This proves the theorem. □

Theorem 5.2. *Let M be an invariant submanifold of a Kenmotsu manifold $\widetilde{M}$ admitting quarter symmetric metric connection. Then M has parallel third fundamental form with respect to quarter symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Let M has parallel third fundamental form with respect to quarter symmetric metric connection. Then we have

$$(\overline{\overline{\widetilde{\nabla}}}_X \overline{\overline{\widetilde{\nabla}}}_Y \sigma)(Z,W) = 0.$$

Taking $W = \xi$ and using (5.2) in the above equation, we have

$$\overline{\nabla}^{\perp}_X((\overline{\overline{\widetilde{\nabla}}}_Y\sigma)(Z,\xi)) - (\overline{\overline{\widetilde{\nabla}}}_Y\sigma)(\overline{\nabla}_X Z,\xi) - (\overline{\overline{\widetilde{\nabla}}}_X\sigma)(Z,\overline{\nabla}_Y\xi) - (\overline{\overline{\widetilde{\nabla}}}_{\overline{\nabla}_X Y}\sigma)(Z,\xi) = 0. \tag{5.5}$$

By using (4.1) and (5.1) in (5.5), we get

$$\begin{aligned}0 = {}& -\overline{\nabla}^{\perp}_X\left\{\sigma(\overline{\nabla}_Y Z,\xi) + \sigma(Z,\overline{\nabla}_Y\xi)\right\} - \overline{\nabla}^{\perp}_Y\sigma(\overline{\nabla}_X Z,\xi) + \sigma(\overline{\nabla}_Y\overline{\nabla}_X Z,\xi) \\ & +2\sigma(\overline{\nabla}_X Z,\overline{\nabla}_Y\xi) - \overline{\nabla}^{\perp}_X\sigma(Z,\overline{\nabla}_Y\xi) + \sigma(Z,\overline{\nabla}_X\overline{\nabla}_Y\xi) + \sigma(\overline{\nabla}_{\overline{\nabla}_X Y}Z,\xi) + \sigma(Z,\overline{\nabla}_{\overline{\nabla}_X Y}\xi).\end{aligned} \tag{5.6}$$

In view of (1.8), (3.1), (3.4) and (4.1) the above result (5.6) gives

$$\begin{aligned}0 = {}& -2\overline{\nabla}^{\perp}_X\sigma(Z,Y) + 2\sigma(\nabla_X Z,Y) - 2\eta(X)\sigma(\phi Z,Y) + 2\sigma(Z,\nabla_X Y) \\ & -2\eta(X)\sigma(Z,\phi Y) - \sigma(Z,\nabla_X\eta(Y)\xi).\end{aligned} \tag{5.7}$$

Put $Z = \xi$ and use (3.4), (4.1) in (5.7) to obtain $\sigma(X,Y) = 0$, i.e., M is totally geodesic. The converse statement is trivial. This proves the theorem. □

Corollary 5.3. *Let M be an invariant submanifold of a Kenmotsu manifold $\widetilde{M}$ admitting quarter symmetric metric connection. Then σ is 2-recurrent with respect to quarter symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Let σ be 2-recurrent with respect to quarter symmetric metric connection. From (1.11), we have

$$(\overline{\tilde{\overline{\nabla}}}_X\overline{\tilde{\overline{\nabla}}}_Y\sigma)(Z,W) = \sigma(Z,W)\phi(X,Y).$$

Taking $W=\xi$ and using (5.2) in the above equation, we have

$$\begin{aligned}&\overline{\nabla}^{\perp}_X((\overline{\tilde{\overline{\nabla}}}_Y\sigma)(Z,\xi)) - (\overline{\tilde{\overline{\nabla}}}_Y\sigma)(\overline{\nabla}_X Z,\xi) - (\overline{\tilde{\overline{\nabla}}}_X\sigma)(Z,\overline{\nabla}_Y\xi)\\ &-(\overline{\tilde{\overline{\nabla}}}_{\overline{\nabla}_X Y}\sigma)(Z,\xi) = \sigma(Z,\xi)\phi(X,Y).\end{aligned} \tag{5.8}$$

In view of (4.1) and (5.1) we write (5.8) in the form

$$\begin{aligned}0 &= -\overline{\nabla}^{\perp}_X\left\{\sigma(\overline{\nabla}_Y Z,\xi)+\sigma(Z,\overline{\nabla}_Y\xi)\right\} - \overline{\nabla}^{\perp}_Y\sigma(\overline{\nabla}_X Z,\xi) + \sigma(\overline{\nabla}_Y\overline{\nabla}_X Z,\xi)\\ &+2\sigma(\overline{\nabla}_X Z,\overline{\nabla}_Y\xi) - \overline{\nabla}^{\perp}_X\sigma(Z,\overline{\nabla}_Y\xi) + \sigma(Z,\overline{\nabla}_X\overline{\nabla}_Y\xi) + \sigma(\overline{\nabla}_{\overline{\nabla}_X Y}Z,\xi) + \sigma(Z,\overline{\nabla}_{\overline{\nabla}_X Y}\xi).\end{aligned} \tag{5.9}$$

Using (1.8), (3.1), (3.4) and (4.1) in (5.9), we get

$$\begin{aligned}0 \;=\; &-2\overline{\nabla}^{\perp}_X\sigma(Z,Y) + 2\sigma(\nabla_X Z,Y) - 2\eta(X)\sigma(\phi Z,Y) + 2\sigma(Z,\nabla_X Y)\\ &-2\eta(X)\sigma(Z,\phi Y) - \sigma(Z,\nabla_X\eta(Y)\xi).\end{aligned} \tag{5.10}$$

Taking $Z=\xi$ and using (3.4), (4.1) in (5.10), we obtain $\sigma(X,Y)=0$, i.e., M is totally geodesic. The converse statement is trivial. This proves the theorem. □

Theorem 5.4. *Let M be an invariant submanifold of a Kenmotsu manifold $\widetilde{M}$ admitting quarter symmetric metric connection. Then σ is generalized 2-recurrent with respect to quarter symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Let σ be generalized 2-recurrent with respect to quarter symmetric metric connection. From (1.12), we have

$$(\overline{\tilde{\overline{\nabla}}}_X\overline{\tilde{\overline{\nabla}}}_Y\sigma)(Z,W) = \psi(X,Y)\sigma(Z,W) + \phi(X)(\overline{\tilde{\overline{\nabla}}}_Y\sigma)(Z,W), \tag{5.11}$$

where ψ and ϕ are 2-recurrent and 1-form respectively. Taking $W=\xi$ in (5.11) and using (4.1), we get

$$(\overline{\tilde{\overline{\nabla}}}_X\overline{\tilde{\overline{\nabla}}}_Y\sigma)(Z,\xi) = \phi(X)(\overline{\tilde{\overline{\nabla}}}_Y\sigma)(Z,\xi).$$

Using (4.1) and (5.2) in above equation, we get

$$\begin{aligned}&\overline{\nabla}^{\perp}_X((\overline{\tilde{\overline{\nabla}}}_Y\sigma)(Z,\xi)) - (\overline{\tilde{\overline{\nabla}}}_Y\sigma)(\overline{\nabla}_X Z,\xi) - (\overline{\tilde{\overline{\nabla}}}_X\sigma)(Z,\overline{\nabla}_Y\xi)\\ &-(\overline{\tilde{\overline{\nabla}}}_{\overline{\nabla}_X Y}\sigma)(Z,\xi) = -\phi(X)\left\{\sigma(\overline{\nabla}_Y Z,\xi)+\sigma(Z,\overline{\nabla}_Y\xi)\right\}.\end{aligned} \tag{5.12}$$

In view of (4.1) and (5.1) the above result (5.12) gives

$$-\overline{\nabla}^{\perp}_{X}\left\{\sigma(\overline{\nabla}_{Y}Z,\xi)+\sigma(Z,\overline{\nabla}_{Y}\xi)\right\}-\overline{\nabla}^{\perp}_{Y}\sigma(\overline{\nabla}_{X}Z,\xi)+\sigma(\overline{\nabla}_{Y}\overline{\nabla}_{X}Z,\xi) \tag{5.13}$$
$$+2\sigma(\overline{\nabla}_{X}Z,\overline{\nabla}_{Y}\xi)-\overline{\nabla}^{\perp}_{X}\sigma(Z,\overline{\nabla}_{Y}\xi)+\sigma(Z,\overline{\nabla}_{X}\overline{\nabla}_{Y}\xi)+\sigma(\overline{\nabla}_{\overline{\nabla}_{X}Y}Z,\xi)$$
$$+\sigma(Z,\overline{\nabla}_{\overline{\nabla}_{X}Y}\xi)=-\phi(X)\left\{\sigma(\overline{\nabla}_{Y}Z,\xi)+\sigma(Z,\overline{\nabla}_{Y}\xi)\right\}.$$

Using (1.8), (3.1), (3.4) and (4.1) in (5.13), we get

$$-2\overline{\nabla}^{\perp}_{X}\sigma(Z,Y)+2\sigma(\nabla_{X}Z,Y)-2\eta(X)\sigma(\phi Z,Y)+2\sigma(Z,\nabla_{X}Y) \tag{5.14}$$
$$-2\eta(X)\sigma(Z,\phi Y)-\sigma(Z,\nabla_{X}\eta(Y)\xi)=-\phi(X)\sigma(Z,Y).$$

Choosing $Z=\xi$ and using (3.4), (4.1) in (5.14), we obtain $\sigma(X,Y)=0$, i.e., M is totally geodesic. The converse statement is trivial. This proves the theorem. □

6. Semiparallel, pseudoparallel and Ricci-generalized pseudoparallel Invariant submanifolds of Kenmotsu manifolds admitting Quarter symmetric metric connection

We consider invariant submanifolds of Kenmotsu manifolds admitting quarter symmetric metric connection satisfying the conditions $\overline{\overline{R}}\cdot\sigma=0$, $\overline{\overline{R}}\cdot\sigma=L_1Q(g,\sigma)$, $\overline{\overline{R}}\cdot\sigma=L_2Q(S,\sigma)$.

Theorem 6.1. *Let M be an invariant submanifold of a Kenmotsu manifold $\widetilde{M}$ admitting quarter symmetric metric connection. Then M is semiparallel with respect to quarter symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Let M be semiparallel satisfying $\overline{\overline{R}}\cdot\sigma=0$. Put $X=V=\xi$ and use (3.1), (3.4) and (4.1) in (4.13) to get

$$0 \;=\; -\sigma(U,R(\xi,Y)\xi)-\sigma(\overline{\overline{\nabla}}_{\xi}\sigma(Y,U),\xi)+\sigma(\overline{\overline{\nabla}}_{\xi}\eta(Y)\phi U,\xi) \tag{6.1}$$
$$-\sigma(\overline{\overline{\nabla}}_{Y}\phi U,\xi)+\sigma(U,\phi\nabla_{Y}\xi).$$

Using (1.8), (2.1), (3.1) (3.4), (3.8) and (4.1) in (6.1), we get

$$-\sigma(U,Y)+\sigma(U,\phi Y)-\sigma(\overline{\overline{\nabla}}_{\xi}\sigma(Y,U),\xi)=0. \tag{6.2}$$

By definition σ is a vector valued covariant tensor and so $\sigma(U,Y)$ is a vector. Therefore $\overline{\overline{\nabla}}_{\xi}\sigma(Y,U)$ is a vector and hence by (4.1), we have

$$\sigma(\overline{\overline{\nabla}}_{\xi}\sigma(Y,U),\xi)=0. \tag{6.3}$$

Then from (6.2), we get

$$-\sigma(U,Y)+\sigma(U,\phi Y)=0. \tag{6.4}$$

Replacing Y by ϕY and using (3.1) and (4.1) in (6.4), we get

$$-\sigma(U,\phi Y)-\sigma(U,Y)=0. \tag{6.5}$$

Adding (6.4) and (6.5), we obtain $\sigma(U,Y)=0$, i.e., M is totally geodesic. The converse statement is trivial. □

Theorem 6.2. *Let M be an invariant submanifold of a Kenmotsu manifold $\widetilde{M}$ admitting quarter symmetric metric connection. Then M is pseudoparallel with respect to quarter symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Let M be pseudoparallel satisfying $\overline{\overline{R}}\cdot\sigma=L_1Q(g,\sigma)$. Put $X=V=\xi$ and use (3.1), (3.4) and (4.1) in (2.6) and (4.13) to get

$$\begin{aligned}&-\sigma(U,R(\xi,Y)\xi)-\sigma(\overline{\overline{\nabla}}_\xi\sigma(Y,U),\xi)+\sigma(\overline{\overline{\nabla}}_\xi\eta(Y)\phi U,\xi)-\sigma(\overline{\overline{\nabla}}_Y\phi U,\xi)\\&+\sigma(U,\phi\nabla_Y\xi)=-L_1\sigma(U,Y).\end{aligned} \tag{6.6}$$

Using (1.8), (2.1), (3.1) (3.4), (3.8) and (4.1) in (6.6), we get

$$-\sigma(U,Y)+\sigma(U,\phi Y)-\sigma(\overline{\overline{\nabla}}_\xi\sigma(Y,U),\xi)=-L_1\sigma(U,Y). \tag{6.7}$$

Now by using (6.3) in (6.7), we get

$$(L_1-1)\sigma(U,Y)+\sigma(U,\phi Y)=0. \tag{6.8}$$

Replacing Y by ϕY and using (3.1) and (4.1) in (6.8), we get

$$(L_1-1)\sigma(U,\phi Y)-\sigma(U,Y)=0. \tag{6.9}$$

Multiplying (6.8) by (L_1-1) and (6.9) by 1 and subtracting these two equations, we obtain $((L_1-1)^2+1)\sigma(U,Y)=0$ and hence if $L_1\neq(1\pm i)$, we have $\sigma(U,Y)=0$, i.e., M is totally geodesic. The converse statement is trivial. □

Theorem 6.3. *Let M be an invariant submanifold of a Kenmotsu manifold $\widetilde{M}$ admitting quarter symmetric metric connection. Then M is Ricci-generalized pseudoparallel with respect to quarter symmetric metric connection if and only if it is totally geodesic with respect to Levi-Civita connection.*

Proof. Let M be Ricci-generalized pseudoparallel satisfying $\overline{\overline{R}}\cdot\sigma = L_2 Q(S,\sigma)$. Put $X = V = \xi$ and use (3.1), (3.4), (3.9) and (4.1) in (2.6) and (4.13) to get

$$\begin{aligned} &-\sigma(U, R(\xi, Y)\xi) - \sigma(\overline{\overline{\nabla}}_\xi \sigma(Y,U), \xi) + \sigma(\overline{\overline{\nabla}}_\xi \eta(Y)\phi U, \xi) \\ &-\sigma(\overline{\overline{\nabla}}_Y \phi U, \xi) + \sigma(U, \phi\nabla_Y \xi) = L_2(n-1)\sigma(U,Y). \end{aligned} \tag{6.10}$$

Using (1.8), (2.1), (3.1), (3.4), (3.8) and (4.1) in (6.10), we get

$$-\sigma(U,Y) + \sigma(U,\phi Y) - \sigma(\overline{\overline{\nabla}}_\xi \sigma(Y,U), \xi) = L_2(n-1)\sigma(U,Y). \tag{6.11}$$

Now by using (6.3) in (6.11), we get

$$(-L_2(n-1) - 1)\sigma(U,Y) + \sigma(U,\phi Y) = 0. \tag{6.12}$$

Replacing Y by ϕY and using (3.1) and (4.1) in (6.12), we get

$$(-L_2(n-1) - 1)\sigma(U,\phi Y) - \sigma(U,Y) = 0. \tag{6.13}$$

Multiplying (6.12) by $(-L_2(n-1)-1)$ and (6.13) by 1 and subtracting these two equations, we obtain $((-L_2(n-1)-1)^2+1)\sigma(U,Y) = 0$ and hence if $L_2 \neq \frac{(-1\pm i)}{(n-1)}$, we have $\sigma(U,Y) = 0$, i.e., M is totally geodesic. The converse statement is trivial. □

Using Theorems and corollary 5.1 to 5.3, 6.4 to 6.6, we have the following result

Corollary 6.4. *Let M be an invariant submanifold of a Kenmotsu manifold $\widetilde{M}$ admitting quarter symmetric metric connection. Then the following statements are equivalent.*

(1) *σ is recurrent.*
(2) *σ is 2-recurrent.*
(3) *σ is generalized 2-recurrent.*
(4) *M has parallel third fundamental form.*
(5) *M is semiparallel.*
(6) *M is pseudoparallel, if $L_1 \neq (1 \pm i)$.*
(7) *M is Ricci-generalized pseudoparallel, if $L_2 \neq \frac{(-1\pm i)}{(n-1)}$.*
(8) *M is totally geodesic.*

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