BULLETIN OF MATHEMATICAL ANALYSIS AND APPLICATIONS
ISSN: 1821-1291, URL: HTTP://WWW.BMATHAA.ORG
VOLUME 4 ISSUE 3 (2012.), PAGES 43-47

RELATED FIXED POINT THEOREMS FOR TWO PAIRS OF MAPPINGS ON TWO SYMMETRIC SPACES

(COMMUNICATED BY PROFESSOR V. MULLER)

M.AAMRI, S.BENNANI, D. EL MOUTAWAKIL

ABSTRACT. Some new related fixed point results for two pairs of mappings on two symmetric spaces are established.

1. INTRODUCTION

In 1997, B. Fisher et P.P. Murthy presented in [2] the following related fixed point Theorem in metric spaces

Theorem 1.1. *Let (X,d) and (Y,δ) be complete metric spaces. let A,B be mappings of X into Y and let S,T be mappings of Y into X satisfying the inequalities :*

$$(1)\delta(SAx,TBx') \leq c\max\{d(x,x'),d(x,SAx),d(x',TBx'),\delta(Ax,Bx')\}$$

$$(2)d(BSy,ATy') \leq c\max\{\delta(y,y'),\delta(y,BSy),\delta(y',ATy'),d(Sy,Ty')\}$$

for all x,x' in X and y,y' in Y, where $0 \leq c < 1$

If one of the mappings A,B,S and T is continuous, then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, $Az = Bz = w$ and $Sw = Tw = z$

Our purpose here is to give a generalization of this Theorem for two symmetric spaces (X,d) and (Y,δ). We begin by recalling some basic concepts of the theory of symmetric spaces. A symmetric function on a set X is a non negative real valued function d on $X \times X$ such that
$(1)d(x,y) = 0$, if and only if $x = y$.
$(2)d(x,y) = d(y,x)$.
Let d a symmetric on a set X and for $r > 0$ and $x \in X$, let $B(x,r) = \{y \in X : d(x,y) < r\}$. A topology t(d) on X is given by $U \in t(d)$ if and only if for each $x \in U$, $B(x,r) \subseteq U$.
A symmetric d is semi-metric if for each $x \in X$ and for each $r > 0$, $B(x,r)$ is a

Date: June 28, 2012.
[0]2000 Mathematics Subject Classification: $54H25$.
Keywords and phrases. Fixed point - Related fixed point - Symmetric spaces.

Submitted 22 May, 2011. Accepted June 28, 2012.

neighborhood of x in the topology $t(d)$. Note that $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if $\lim_{n\to\infty} x_n = x$ in the topology $t(d)$.
The following axioms are available in [3], [4] and [5]:
(W_3)[5] Given $\{x_n\}$, x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply x=y.
(W_4)[5] Given $\{x_n\}$,$\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$ imply that $\lim_{n\to\infty} d(y_n, x) = 0$.
$(1C)$[3] A symmetric d on a set X is said to be 1-continuous if $\lim_{n\to\infty} d(x_n, x) = 0$ implies $\lim_{n\to\infty} d(x_n, y) = d(x, y)$, for all $y \in X$.
It is easy to see that for a semi metric d, if t(d)is Hausdorff, then (W_3) holds. Also (W_4) implies (W_3) and $(1C)$ implies (W_3) but converse implications are not true.
A sequence in X is d- Cauchy if it satisfies the usual metric condition with respect to d. There are several concepts of completeness in this setting (see [4])
1) (X, d) is d-Cauchy complete if for every d-Cauchy sequence $\{x_n\}$ there exists $x \in X$ with $x_n \to x$ in the topology t(d).
2) (X, d) is S-complete if for every d-Cauchy sequence $\{x_n\}$ there exists $x \in X$ with $\lim_{n\to\infty} d(x_n, x) = 0$.
3) (X, d) is $(\sum)$ d-complete if for every sequence $\{x_n\}$, $\sum_{n=1}^{+\infty} d(x_n, x_{n+1}) < \infty$ implies that $\{x_n\}$ is convergent in the topology $t(d)$.

2. Main result

Theorem 2.1. *Let (X, d) and (Y, δ) be two 1-continuous semi-metric spaces. let A,B be mappings of X into Y, and let S,T be mappings of Y into X satisfying*

$$(i)\ d(SAx, TBx') \le c\max\{d(x, x'), d(x, SAx), d(x', TBx'), \delta(Ax, Bx')\}$$

$$(ii)\ \delta(BSy, ATy') \le c\max\{\delta(y, y'), \delta(y, BSy), \delta(y', ATy'), d(Sy, Ty')\}$$

for all x,x' in X and y,y' in Y, where $0 \le c < 1$
If either X is $(\sum)$ d-complete and Y satisfies (W_4) or Y is $(\sum)$ δ-complete and X satisfies (W_4), and one of the mappings A, B, S and T is continuous then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y . Further, $Az = Bz = w$ and $Sw = Tw = z$.

Proof : Let x be an arbitrary point in X. Define the sequences $\{x_n\}$ and $\{y_n\}$ in X and Y, respectively, as follows: $y_1 = Ax$, $x_1 = Sy_1$, $y_2 = Bx_1$, $x_2 = Ty_2$, $y_3 = Ax_2$....
In general, we define $x_{2n-1} = Sy_{2n-1}$, $y_{2n} = Bx_{2n-1}$, $x_{2n} = Ty_{2n}$ and $y_{2n+1} = Ax_{2n}$ for $n = 1, 2, ...$
On the one hand, using inequality (i) we get

$$\begin{array}{lcc} d(x_{2n}, x_{2n+1}) & = & d(TBx_{2n-1}, SAx_{2n}) \\ & \le & c\max\{d(x_{2n}, x_{2n-1}), d(x_{2n}, SAx_{2n}), d(x_{2n-1}, TBx_{2n-1}), \delta(Ax_{2n}, Bx_{2n-1})\} \\ & \le & c\max\{d(x_{2n-1}, x_{2n}), d(x_{2n}, x_{2n+1}), \delta(y_{2n}, y_{2n+1})\} \end{array}$$

Then

$$d(x_{2n}, x_{2n+1}) \le c\max\{d(x_{2n-1}, x_{2n}), \delta(y_{2n}, y_{2n+1})\}$$

Similarly, using inequality (i), we get

$$d(x_{2n-1}, x_{2n}) \leq c \max\{d(x_{2n-1}, x_{2n-2}), \delta(y_{2n-1}, y_{2n})\}$$

which imply

$$d(x_n, x_{n+1}) \leq c \max\{d(x_{n-1}, x_n), \delta(y_n, y_{n+1})\}$$

On the other hand, applying inequality (ii) we get

$$\delta(y_{2n}, y_{2n+1}) \leq c \max\{d(x_{2n-1}, x_{2n}), \delta(y_{2n-1}, y_{2n})\}$$

and

$$\delta(y_{2n-1}, y_{2n}) \leq c \max\{d(x_{2n-1}, x_{2n-2}), \delta(y_{2n-1}, y_{2n-2})\}$$

which imply

$$\delta(y_n, y_{n+1}) \leq c \max\{d(x_{n-1}, x_n), \delta(y_{n-1}, y_n)\}$$

It follows that

$$\max\{d(x_n, x_{n+1}), \delta(y_n, y_{n+1})\} \leq c^{n-1} \max\{d(x_1, x_2), \delta(y_1, y_2)\} = c^{n-1} M_{d,\delta}$$

where $M_{d,\delta} = \max\{d(x_1, x_2), \delta(y_1, y_2)\}$.
Therefore, we get $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = \lim\limits_{n\to\infty} \delta(y_n, y_{n+1}) = 0$.
Suppose that X is $(\sum)$ d-complete. We have

$$\sum_{k=1}^{k=n} d(x_k, x_{k+1}) \leq M_{d,\delta} \sum_{k=1}^{k=n} c^{k-1}, \quad n \geq 1$$

which implies $\sum\limits_{k=1}^{+\infty} d(x_k, x_{k+1}) < \infty$. Therefore $x_n \to z$ for some $z \in X$. Let $w = Az$ and suppose that A is continuous. Then $\lim\limits_{n\to\infty} \delta(y_{2n+1}, w) = \lim\limits_{n\to\infty} \delta(Ax_{2n}, Az) = 0$ and therefore $\lim\limits_{n\to\infty} \delta(y_{2n}, w) = 0$ since $\lim\limits_{n\to\infty} \delta(y_{2n}, y_{2n+1}) = 0$ and Y satisfies (W_4).
Hence $\lim\limits_{n\to\infty} \delta(y_n, w) = 0$.
Using inequality (i) we get

$$\begin{aligned} d(Sw, x_{2n}) &= d(SAz, TBx_{2n-1}) \\ &\leq c \max\{d(z, x_{2n-1}), d(z, SAz), d(x_{2n-1}, TBx_{2n-1}), \delta(Az, Bx_{2n-1})\} \end{aligned}$$

Letting n tend to infinity, on using the 1-continuity of d, we get $d(Sw, z) \leq cd(Sw, z)$ and therefore $Sw = z = SAz$. Applying inequality (ii) we get

$$\begin{aligned} \delta(Bz, y_{2n+1}) &= d(BSw, ATy_{2n}) \\ &\leq c \max\{\delta(w, y_{2n}), \delta(w, BSw), \delta(y_{2n}, ATy_{2n}), d(Sw, Ty_{2n})\} \end{aligned}$$

Letting n tend to infinity, on using the 1-continuity of δ, we obtain $\delta(Bz, w) \leq c\delta(Bz, w)$ and therefore $Bz = w = BSw$. Using inequality (i) we have

$$\begin{aligned} d(z, Tw) &= d(SAz, TBz) \\ &\leq c \max\{d(z, z), d(z, SAz), d(z, TBz), \delta(Az, Bz)\} \\ &\leq cd(z, Tw) \end{aligned}$$

from which it follows that $Tw = z = TBz$.
The same results of course hold if one of the mappings B, S, T is continuous instead

of A. To prove uniqueness, suppose that SA and TB have a second fixed point z'. On using uniquality (i) we get

$$\begin{array}{rcl} d(z,z') & = & d(SAz,TBz') \\ & \leq & c\max\{d(z,z'),d(z,SAz),d(z',TBz'),\delta(Az,Bz')\} \\ & \leq & c\max\{d(z,z'),\delta(w,w')\} \end{array}$$

Therefore $d(z,z') \leq c\delta(w,w')$. Similarly, using inequality (ii) we get

$$\begin{array}{rcl} \delta(w,w') & = & \delta(BSw,ATw') \\ & \leq & c\max\{\delta(w,w'),\delta(w',BSw'),\delta(w',ATw'),d(Swz,Tw')\} \\ & \leq & c\max\{\delta(w,w'),d(z,z')\} \end{array}$$

and so $\delta(w,w') \leq cd(z,z')$ and therefore $d(z,z') \leq c\delta(w,w') \leq c^2 d(z,z')$. Hence $z = z'$.
Similarly, we prove that w is the unique fixed point of BS and AT. The same results of course hold if Y is supposed $(\sum)$ δ-complete. This completes the proof of the Theorem.

For a metric space and a semi-metric space, we have the following new results

Corollary 2.1. *Let (X,d) be a 1-continuous semi-metric space and (Y,δ) be a metric space. let A,B be mappings of X into Y, and let S,T be mappings of Y into X satisfying*

$$(i)\ d(SAx,TBx') \leq c\max\{d(x,x'),d(x,SAx),d(x',TBx'),\delta(Ax,Bx')\}$$

$$(ii)\ \delta(BSy,ATy') \leq c\max\{\delta(y,y'),\delta(y,BSy),\delta(y',ATy'),d(Sy,Ty')\}$$

for all x,x' in X and y,y' in Y, where $0 \leq c < 1$
If either X is $(\sum)$ d-complete or Y is complete and X satisfies (W_4), and one of the mappings A,B,S and T is continuous then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y . Further, $Az = Bz = w$ and $Sw = Tw = z$.

Corollary 2.2. *Let (X,d) be a metric space and (Y,δ) be a 1-continuous semi-metric space. let A,B be mappings of X into Y, and let S,T be mappings of Y into X satisfying*

$$(i)\ d(SAx,TBx') \leq c\max\{d(x,x'),d(x,SAx),d(x',TBx'),\delta(Ax,Bx')\}$$

$$(ii)\ \delta(BSy,ATy') \leq c\max\{\delta(y,y'),\delta(y,BSy),\delta(y',ATy'),d(Sy,Ty')\}$$

for all x,x' in X and y,y' in Y, where $0 \leq c < 1$
If either X is complete and Y satisfies (W_4) or Y is $(\sum)$ δ-complete, and one of the mappings A,B,S and T is continuous then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y . Further, $Az = Bz = w$ and $Sw = Tw = z$.

When (X,d) and (Y,δ) are metric spaces, Theorem 2.1 gives a generalization of Theorem 2 in [1] in the following way

Corollary 2.3. *Let (X,d) and (Y,δ) be two metric spaces. let A,B be mappings of X into Y, and let S,T be mappings of Y into X satisfying*

$$(i)\ d(SAx,TBx') \leq c\max\{d(x,x'),d(x,SAx),d(x',TBx'),\delta(Ax,Bx')\}$$

$$(ii)\ \delta(BSy,ATy') \leq c\max\{\delta(y,y'),\delta(y,BSy),\delta(y',ATy'),d(Sy,Ty')\}$$

for all x,x' in X and y,y' in Y, where $0 \leq c < 1$
If either X *or* Y *is complete, and one of the mappings* A, B, S *and* T *is continuous then* SA *and* TB *have a unique common fixed point* z *in* X *and* BS *and* AT *have a unique common fixed point* w *in* Y *. Further,* $Az = Bz = w$ *and* $Sw = Tw = z$*.*

Acknowledgments. The authors are very grateful to the referee for his comments and helpful suggestions.

References

[1] A.H. Soliman, M. Imdad, and M. Hasan, Proving unifiedcommon fixed point Theorems via common property (E-A) in symmetric spaces, Commun. Korean Math. Soc. 25 (2010), No. 4, pp. 629-645.
[2] Fisher B. and Murthy P.P., Related fixed points theorems for two pairs of mappings on two metric spaces. Kyungpook Math. J. 37(1997), 343-347.
[3] S. H. Cho, G. Y. Lee, and J. S. Bae, On coincidence and fixed-point theorems in sym- metric spaces, Fixed Point Theory Appl. 2008 (2008), Art. ID 562130, 9 pp.
[4] T. L. Hicks, Rhoades, Fixed points theory in symmetric spaces with applications to probabilistic spaces, Nonlinear Analysis 36(1999), 331-344.
[5] W. A. Wilson, On semi-metric spaces, Amer. J. Math. 53(1931)361-373.

MOHAMED AAMRIi, SAMIA BENNANI
Université Hassan II-Mohammédia,
Faculté des Sciences Ben M'sik,
Département de Mathématiques, BP. 7955, Sidi Othmane,
Casablanca, Maroc.

DRISS EL MOUTAWAKILl,
Université Hassan 1er,
Faculté polydisciplinaire de Khouribga, BP. 145,
Khouribga, Maroc.

E-mail address: d.elmoutawakil@gmail.com