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ON SOME QI TYPE INEQUALITIES USING FRACTIONAL q-INTEGRAL (COMMUNICATED BY NAIM BRAHA)

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Abstract. In this paper, we provide some Qi type inequalities using a fractional q-integral.

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1. introduction

In [12], Feng Qi proposed the following problem:
Under what conditions does the inequality:

$$\int_a^b [f(x)]^t dx \geq \left[\int_a^b f(x)dx\right]^{t-1} \tag{1}$$

hold for $t > 1$?

In view of the interest in this type of inequalities, many attentions have been payed to the problem and many authors have extended the inequality to more general cases (see [1, 2, 3, 4, 9, 8, 11, 14]). In this paper, we establish some inequalities of F. Qi type by using a fractional q-integral and we will generalize the inequalities given in [3].
This paper is organized as follows: In Section 2, we present some definitions and facts from the q-calculus necessary for understanding this paper. In Section 3, we discuss some generalizations of Qi's inequality.

2. Notations and preliminaries

Throughout this paper, we will fix $q \in (0,1)$. For the convenience of the reader, we provide in this section a summary of the mathematical notations and definitions used in this paper (see [5, 7, 13]). We write for $a \in \mathbb{C}$,

$$[a]_q = \frac{1-q^a}{1-q}, \ (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \ n = 1, 2, ...\infty,$$

$$[0]_q! = 1, \qquad [n]_q! = [1]_q[2]_q...[n]_q, \quad n = 1, 2, ...$$

and

$$(x-a)^{(n)} = \begin{cases} 1, & n = 0 \\ (x-a)(x-qa)...(x-q^{n-1}a), & n \neq 0 \end{cases}$$

Their natural expansions to the reals are

$$(a-b)^{(\alpha)} = a^{\alpha}\frac{(\frac{b}{a};q)_\infty}{(q^\alpha \frac{b}{a};q)_\infty},\ (a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}.$$

Notice that

$$(a-b)^{(\alpha)} = a^{\alpha}(\frac{b}{a};q)_\alpha.$$

The q-derivative $D_q f$ of a function f is given by

$$(D_q f)(x) = \frac{f(x)-f(qx)}{(1-q)x},\ \text{if } x \neq 0, \tag{2}$$

$(D_q f)(0) = f'(0)$ provided $f'(0)$ exists.
Clearly

$$D_{q,t}(b-t)^{(\alpha)} = -[\alpha]_q (b-qt)^{(\alpha-1)}. \tag{3}$$

The q-Jackson integrals from 0 to a is defined by (see [6])

$$\int_0^a f(x)d_q x = (1-q)a\sum_{n=0}^{\infty} f(aq^n)q^n, \tag{4}$$

provided the sum converges absolutely.
The q-Jackson integral in a generic interval $[a,b]$ is given by (see [6])

$$\int_a^b f(x)d_q x = \int_0^b f(x)d_q x - \int_0^a f(x)d_q x. \tag{5}$$

For any function f, we have (see [7])

$$D_q\left(\int_a^x f(t)d_q t\right) = f(x). \tag{6}$$

The fractional q-integral of the Riemann-Liouville type is[13]

$$I^{\alpha}_{q,a}(f)(x) = I^{\alpha}_{q,a}(f(t))(x) = \frac{1}{\Gamma_q(\alpha)}\int_a^x (x-qt)^{(\alpha-1)} f(t)d_q t, \qquad \alpha > 0. \tag{7}$$

Finally, for $b > 0$ and $a = bq^n$, $\quad n = 1, 2, \dots, \infty$, we write

$$[a,\ b]_q = \{bq^k:\ 0 \le k \le n\} \quad \text{and} \quad (a,\ b]_q = [q^{-1}a,\ b]_q.$$

3. Fractional q-Integral Inequalities of Qi type

Let us begin by recalling the following useful result[3]:

Lemma 1. *Let $p \ge 1$ be a real number and g be a nonnegative and monotonous function on $[a,b]_q$. Then*

$$pg^{p-1}(qx)D_q g(x) \le D_q[g^p(x)] \le pg^{p-1}(x)D_q g(x), \quad x \in (a,b]_q. \tag{8}$$

Proposition 1. *Let f be a function defined on $[a,b]_q$ satisfying*

$$f(a) \ge 0 \quad \text{and} \quad D_q f(x) \ge (\beta-2)C_\alpha^{\beta-2}(x-a)^{\beta-3} \quad \text{for} \quad x \in (a,b]_q \quad \text{and} \quad \beta \ge 3.$$

Then

$$I^{\alpha}_{q,a}(f^{\beta}(t))(b) \ge \Gamma_q^{\beta-2}(\alpha)\left[I^{\alpha}_{q,a}(f(qt))(b)\right]^{\beta-1}, \tag{9}$$

where $C_\alpha = \sup\limits_{t\in[a,b]_q}(b-qt)^{(\alpha-1)} = \begin{cases} (b-qa)^{(\alpha-1)}, & \text{if} \quad \alpha \ge 1 \\ b^{\alpha-1}(q,q)_{\alpha-1}, & \text{if} \quad 0 < \alpha < 1. \end{cases}$

Proof. For $x \in (a,b]_q$, we put

$$g(x) = \int_a^x (b-qt)^{(\alpha-1)} f(qt) d_qt \quad \text{and } F(x) = \int_a^x (b-qt)^{(\alpha-1)} [f(t)]^\beta d_qt - \left(\int_a^x (b-qt)^{(\alpha-1)} f(qt) d_qt\right)^{\beta-1}.$$

We have

$$D_qF(x) = (b-qx)^{(\alpha-1)} f^\beta(x) - D_q[g^{\beta-1}](x).$$

Since f and g increase on $[a,b]_q$, we obtain from Lemma 1,

$$\begin{aligned} D_qF(x) &\geq (b-qx)^{(\alpha-1)} f^\beta(x) - (\beta-1) g^{\beta-2}(x)(b-qx)^{(\alpha-1)} f(qx) \\ &\geq (b-qx)^{(\alpha-1)} f^\beta(x) - (\beta-1) g^{\beta-2}(x)(b-qx)^{(\alpha-1)} f(x) \geq (b-qx)^{(\alpha-1)} f(x) h(x), \end{aligned}$$

where $h(x) = f^{\beta-1}(x) - (\beta-1) g^{\beta-2}(x)$.
On the other hand, we have

$$D_qh(x) = D_q[f^{\beta-1}](x) - (\beta-1) D_q[g^{\beta-2}](x).$$

Using Lemma 1 once again leads to

$$\begin{aligned} D_qh(x) &\geq (\beta-1) f^{\beta-2}(qx) D_qf(x) - (\beta-1)(\beta-2) g^{\beta-3}(x) D_qg(x) \\ &\geq (\beta-1) f^{\beta-2}(qx) D_qf(x) - (\beta-1)(\beta-2) g^{\beta-3}(x)(b-qx)^{(\alpha-1)} f(qx) \\ &\geq (\beta-1) f(qx) \left[f^{\beta-3}(qx) D_qf(x) - (\beta-2)(b-qx)^{(\alpha-1)} g^{\beta-3}(x)\right]. \end{aligned}$$

The use of the inequality,

$$\int_a^x (b-qt)^{(\alpha-1)} f(qt) d_qt \quad \leq \quad C_\alpha f(qx)(x-a)$$

gives

$$\begin{aligned} D_qh(x) &\geq (\beta-1) f^{\beta-2}(qx) \left[D_qf(x) - (\beta-2)(b-qx)^{(\alpha-1)} C_\alpha^{\beta-3} (x-a)^{\beta-3}\right] \\ &\geq (\beta-1) f^{\beta-2}(qx) \left[D_qf(x) - (\beta-2) C_\alpha^{\beta-2} (x-a)^{\beta-3}\right] \geq 0. \end{aligned}$$

From the fact $h(a) = f^{\beta-1}(a) \geq 0$, we get $h(x) \geq 0, \quad x \in [a,b]_q$.
Therefore $D_qF(x) \geq 0$, and so $F(x) \geq 0$ for all $x \in [a,b]_q$, in particular

$$F(b) = \int_a^b (b-qt)^{(\alpha-1)} [f(t)]^\beta d_qt - \left(\int_a^b (b-qt)^{(\alpha-1)} f(qt) d_qt\right)^{\beta-1} \geq 0,$$

hence

$$\frac{1}{\Gamma_q(\alpha)} \int_a^b (b-qt)^{(\alpha-1)} [f(t)]^\beta d_qt \geq \Gamma_q^{\beta-2}(\alpha) \left(\frac{1}{\Gamma_q(\alpha)} \int_a^b (b-qt)^{(\alpha-1)} f(qt) d_qt\right)^{\beta-1}.$$

The proof is complete. ■

Remark: We can see that for $\alpha = 1$, we obtain the proposition 3.2 of [3].

Corollary 1. *Let n be a positive integer and f be a function defined on $[a,b]_q$ satisfying*

$$f(a) \geq 0 \quad \textit{and} \quad D_qf(x) \geq nC_\alpha^n (x-a)^{n-1}, \quad x \in (a,b]_q.$$

Then

$$I_{q,a}^\alpha (f^{n+2}(t))(b) \geq \Gamma_q^n(\alpha) \left[I_{q,a}^\alpha (f(qt))(b)\right]^{n+1}.$$

Proof. It suffices to take $\beta = n+2$ in Proposition 1 and the result follows. ■

Corollary 2. *Let n be a positive integer and f be a function defined on $[a,b]_q$ satisfying*

$$D_q^i f(a) \geq 0, \quad 0 \leq i \leq n-1 \quad \textit{and} \quad D_q^n f(x) \geq n[n-1]_q! C_\alpha^n; \quad x \in (a,b]_q.$$

Then

$$I_{q,a}^\alpha (f^{n+2}(t))(b) \geq \Gamma_q^n(\alpha) \left[I_{q,a}^\alpha (f(qt))(b)\right]^{n+1}.$$

Proof. Since $D_q^n f(x) \geq n[n-1]_q! C_\alpha^n$, then by q-integrating $n-1$ times over $[a,x]$, we get $D_q f(x) \geq nC_\alpha^n (x-a)^{(n-1)} \geq nC_\alpha^n (x-a)^{n-1}$. The result turns out from Corollary 1. ■

Proposition 2. *Let $p \geq 1$ be a real number and f be a function defined on $[a,b]_q$ satisfying*

$$f(a) \geq 0, \qquad D_q f(x) \geq pC_\alpha^p,\ x \in (a,b]_q. \tag{10}$$

Then

$$I_{q,a}^\alpha (f^{p+2}(t))(b) \geq \frac{[\Gamma_q(\alpha)]^p}{(b-a)^{p-1}} \left[I_{q,a}^\alpha (f(qt))(b) \right]^{p+1}. \tag{11}$$

Proof. Define $g(x) = \int_a^x (b-qt)^{(\alpha-1)} f(qt) d_q t$ and

$$H(x) = \int_a^x (b-qt)^{(\alpha-1)} f^{p+2}(t) d_q t - \frac{1}{(b-a)^{p-1}} \left[\int_a^x (b-qt)^{(\alpha-1)} f(qt) d_q t \right]^{p+1}, \quad x \in [a,b]_q.$$

We have

$$D_q H(x) = (b-qx)^{(\alpha-1)} f^{p+2}(x) - \frac{1}{(b-a)^{p-1}} D_q[g^{p+1}](x), \quad x \in (a,b]_q.$$

Since f and g increase on $[a,b]_q$, we obtain according to Lemma 1,

$$\begin{aligned} D_q H(x) &\geq (b-qx)^{(\alpha-1)} f^{p+2}(x) - \frac{1}{(b-a)^{p-1}} (p+1) g^p(x) D_q g(x) \\ &\geq (b-qx)^{(\alpha-1)} f^{p+2}(x) - \frac{1}{(b-a)^{p-1}} (p+1) g^p(x) (b-qx)^{(\alpha-1)} f(qx) \\ &\geq \left(f^{p+1}(x) - \frac{1}{(b-a)^{p-1}} (p+1) g^p(x) \right) (b-qx)^{(\alpha-1)} f(x) = (b-qx)^{(\alpha-1)} f(x) h(x), \end{aligned}$$

where $h(x) = f^{p+1}(x) - \frac{1}{(b-a)^{p-1}} (p+1) g^p(x)$.
On the other hand, we have

$$D_q h(x) = D_q[f^{p+1}](x) - \frac{1}{(b-a)^{p-1}} (p+1) D_q[g^p](x).$$

In virtue of Lemma 1, it follows that

$$\begin{aligned} D_q h(x) &\geq (p+1) f^p(qx) D_q f(x) - \frac{(p+1)p}{(b-a)^{p-1}} g^{p-1}(x) (b-qx)^{(\alpha-1)} f(qx) \\ &\geq (p+1) f(qx) \left[f^{p-1}(qx) D_q f(x) - \frac{p}{(b-a)^{p-1}} g^{p-1}(x) (b-qx)^{(\alpha-1)} \right]. \end{aligned}$$

Since f is non-negative and increasing function, then

$$g(x) = \int_a^x (b-qt)^{(\alpha-1)} f(qt) d_q t \quad \leq \quad f(qx) C_\alpha (x-a),$$

hence

$$D_q h(x) \quad \geq \quad (p+1) f^p(qx) \left[D_q f(x) - pC_\alpha^p \right] \geq 0,$$

which implies that h increases on $[a,b]_q$.
Finally, since $h(a) = f^{p+1}(a) \geq 0$, then H increases and $H(b) \geq H(a) \geq 0$.
The proof is complete. ■

In what follows, we will adopt the terminology of the following definition.

Definition 1. *Let $b>0$ and $a=bq^n$, n be a positive integer. For each real number r, we denote by $E^{\alpha}_{q,r}([a,b])$ the set of functions defined on $[a,b]_q$ such that*

$$f(a)\geq 0 \qquad \text{and} \qquad D_qf(x)\geq [r]_qK_\alpha, \quad \forall x\in(a,b]_q,$$

where $K_\alpha=\sup_{t\in[a,b]_q}(b-q^2t)^{(\alpha-1)}=\begin{cases}(b-q^2a)^{(\alpha-1)} & \text{if } \alpha\geq 1\\ b^{\alpha-1}(q^2,q)_{\alpha-1} & \text{if } 0<\alpha<1.\end{cases}$

Proposition 3. *Let $f\in E^{\alpha}_{q,2}([a,b])$. Then for all $p>0$, we have*

$$I^{\alpha}_{q,a}\left(f^{2p+1}(t)\right)(b)>\Gamma_q(\alpha)\left[I^{\alpha}_{q,a}\left(f^p(t)\right)(b)\right]^2. \tag{12}$$

Proof. For $x\in[a,b]_q$, let

$$F(x)=\int_a^x(b-qt)^{(\alpha-1)}f^{2p+1}(t)d_qt-\left[\int_a^x(b-qt)^{(\alpha-1)}f^p(t)d_qt\right]^2$$

and $g(x)=\int_a^x(b-qt)^{(\alpha-1)}f^p(t)d_qt$.

We have for $x\in[a,b]_q$,

$$\begin{aligned}D_qF(x) &= (b-qx)^{(\alpha-1)}f^{2p+1}(x)-(b-qx)^{(\alpha-1)}f^p(x)(g(x)+g(qx))\\ &= (b-qx)^{(\alpha-1)}f^p(x)G(x),\end{aligned}$$

where $G(x)=f^{p+1}(x)-[g(x)+g(qx)]$.

On the other hand, we have

$$\begin{aligned}D_qG(x) &= \frac{f^{p+1}(x)-f^{p+1}(qx)}{(1-q)x}-\Big((b-qx)^{(\alpha-1)}f^p(x)+q(b-q^2x)^{(\alpha-1)}f^p(qx)\Big)\\ &= f^p(x)\frac{f(x)-(1-q)x(b-qx)^{(\alpha-1)}}{(1-q)x}-f^p(qx)\frac{f(qx)+q(b-q^2x)^{(\alpha-1)}(1-q)x}{(1-q)x}.\end{aligned}$$

From the relation $D_qf(x)\geq K_\alpha[2]_q\geq(b-q^2x)^{(\alpha-1)}[2]_q$, we obtain
$f(x)\geq f(qx)+(1-q^2)x(b-q^2x)^{(\alpha-1)}$,
hence

$$D_qG(x)\geq\frac{f^p(x)-f^p(qx)}{(1-q)x}[f(qx)+q(b-q^2x)^{(\alpha-1)}(1-q)x]>0 \quad x\in(a,b]_q. \tag{13}$$

Therefore G is strictly increasing on $[a,b]_q$. Moreover, we have

$$G(a)=[f(a)]^{p+1}+(b-qa)^{(\alpha-1)}(f(a))^p\geq 0,$$

then for all $x\in(a,b]_q, G(x)>G(a)\geq 0$,which proves that $D_qF(x)>0$, and so F is strictly increasing on $[a,b]_q$. In particular, $F(b)>F(a)=0$. ■

Corollary 3. *Let $\beta>0$ and $f\in E^{\alpha}_{q,2}([a,b])$. Then for all positive integers m, we have*

$$I^{\alpha}_{q,a}\left(f^{(\beta+1)2^m-1}(t)\right)(b)>\Gamma_q^{2^m-1}(\alpha)\left[I^{\alpha}_{q,a}\left(f^{\beta}(t)\right)(b)\right]^{2^m}. \tag{14}$$

Proof. We suggest here a proof by induction. For this purpose, we note

$$p_m(\beta)=(\beta+1)2^m-1.$$

We have

$$p_m(\beta)>0 \quad \text{and} \quad p_{m+1}(\beta)=2p_m(\beta)+1. \tag{15}$$

From Proposition 3, we deduce that the inequality (14) is true for $m=1$.
Suppose that (14) holds for an integer m and let us prove it for $m+1$.
By using the relation (15) and Proposition 3, we obtain

$$I^{\alpha}_{q,a}(f^{(\beta+1)2^{m+1}-1}(t))(b)>\Gamma_q(\alpha)\left[I^{\alpha}_{q,a}(f^{(\beta+1)2^m-1}(t))(b)\right]^2. \tag{16}$$

And by assumption, we have

$$I_{q,a}^{\alpha}\left(f^{(\beta+1)2^m-1}(t)\right)(b) > \Gamma_q^{2^m-1}(\alpha)\left[I_{q,a}^{\alpha}\left(f^{\beta}(t)\right)(b)\right]^{2^m}. \tag{17}$$

Finally, the relations (16) and (17) imply that the inequality (14) is true for $m+1$. This completes the proof. ■

Corollary 4. *Let $f \in E_{q,2}^{\alpha}([a,b])$ and $\beta > 0$. For $m \in \mathbb{N}$, we have*

$$\left[I_{q,a}^{\alpha}\left(f^{(\beta+1)2^{m+1}-1}(t)\right)(b)\right]^{\frac{1}{2^{m+1}}} > [\Gamma_q(\alpha)]^{\frac{1}{2^{m+1}}}\left[I_{q,a}^{\alpha}\left(f^{(\beta+1)2^m-1}(t)\right)(b)\right]^{\frac{1}{2^m}}. \tag{18}$$

Proof. Since, from Proposition 3,

$$I_{q,a}^{\alpha}(f^{(\beta+1)2^{m+1}-1}(t))(b) > \Gamma_q(\alpha)\left[I_{q,a}^{\alpha}(f^{(\beta+1)2^m-1}(t))(b)\right]^2, \tag{19}$$

then

$$\left[I_{q,a}^{\alpha}\left(f^{(\beta+1)2^{m+1}-1}(t)\right)(b)\right]^{\frac{1}{2^{m+1}}} > [\Gamma_q(\alpha)]^{\frac{1}{2^{m+1}}}\left[I_{q,a}^{\alpha}\left(f^{(\beta+1)2^m-1}(t)\right)(b)\right]^{\frac{1}{2^m}}. \tag{20}$$

■

Corollary 5. *Let $f \in E_{q,2}^{\alpha}([a,b])$. For all integers $m \geq 2$, we have*

$$I_{q,a}^{\alpha}(f^{2^{m+1}-1}(t))(b) > [\Gamma_q(\alpha)]^{2^m-1}\left[I_{q,a}^{\alpha}(f(t))(b)\right]^{2^m}. \tag{21}$$

Proof. By using Proposition 3 for $\beta = 1$, we obtain the result. ■

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