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FIXED POINTS OF MAPPINGS DEFINED ON PROBABILISTIC MODULAR SPACES

(COMMUNICATED BY LAHCENE MEZRAG)

FATEMEH LAEL[1], KOUROSH NOUROUZI[2]

Abstract. In the present paper we give some results on the fixed points of mappings defined on probabilistic modular spaces.

1. Introduction and Preliminaries

A reformulation of the notion of a modular space [3, 4, 6] is given in [2] under the name of "probabilistic modular space" according to Menger's probabilistic approach [5] to generalise the notion of distance. As Menger's interpretation, in this notion the values of the modulars are probability distribution functions rather than numbers and therefore a probabilistic modular space is a natural generalisation of a modular space.

In this paper after introducing the prerequisites we shall present some results on fixed points of mappings defined between probabilistic modular spaces.

We will denote the set of all non-decreasing functions $f : \mathbb{R} \to {\mathbb{R}_0}^+$ satisfying $\inf_{t\in\mathbb{R}} f(t) = 0$, and $\sup_{t\in\mathbb{R}} f(t) = 1$ by Δ. We also denote the function *min* by $\wedge$.

Definition 1. [2] A *probabilistic modular space* (briefly, $\mathcal{PM}$*-space*) is a pair (X, μ) in which X is a real vector space, μ is a mapping from X into Δ (for $x \in X$, the function $\mu(x)$ is denoted by μ_x, and $\mu_x(t)$ is the value μ_x at $t \in \mathbb{R}$) satisfying the following conditions:

(1) $\mu_x(0) = 0$,
(2) $\mu_x(t) = 1$ for all $t > 0$ iff $x = 0$,
(3) $\mu_{-x}(t) = \mu_x(t)$ for all $x \in X$,
(4) $\mu_{\alpha x+\beta y}(s+t) \geq \mu_x(s) \wedge \mu_y(t)$ for all $x, y \in X$, and $\alpha, \beta, s, t \in {\mathbb{R}_0}^+$, $\alpha+\beta = 1$.

A $\mathcal{PM}$-space (X, μ) is said to satisfy Δ_2-condition if there is a constant $c > 0$ such that $\mu_{2x}(t) \geq \mu_x(\frac{t}{c})$, for all $x \in X$ and $t > 0$.

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A probabilistic modular μ defines a corresponding probabilistic modular vector space (briefly, $\mathcal{PMV}$-space) X_μ given by

$$X_\mu = \{x \in X : \lim_{\lambda \to 0} \mu_{\lambda x}(t) = 1, \quad \text{for all } t > 0\}.$$

Definition 2. [2] Let (X, μ) be a $\mathcal{PM}$-space.

1. A sequence $\{x_n\}$ in X is said to be *μ-convergent* to a point $x \in X$ and denoted by $x_n \to x$, if for every $t > 0$ and $r \in (0,1)$, there exists a positive integer k such that $\mu_{x_n - x}(t) > 1 - r$ for all $n \geq k$ (This limit is unique [2]).
2. The *μ-closure* of a subset A of X is denoted by $\overline{A}$ and defined by the set of all $x \in X$ such that there is a sequence $\{x_n\}$ of elements of A with $x_n \to x$. We say that A is *μ-closed* if $A = \overline{A}$
3. A sequence $\{x_n\}$ in X is said to be *μ-Cauchy* sequence, if for every $t > 0$ and $r \in (0,1)$, there exists a positive integer k such that $\mu_{x_n - x_m}(t) > 1 - r$ for all $n, m \geq k$.
4. A subset C of X is said to be *μ-complete* if each μ-Cauchy sequence in C is μ-convergent to a point of C.
5. The $\mathcal{PM}$- space (X, μ) is said to be *sequentially μ-compact* if every sequence in X has a μ-convergent subsequence in X.

Definition 3. [2] A mapping T from (X, μ) to (Y, ν) is said to be sequentially *$\mathcal{PM}$-continuous* at x_0, if any sequence $\{x_n\}$, $x_n \in X$ with $x_n \xrightarrow{\mu} x$ imply $T(x_n) \xrightarrow{\nu} T(x)$. If T is sequentially $\mathcal{PM}$-continuous at each point of X, then T is said to be sequentially $\mathcal{PM}$-continuous on X.

2. Fixed Points in $\mathcal{PM}$-spaces

Fixed point theory from the probabilistic point of view constitutes a part of probabilistic analysis [1]. In this section, we shall give a probabilistic modular version of some fixed point theorems.

Definition 4. Let C be a subset of a $\mathcal{PM}$-space (X, μ). A mapping $T : C \to C$ is said to be *μ-Lipschitzian* if there exist constants $k, l > 0$ such that

$$\mu_{T(x) - T(y)}(t) \geq \mu_{l(x-y)}(\frac{t}{k}),$$

for every $x, y \in X$, and $t \in \mathbb{R}$.

If $k, l \in (0,1)$, we say that T is a *μ-contraction*, whereas if $k = l = 1$, we say that T is *μ-nonexpansive*. It can be easily seen that every μ-contraction map is sequentially $\mathcal{PM}$-continuous.

Our first result is the probabilistic modular formulation of well known Banach contraction principle.

Lemma 1. *Let C be a μ-complete subset of (X, μ), μ satisfy Δ_2-condition, and $T : C \to C$ be a μ-contraction with μ-Lipschitzian constants k, l. Then T has a unique fixed point $z \in C$ and $\lim_{n \to \infty} T^n(x) = z$, for every $x \in C$.*

Proof. Choose $x \in C$ arbitrarily. The sequence $\{T^n(x)\}$ is a μ-Cauchy sequence. In fact, for $t > 0$ and $p = 1, 2, 3, \ldots$ we have

$$\mu_{T^n(x) - T^{n+p}(x)}(t) \geq \mu_{l(T^{n-1}(x) - T^{n+p-1}(x))}(\frac{t}{k}) \geq \ldots \geq \mu_{l(x - T^p(x))}(\frac{t}{k^n}).$$

Let $l + 1/\alpha = 1$. Moreover

$$\begin{aligned}
\mu_{l(x-T^p(x))}(\tfrac{t}{k^n}) &= \mu_{l(x-T(x))+l(T(x)-T^p(x))}(\tfrac{t}{k^n}),\\
&= \mu_{\frac{\alpha l}{\alpha}(x-T(x))+l(T(x)-T^p(x))}(\tfrac{kt+(1-k)t}{k^n}),\\
&\geq \mu_{\alpha l(x-T(x))}(\tfrac{(1-k)t}{k^n}) \wedge \mu_{T(x)-T^p(x)}(\tfrac{kt}{k^n}),\\
&\geq \mu_{\alpha l(x-T(x))}(\tfrac{(1-k)t}{k^n}) \wedge \mu_{l(x-T^{p-1}(x))}(\tfrac{t}{k^n}).
\end{aligned}$$

By induction, we have

$$\mu_{l(x-T^p(x))}(\frac{t}{k^n}) \geq \mu_{\alpha l(x-T(x))}(\frac{(1-k)t}{k^n}) \wedge \ldots \wedge \mu_{\alpha l(x-T(x))}(\frac{(1-k)t}{k^n}),$$

and therefore $\lim_{n\to\infty} \mu_{T^n(x)-T^{n+p}(x)}(t) = 1$. Since X is μ-complete, there exists $z \in X$ with $\lim_{n\to\infty} T^n(x) = z$. Moreover the μ-contractivity of T yields sequentially $\mathcal{PM}$-continuity. Therefore $z = \lim_{n\to\infty} T^{n+1}(x) = \lim_{n\to\infty} T(T^n(x)) = T(z)$, that is z is a fixed point of T.

To see the uniqueness, suppose on the contrary that there exist distinct elements $z, y \in C$ with $z = T(z)$ and $y = T(y)$. Thus, for each n, $T^n(z) = z$ and $T^n(y) = y$, and there exists $s > 0$ such that $\mu_{y-z}(s) = a \in [0, 1)$. Then for every $n \in \mathbb{N}$,

$$a = \mu_{y-z}(s) = \mu_{T^n(y)-T^n(z)}(s) \geq \mu_{y-z}(\frac{s}{k^n}),$$

and when $n \to \infty$ we obtain $a = 1$ which contradicts $a < 1$. Therefore $z = y$. □

In Lemma 1, as its classical form, the μ-contractivity is essential. Indeed, any (nonlinear) affine function defined on $\mathcal{PM}$-space $(\mathbb{R}, \frac{t}{t+|x|})$ which is μ-nonexpansive has no fixed point. Also, a μ-contraction T from an incomplete subset C of $\mathcal{PM}$-space X into itself need not have a fixed point. In light of Lemma 1, this can be easily seen via removing the only fixed point from the domain. Furthermore, a μ-contraction map defined on an incomplete $\mathcal{PM}$-space need not have any fixed point. To see this, consider the real vector space ℓ_0 consisting of all infinite sequences of real numbers which have only finitely many non-zero terms. By the modular μ which defined as

$$\mu_x(t) = \begin{cases} 1 & t > \sup|x_n| : n \in \mathbb{N} \\ \frac{1}{2} & \sup\{|\frac{x_n}{n}| : n \in \mathbb{N}\} < t \leq \sup\{|x_n| : n \in \mathbb{N}\} \\ 0 & t \leq \sup\{|\frac{x_n}{n}| : n \in \mathbb{N}\}, \end{cases}$$

for any $x = (x_1, \cdots, x_n, \cdots) \in \ell_0$, the μ-contraction mapping $T : \ell_0 \to \ell_0$ defined by $T(x) = \frac{1}{2}(1, x_1, x_2, \ldots, x_n, \ldots)$ has no fixed point.

A $\mathcal{PM}$-space (X, μ) possesses *Fatou's property* if for any sequence $\{x_n\}$ of X converging to x, we have

$$\mu_x(t) \geq \limsup_{n\geq 1} \mu_{x_n}(t),$$

for each $t > 0$.

Theorem 1. *Let (X, μ) be a μ-complete $\mathcal{PM}$-space having Fatou's property and satisfying Δ_2-condition with Δ_2-constant c. Let $A = \{x \in X : \ \mu_{x-x_0}(t_0) \geq 1-r_0\}$, where $x_0 \in X$ and $r_0, t_0 > 0$. Suppose $T : A \to X$ is a μ-contraction with μ-Lipschitzian constant L and*

$$\mu_{T(x_0)-x_0}((\frac{1}{c} - L)t_0) > 1 - r_0,$$

for all $x, y \in A$. Then T has a unique fixed point in A.

Proof. We see that A is μ-closed. In fact, if $y \in \overline{A}$, then by Fatou's property

$$\mu_{y-x_0}(t_0) \geq \limsup_n \mu_{x_n-x_0}(t_0) \geq 1 - r_0,$$

therefore A is μ-closed. Now if $x \in A$, then

$$\begin{aligned}
\mu_{T(x)-x_0}(t_0) &\geq \mu_{T(x)-T(x_0)+T(x_0)-x_0}(t_0), \\
&\geq \mu_{x-x_0}(t_0) \wedge \mu_{T(x_0)-x_0}((\tfrac{1}{c} - L)t_0), \\
&\geq 1 - r_0 \wedge 1 - r_0, \\
&\geq 1 - r_0.
\end{aligned}$$

It follows that $T(A) \subseteq A$. We can now apply Lemma 1 to deduce that T has a unique fixed point in A. □

Let (X, μ) be a $\mathcal{PM}$-space. For $x \in X$, $t > 0$, and $0 < r < 1$, we suppose that

$$B(x, r, t) = \{y \in X : \mu_{x-y}(t) > 1 - r\}.$$

Theorem 2. *Let C be a μ-complete subset of $\mathcal{PM}$-space(X, μ) satisfying Δ_2-condition with constant c and $T : C \to C$ a map. Suppose further that for each $\epsilon > 0$ and $t > 0$ there is a $\delta > 0$ such that if $\mu_{x-T(x)}(t) > \delta$, then $T(B(x, t, \epsilon)) \subseteq B(x, t, \epsilon)$. If for some $u \in X$ we have $\lim_{n\to\infty} \mu_{T^n(u)-T^{n+1}(u)}(t) = 1$ for all $t > 0$, then the sequence $\{T^n(u)\}$ converges to a fixed point of T.*

Proof. Consider u as $\lim_{n\to\infty} \mu_{T^n(u)-T^{n+1}(u)}(t) = 1$, for all $t > 0$ and let $T^n(u) = u_n$. The sequence $\{u_n\}$ is μ-Cauchy. Since if $\epsilon > 0$ and $t_0 > 0$ are given, for the corresponding $\delta_{(\epsilon, \frac{t_0}{2c})}$, we can choose N large enough so that $\mu_{u_n-u_{n+1}}(\frac{t_0}{2c}) > \delta$, for all $n \geq N$. Because $\mu_{u_n-T(u_n)}(\frac{t_0}{2c}) > \delta$, by assumption we get

$$T(B(u_N, \frac{t_0}{2c}, \epsilon)) \subseteq B(u_N, \frac{t_0}{2c}, \epsilon),$$

and so $T(u_N) = u_{N+1} \in B(u_N, \frac{t_0}{2c}, \epsilon)$. Now by induction

$$T^k(u_N) = u_{N+k} \in B(u_N, \frac{t_0}{2c}, \epsilon),$$

for all $k \in \{0, 1, \ldots\}$. Thus

$$\mu_{u_k-u_l}(t_0) \geq \mu_{u_k-u_N}(\frac{t_0}{2c}) \wedge \mu_{u_N-u_l}(\frac{t_0}{2c}) > 1 - \epsilon,$$

for all $k, l \geq N$. Therefore $\{u_n\}$ is a μ-Cauchy sequence and so there exists $y \in C$ with $\lim_{n\to\infty} u_n = y$. This y, in fact, is a fixed point of T. Otherwise, $\mu_{y-T(y)}(t_1) = \gamma < 1$ for some $t_1 > 0$. We can now choose a $u_n \in B(y, \frac{t_1}{2c}, 1 - \gamma')$ where $\gamma < \gamma' < 1$ with

$$\mu_{u_n-u_{n+1}}(\frac{t_1}{2c}) > \delta_{(1-\gamma', \frac{t_1}{2c})}.$$

Again by assumption, we have

$$T(B(u_n, \frac{t_1}{2c}, 1 - \gamma')) \subseteq B(u_n, \frac{t_1}{2c}, 1 - \gamma'),$$

and consequently

$$T(y) \in B(u_n, \frac{t_1}{2c}, 1 - \gamma').$$

This is a contradiction since

$$\mu_{y-T(y)}(t_1) \geq \mu_{y-u_n}(\frac{t_1}{2c}) \wedge \mu_{u_n-T(y)}(\frac{t_1}{2c}) \geq \gamma' > \gamma.$$

$\square$

Definition 5. Let (X,μ) and (Y,ν) be two $\mathcal{PM}$-spaces. A mapping $T:(X,\mu) \to (Y,\nu)$ is said to be *μ-compact* if $T(X)$ is contained in a μ-compact subset of Y.

Let D be a subset of a $\mathcal{PM}$-space X and $T: D \to X$ a map. Given $\epsilon > 0$ and $\lambda \in (0,1)$, a point $d \in D$ with $\mu_{d-T(d)}(\epsilon) > 1-\lambda$ is called an (ϵ,λ)-fixed point of T.

Theorem 3. *Let (X,μ) be a $\mathcal{PM}$-space satisfying Δ_2-condition, D a μ-closed subset of X, and $T: D \to X$ be a μ-compact, sequentially $\mathcal{PM}$-continuous map. Then T has a fixed point if and only if T has an (ϵ,λ)-fixed point for each $\epsilon > 0$ and $\lambda \in (0,1)$.*

Proof. Assume that T has an (ϵ,λ)-fixed point for each $\epsilon > 0$ and $\lambda \in (0,1)$. Now for each $n > 1$, let d_n be a $(\frac{1}{n},\frac{1}{n})$-fixed point of T, that is,

$$\mu_{d_n-T(d_n)}(\frac{1}{n}) > 1-\frac{1}{n}.$$

Since T is μ-compact, $T(D)$ is contained in a μ-compact subset K of X and therefore there exists a subsequence S of integers and $x \in K$ such that $T(d_n) \to x \in K$, as $n \to \infty$ in S. Now, if c is Δ_2-constant and $t > 0$, then the inequality above implies that

$$\mu_{d_n-x}(t) \geq \mu_{d_n-T(d_n)}(\frac{t}{2c}) \wedge \mu_{T(d_n)-x}(\frac{t}{2c}) > 1-\frac{1}{n},$$

for all large n. Therefore $d_n \to x$ as $n \to \infty$ in S and since D is μ-closed we have that $x \in D$. Now the sequentially continuity of T implies that $T(d_n) \to T(x)$ in S and this together with the uniqueness of μ-limit implies $T(x) = x$. $\square$

Lemma 2. *Let X_μ be a $\mathcal{PMV}$-space, $A \subseteq X_\mu$ and $\bar{A}$ be sequentially μ-compact. If $\lambda_k \to 0$, as $k \to \infty$ and $\{x_k\} \subseteq A$, then $\mu_{\lambda_k x_k}(t) \to 1$, for all $t > 0$.*

Proof. Let $\{k_n\}$ be any increasing sequence of indices. There exists $x \in X_\mu$ and a subsequence $\{k_{n_i}\}$ of $\{k_n\}$ such that $\mu_{x_{k_{n_i}}-x}(t) \to 1$ as $i \to \infty$. Taking i so large that $2\lambda_{k_{n_i}} < 1$ and $t > 0$, we obtain

$$\begin{aligned} \mu_{\lambda_{n_{k_i}} x_{n_{k_i}}}(t) &\geq \mu_{2\lambda_{n_{k_i}}(x_{n_{k_i}}-x)}(\tfrac{t}{2}) \wedge \mu_{2\lambda_{n_{k_i}}x}(\tfrac{t}{2}), \\ &\geq \mu_{x_{n_{k_i}}-x}(\tfrac{t}{2}) \wedge \mu_{2\lambda_{n_{k_i}}x}(\tfrac{t}{2}), \\ &> 1-\lambda. \end{aligned}$$

for $\lambda \in (0,1)$. This implies that each subsequence of $\{\lambda_k x_k\}$ has a subsequence which is μ-converges to 0. Thus $\lim_{k\to\infty} \lambda_k x_k = 0$. $\square$

Definition 6. A $\mathcal{P}$-modular μ is said to satisfy *regular increasing* condition if for all $k \in [0,1)$,

$$W_\mu(k) = \inf\{L > 0 : \ \mu_{kx}(t) \geq \mu_x(\frac{t}{L}),\, x \in X_\mu, t > 0\} < 1.$$

Lemma 3. [2] Let (X,μ) be a $\mathcal{PM}$-space satisfying Δ_2-condition. If $\{x_n\}$ is a μ-Cauchy sequence having a μ-convergent subsequence, then $\{x_n\}$ is μ-convergent.

A subset B of vector space X is said to be star-shaped if there exists $z \in B$ such that for all $x \in B$, $\alpha z + \beta x \in B$, whenever α, $\beta \in \mathbb{R}^+$ with $\alpha + \beta = 1$. Such a point z is called a center of B.

Theorem 4. *Let X_μ be a $\mathcal{PMV}$-space satisfying Δ_2 and the regular increasing conditions. Then every μ-compact, μ-nonexpansive map T on a star-shaped subset C of X_μ has at least one fixed point.*

Proof. Let $x_0 \in C$ be a center of C. For $n = 2, 3, \dots$, define

$$T_n = (1 - \frac{1}{n})T + \frac{1}{n}x_0.$$

Since C is star-shaped, we see that $T_n : C \to C$. Let $\lambda \in (1, \frac{1}{1-\frac{1}{n}})$. Because $W_\mu(\lambda(1 - \frac{1}{n})) < 1$, there exists $L < 1$ such that for all $x, y \in C$ and $t > 0$,

$$\begin{aligned}
\mu_{\lambda(T_n(x)-T_n(y))}(t) &= \mu_{\lambda(1-\frac{1}{n})(T(x)-T(y))}(t), \\
&\geq \mu_{T(x)-T(y)}(\tfrac{t}{L}), \\
&\geq \mu_{x-y}(\tfrac{t}{L}),
\end{aligned}$$

Without less of generality we can assume

$$\mu_{T_n(x)-T_n(y)}(t) \geq \mu_{\frac{1}{\lambda}(x-y)}(\frac{t}{L})$$

Hence $T_n : C \to C$ is a μ-contraction for all $n \in \mathbb{N}$. Since $\overline{T_n(C)}$ is μ-compact for all n, Lemma 1 implies that they are also μ-complete. By Lemma1 we conclude that each operator $T_n : \overline{T_n(C)} \to \overline{T_n(C)}$ has a unique fixed point, that is,

$$x_n = (1 - \frac{1}{n})T(x_n) + \frac{1}{n}x_0.$$

Therefore

$$\begin{aligned}
\mu_{T(x_n)-x_n}(t) &= \mu_{T(x_n)-\frac{1}{n}x_0-(1-\frac{1}{n})T(x_n)}(t), \\
&\geq \mu_{\frac{2}{n}T(x_n)}(\tfrac{t}{2}) \wedge \mu_{\frac{2}{n}x_0}(\tfrac{t}{2}).
\end{aligned}$$

Lemma 2 implies that for all $t > 0$, we have $\mu_{T(x_n)-x_n}(t) \to 1$ as $n \to \infty$. And an application of Theorem 3 says that T has a fixed point. $\square$

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[1,2] Department of Mathematics, K.N. Toosi University of Technology, P.O. Box 16315-1618, Tehran, Iran.

E-mail address: `nourouzi@.kntu.ac.ir`