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SOME PROPERTIES OF LORENTZIAN SASAKIAN MANIFOLDS WITH TANAKA-WEBSTER CONNECTION

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Abstract. In this work we study some curvature properties of Lorentzian Sasakian manifolds with respect to the Tanaka-Webster connection and obtain some results about the slant curves of the 3-dimensional Lorentzian Sasakian manifold with Tanaka-Webster connection.

1. Introduction

If a differentiable manifold has a Lorentzian metric g, i.e., a symmetric non-degenerated (0,2) tensor field of index 1, then it is called a Lorentzian manifold. Generally, a differentiable manifold has a Lorentzian metric if and only if it has a 1-dimensional distribution. Hence odd dimensional manifold is able to have a Lorentzian metric. It is very natural and interesting to define both a Sasakian structure and a Lorentzian metric on an odd dimensional manifold. In fact, odd dimensional de Sitter space and Goedell Universe, that are important examples on relativity theory, have Sasakian structure with Lorentzian metric, [6], [8], [15].

In this paper, we will define the Tanaka-Webster connection on a Lorentzian Sasakian manifold and investigate some of its properties like curvature tensor, projective curvature tensor and locally ϕ-symmetry, see [12], [14]. As is well known, the unit 3-sphere S^3 is a typical example of a Sasakian manifold. In 3-dimensional contact metric geometry, Legendre curves play a fundamental role. As a generalisation of Legendre curves, in this paper, we will also study slant curves of a Lorentzian Sasakian 3-manifold M with the Tanaka-Webster connection, [1].

A curve on a manifold is said to be *slant* if its tangent vector field has constant angle with the Reeb vector field ξ. It is well known that biharmonic curves in 3-dimensional Sasakian space forms are slant helices, see, [4], [5].

2. Preliminaries

2.1. **Sasakian manifolds with Lorentzian metric.** Let M be a differentiable manifold of class C^∞ and ϕ, ξ, η be a tensor field of type (1.1), a vector field, a

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1-form on M, respectively, such that

$$\begin{gathered}\phi^2(X) = -X + \eta(X)\xi, \phi\xi = 0, \\ \eta(\phi X) = 0, \eta(\xi) = 1\end{gathered} \tag{2.1}$$

for any vector field X on M. Then M is said to have an almost contact structure (ϕ, ξ, η) and is called an almost contact manifold.The almost contact structure is said to be normal if $N + 2d\eta \otimes \xi = 0$, where

$$N_\phi(X, Y) = [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + \phi^2[X, Y], \forall X, Y \in \varkappa(M),$$

is the Nijenhuis tensor field of ϕ and $\varkappa(M)$ denotes the Lie algebra of all smooth vector fields on M, [2].

Since M has a globally defined unique vector field ξ which is also called the *Reeb vector field*, it is able to have a Lorentzian metric g such that $g(\xi, \xi) = -1$, see [7], [8]. If M has the normal almost contact structure (ϕ, ξ, η) and the Lorentzian metric g with

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), (\bigtriangledown_X \eta)(Y) = g(\phi X, Y), X, Y \in \varkappa(M) \tag{2.2}$$

where $\bigtriangledown$ is the covariant derivative with respect to g, then M is called a *Sasakian manifold with the Lorentzian metric*.

In a Sasakian manifold with Lorentzian metric, we have

$$\begin{gathered}\eta(X) = -g(\xi, X), \bigtriangledown_X \xi = -\phi X, \bigtriangledown_X \phi(Y) = -\eta(Y)X - g(X, Y)\xi, \\ X, Y \in \varkappa(M).\end{gathered} \tag{2.3}$$

The formulas (2.3) imply that an almost contact manifold is Sasakian if and only if its Reeb vector field ξ is a Killing vector field. A Frenet curve parametrised by arc length s is said to be a *slant curve* if its contact angle defined by $cos\theta(s) = g(T(s), \xi)$ is constant, where $T(s)$ is the tangent vector field of the curve. The Riemann curvature tensor R of a Sasakian manifold with Lorentzian metric satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y = g(\xi, X)Y - g(\xi, Y)X. \tag{2.4}$$

If D is the contact distribution in a contact manifold (M, ϕ, ξ, η), defined by the subspaces $D_x = \{X \in T_xM \mid \eta(X) = 0\}$, then a one-dimensional integral submanifold of D will be called a Legendre curve. These curves are the slant curves of contact angle $\frac{\pi}{2}$. A curve $\gamma : I \to M$, parametrized by its arc length is a Legendre curve if and only if $\eta(\gamma') = 0$, [1].

A plane section in T_pM is called a ϕ-section if there exists a vector $X \in T_pM$ orthogonal to ξ such that $\{X, \phi X\}$ span the section. The sectional curvature, $K(X, \phi X)$, is called ϕ-sectional curvature. A Sasakian manifold of constant ϕ-sectional curvature with Lorentzian metric g will be called a Lorentzian Sasakian space form and denoted by $M(c)$. The curvature tensor of a Sasakian space form with Lorentzian metric is given by

$$\begin{gathered}4R(X, Y)Z = (c - 3)\{g(Y, Z)X - g(X, Z)Y\} + (c + 1)\{g(X, \phi Z)\phi Y \\ -g(Y, \phi Z)\phi X + 2g(X, \phi Y)\phi Z\} + (c + 1)\{\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\ +g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi\},\end{gathered} \tag{2.5}$$

where c is a constant, [8].

2.2. **Tanaka-Webster connection on a Sasakian manifold.** Now, we review the Tanaka-Webster connection, on a $(2n+1)$- dimensional Sasakian manifold M with Lorentzian metric g, see [16] and [17]. We denote by $\bigtriangledown$ the Lorentzian connection defined by g. Let r be arbitrary fixed real number, and let A be a tensor fields of type (1,2) defined by

$$A(X)Y = g(\phi X, Y)\xi + r\eta(X)\phi(Y) + \eta(Y)\phi X \tag{2.6}$$

for all vector fields X, Y on M. Then we can define a linear connection D (D-connection, for short) as

$$D_X Y = \bigtriangledown_X Y + A(X)Y. \tag{2.7}$$

The tensor fields ξ, η, g and A are parallel with respect to the D- connection, for the proof, see [8]. If we choose $r = 1$ in (2.6) we get the special form of D-connection which is called the Tanaka-Webster connection and denoted by $\hat{\bigtriangledown}$, that is we will define

$$\hat{\bigtriangledown}_X Y = \bigtriangledown_X Y + g(\phi X, Y)\xi + \eta(X)\phi(Y) + \eta(Y)\phi X. \tag{2.8}$$

We see that the Tanaka-Webster connection $\hat{\bigtriangledown}$ for Sasakian manifold M with Lorentzian metric g has the torsion

$$\hat{T}(X, Y) = -2g(X, \phi Y)\xi. \tag{2.9}$$

Lemma 2.1. *([8]) The tensor A satisfies followings*

$$\begin{aligned} A(A(Z)X)Y &= g(X, \phi Z)\phi Y - g(X, Y)\eta(Z)\xi - g(Y, Z)\eta(X)\xi \\ &\quad -\eta(Y)\eta(Z)X - \eta(X)\eta(Y)Z, \\ A(Z)A(X)Y - A(X)A(Z)Y &= \eta(X)g(Z, Y)\xi - \eta(Z)g(X, Y)\xi + \eta(Y)\eta(X)Z \\ &\quad +g(\phi X, Y)\phi Z - g(\phi Z, Y)\phi X - \eta(Z)\eta(Y)X. \end{aligned}$$

3. Curvature tensors of Tanaka-Webster connection

Since the curvature tensor $\hat{R}$ of the Tanaka-Webster connection and the curvature tensor R of the Lorentzian connection satisfies

$$\hat{R}(X, Y)Z = R(X, Y)Z + A(A(Y)X)Z - A(A(X)Y)Z + A(X)A(Y)Z - A(Y)A(X)Z,$$

from Lemma 2.1, we have the following :

Proposition 3.1. *([8]) Curvature tensors $\hat{R}$ and R satisfies following equation*

$$\begin{aligned} \hat{R}(X, Y)Z = R(X, Y)Z &+ 2g(\phi X, Y)\phi Z + g(Z, Y)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\ &+\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X. \end{aligned} \tag{3.1}$$

As the Reeb vector field ξ is a parallel vector field with respect to the Tanaka-Webster connection, we obtain following

Theorem 3.2. *([8]) Let M be a Sasakian manifold with Lorentzian metric. Then the sectional curvature $\hat{K}(X, \xi)$ of the Tanaka-Webster connection with respect to a section spanned by ξ and X is identically zero.*

From Proposition 3.1, we have the following equations about the Ricci tensors and the scalar curvatures.

Proposition 3.3. *([8]) The Ricci tensor $\hat{Ric}$ of the Tanaka-Webster connection and the Ricci tensor Ric of the Lorentzian connection satisfies*

$$\hat{Ric}(X,Y) = Ric(X,Y) - 2g(X,Y) - 2(n+1)\eta(X)\eta(Y). \tag{3.2}$$

The scalar curvature $\hat{\rho}$ of the Tanaka-Webster connection and the scalar curvature of the Lorentzian connection satisfies $\hat{\rho} = \rho - 2n$. Now, we will prove the following theorem:

Theorem 3.4. *Let M be a Sasakian manifold with Lorentzian metric. If M is of constant curvature c with respect to the Tanaka-Webster connection, then $c = 0$.*
Proof.
From Proposition 3.1, it follows that

$$R(X,Y)Z = cg(Y,Z)X - cg(X,Z)Y - 2g(\phi X,Y)\phi Z - g(Z,Y)\eta(X)\xi$$
$$+g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y - g(\phi X,Z)\phi Y + g(\phi Y,Z)\phi X$$

by virtue of the assumption. Hence, using (2.4), we have

$$R(X,\xi)Z = (1-c)\eta(Z)X + (1-c)g(X,Z)\xi = \eta(Z)X + g(X,Z)\xi,$$

so that $c\{\eta(Z)X + g(X,Z)\xi\} = 0$, for any vectors X and Z. Putting $Z = \xi$ and $\eta(X) = 0$ in this equation, we obtain $c = 0$.

4. Projective curvature tensor on Lorentzian Sasakian manifolds with Tanaka-Webster connection

Let M be an (2n+1)-dimensional Lorentzian Sasakian manifold equipped with a Tanaka-Webster connection. Since the Ricci tensor $\hat{Ric}$ of the Tanaka-Webster connection is symmetric, the projective curvature tensor of the Sasakian manifold with respect to the Tanaka-Webster connection can be defined by

$$\hat{P}(X,Y)Z = \hat{R}(X,Y)Z - \frac{1}{2n}\{\hat{Ric}(Y,Z)X - \hat{Ric}(X,Z)Y\}. \tag{4.1}$$

Using (3.1) and (3.2), (4.1) reduces to

$$\begin{aligned}\hat{P}(X,Y)Z = {} & R(X,Y)Z + 2g(\phi X,Y)\phi Z + g(Z,Y)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ & +\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X \\ & -\frac{1}{2n}\{Ric(Y,Z)X - 2g(Y,Z)X - 2(n+1)\eta(Y)\eta(Z)X \\ & -Ric(X,Z)Y + 2g(X,Z)Y + 2(n+1)\eta(X)\eta(Z)Y\}\end{aligned}$$

or

$$\begin{aligned}\hat{P}(X,Y)Z = {} & P(X,Y)Z + 2ng(\phi X,Y)\phi Z + g(Z,Y)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ & -\eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X + g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X \\ & +g(Y,Z)X - g(X,Z)Y,\end{aligned}$$

where P is the projective curvature tensor with respect to the Lorentzian metric on the manifold, [12]. Putting $Z=\xi$ in the last equation and using (2.3), we have that

$$\hat{P}(X,Y)\xi = 0. \tag{4.2}$$

Definition. *A Sasakian manifold is called ξ-projectively flat if the condition $P(X,Y)\xi = 0$ is satisfied on the manifold.*

So from (4.2) we have that the following

Theorem 4.1. *A Lorentzian Sasakian manifold with Tanaka-Webster connection is ξ-projectively flat with respect to the Tanaka-Webster connection $\hat{\bigtriangledown}$.*

Now we express the following definitions which we will need later.

Definition. *A Sasakian manifold is called ϕ-projectively flat if the condition*

$$\phi^2 P(\phi X, \phi Y)\phi Z) = 0 \tag{4.3}$$

is satisfied on the manifold, [13].

Definition. *A Sasakian manifold is called η-Einstein manifold if it satisfies the condition $Ric(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y)$ for any real numbers α and β, [18].*

Let us assume that M is a ϕ-projectively flat Lorentzian Sasakian manifold with a Tanaka-Webster connection on it. Then, it can be verified that

$$g(\hat{P}(\phi X, \phi Y)\phi Z, \phi W) = 0, \tag{4.4}$$

so from (4.1) we have

$$g(\hat{R}(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{2n}\{\hat{Ric}(\phi Y, \phi Z)g(\phi X, \phi W) - \hat{Ric}(\phi X, \phi Z)g(\phi Y, \phi W)\}.$$

for $X, Y, Z, W \in T(M)$.

Let $\{e_1, e_2, ..., e_{2n}, \xi\}$ be an orthonormal basis of the vector fields in M. Putting $X = W = e_i$ in the last equation and summing up over i, we have

$$\sum_{i=1}^{2n} g(\hat{R}(\phi e_i, \phi Y)\phi Z, \phi e_i) = \frac{1}{2n}\sum_{i=1}^{2n}\{\hat{Ric}(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - \hat{Ric}(\phi e_i, \phi Z)g(\phi Y, \phi e_i\}. \tag{4.5}$$

Using (2.1)-(2.3) and (3.2), it can be easily verified that

$$\begin{aligned}
\sum_{i=1}^{2n} g(\hat{R}(\phi e_i, \phi Y)\phi Z, \phi e_i) &= \sum_{i=1}^{2n} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) + (2n+1)g(Y,Z) + \eta(Y)\eta(Z) \\
&= \sum_{i=1}^{2n} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) + 2ng(Y,Z) + g(\phi Y, \phi Z) \\
&= Ric(Y,Z) + R(\xi, Y, Z, \xi) + 2ng(Y,Z) + g(\phi Y, \phi Z) \\
&= Ric(Y,Z) + \eta(Z)Y - \eta(Y)Z + (2n+1)g(Y,Z) + \eta(Y)\eta(Z) \\
&= \hat{Ric}(Y,Z) + (4n+1)g(Y,Z) - (4n+1)\eta(Y)\eta(Z).
\end{aligned}$$

On the other hand taking into account that

$$\sum_{i=1}^{2n} g(\phi e_i, \phi e_i) = 2n, \sum_{i=1}^{2n} \hat{Ric}(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = \hat{Ric}(\phi Y, \phi Z),$$

from (4.5) we obtain that

$$\begin{aligned}\frac{2n-1}{2n}\hat{Ric}(\phi Y,\phi Z) &= \frac{2n-1}{2n}\{\hat{Ric}(Y,Z)+\eta(Y)\eta(Z)\} \\ &= \hat{Ric}(Y,Z)+(4n+1)g(Y,Z)-(4n+1)\eta(Y)\eta(Z).\end{aligned} \tag{4.6}$$

Using (3.2) and $Ric(X,\xi)=2n\eta(X)$, from (4.6) we get

$$\hat{Ric}(Y,Z)=\alpha g(Y,Z)+\beta\eta(Y)\eta(Z), \tag{4.7}$$

where $\alpha=-2n(4n+1)$ and $\beta=4n(2n+1)-1$.

Hence we can state the following

Theorem 4.2. *If a Lorentzian Sasakian manifold is ϕ-projectively flat with respect to the Tanaka-Webster connection then the manifold is an η-Einstein manifold with respect to the Tanaka-Webster connection.*

5. Locally ϕ-symmetric Lorentzian Sasakian manifolds with Tanaka-Webster connection

Definition. *A Sasakian manifold M is called to be locally ϕ-symmetric if the condition $\phi^2(\bigtriangledown_W R)(X,Y)Z=0$ for all vector fields $X,Y,Z,W\in T(M)$ orthogonal to ξ. This notion was introduced by Takahashi [14].*

Now, we will consider a Lorentzian Sasakian manifold M with Tanaka-Webster connection and define locally ϕ-symmetry on it by

$$\phi^2(\hat{\bigtriangledown}_W\hat{R})(X,Y)Z=0 \tag{5.1}$$

for all vector fields $X,Y,Z,W\in T(M)$ orthogonal to ξ.
From (2.8) we have that

$$\begin{aligned}(\hat{\bigtriangledown}_W\hat{R})(X,Y)Z &= (\bigtriangledown_W\hat{R})(X,Y)Z+g((\phi\hat{R})(X,Y)Z,W)\xi \\ &\quad +\eta(W)(\phi\hat{R})(X,Y)Z+\eta(\hat{R}(X,Y)Z)\phi(W)\end{aligned} \tag{5.2}$$

and from (3.1), we may rewrite the curvature tensor of Tanaka-Webster connection as follows

$$\begin{aligned}\hat{R}(X,Y)Z &= R(X,Y)Z+2d\eta(X,Y)\phi Z+d\eta(X,Z)\phi(Y)-d\eta(Y,Z)\phi(X) \\ &\quad +g(Z,Y)\eta(X)\xi-g(X,Z)\eta(Y)\xi+\eta(X)\eta(Z)Y-\eta(Y)\eta(Z)X.\end{aligned}$$

Differentiating this equation in the direction of W we get

$$\begin{aligned}(\bigtriangledown_W\hat{R})(X,Y)Z &= (\bigtriangledown_W R)(X,Y)Z+2d\eta(X,Y)(\bigtriangledown_W\phi)(Z)+d\eta(X,Z)(\bigtriangledown_W\phi)(Y) \\ &\quad -d\eta(Y,Z)(\bigtriangledown_W\phi)(X)+\{(\bigtriangledown_W\eta)(X)g(Y,Z)-(\bigtriangledown_W\eta)(Y)g(X,Z)\}\xi \\ &\quad +(\bigtriangledown_W\xi)\{\eta(X)g(Y,Z)-\eta(Y)g(X,Z)\}+(\bigtriangledown_W\eta)(X)\eta(Z)Y+(\bigtriangledown_W\eta)(Z)\eta(X)Y \\ &\quad -(\bigtriangledown_W\eta)(Y)\eta(Z)X-(\bigtriangledown_W\eta)(Z)\eta(Y)X.\end{aligned}$$

Then, using (2.2), (2.3) in this equation we have

$$\begin{aligned}(\bigtriangledown_W\hat{R})(X,Y)Z &= (\bigtriangledown_W R)(X,Y)Z-2d\eta(X,Y)\{g(Z,W)\xi+\eta(Z)W\} \\ &\quad -d\eta(X,Z)\{g(Y,W)\xi+\eta(Y)W\}+d\eta(Y,Z)\{g(X,W)\xi+\eta(X)W\} \\ &\quad +\{g(X,\phi W)g(Y,Z)-g(Y,\phi W)g(X,Z)\}\xi-\phi W\{\eta(X)g(Y,Z)-\eta(Y)g(X,Z)\} \\ &\quad +g(X,\phi W)\eta(Z)Y+g(Z,\phi W)\eta(X)Y-g(Y,\phi W)\eta(Z)X-g(Z,\phi W)\eta(Y)X.\end{aligned}$$

Now, applying ϕ^2 both sides of (5.2) we have

$$\phi^2(\hat{\bigtriangledown}_W \hat{R})(X,Y)Z = \phi^2(\bigtriangledown_W \hat{R})(X,Y)Z + \eta(W)\phi^2(\phi\hat{R})(X,Y)Z \\ -\phi(W)\eta(\hat{R}(X,Y)Z)$$

and using (2.1) and this equation in (5.2) we get

$$\begin{aligned}\phi^2(\hat{\bigtriangledown}_W \hat{R})(X,Y)Z &= \eta(W)\phi^2(\phi\hat{R})(X,Y)Z - \phi(W)\eta(\hat{R}(X,Y)Z) + \phi^2(\bigtriangledown_W R)(X,Y)Z \\ &-2d\eta(X,Y)\{\eta(Z)\eta(W)+\eta(Z)W\} - d\eta(X,Z)\{\eta(Y)\eta(W)+\eta(Y)W\} \\ &+d\eta(Y,Z)\{\eta(X)\eta(W)+\eta(X)W\} + \phi W\{\eta(X)g(Y,Z)-\eta(Y)g(X,Z)\} \\ &-\{g(X,\phi W)\eta(Y)\eta(Z)+g(Y,\phi W)\eta(X)\eta(Z)\}\xi - g(X,\phi W)\eta(Z)Y - g(Z,\phi W)\eta(X)Y \\ &+g(Y,\phi W)\eta(Z)X - g(Z,\phi W)\eta(Y)X.\end{aligned}$$

If we take X, Y, Z, W orthogonal to ξ, from the last equation we find that

$$\phi^2(\hat{\bigtriangledown}_W \hat{R})(X,Y)Z = \phi^2(\bigtriangledown_W R)(X,Y)Z.$$

Hence we can state the following:

Theorem 5.1. *For a Lorentzian Sasakian manifold the Lorentzian connection $\bigtriangledown$ is locally ϕ-symmetric if and only if the Tanaka-Webster connection $\hat{\bigtriangledown}$ is locally ϕ-symmetric.*

6. Slant curves in 3-dimensional Lorentzian Sasakian manifolds with Tanaka-Webster connection

Let M be 3-dimensional Lorentzian Sasakian manifold equipped with Tanaka-Webster connection $\hat{\bigtriangledown}$ and γ be a slant curve parametrised by arc length s in M. Then $cos\theta(s)$=$g(\hat{T}(s),\xi)$ is constant and satisfies $cos\theta(s) = -\eta(\hat{T})$, where g is the Lorentzian metric. Since $\hat{\bigtriangledown}$ is a metrical connection, *i.e.*, $\hat{\bigtriangledown}g = 0$, there exists an orthonormal frame field $\{\hat{T}, \hat{N}, \hat{B}\}$ along γ such that $\hat{T} = \gamma^{'}$ and satisfies the following Frenet-Serret equation with respect to the Tanaka-Webster connection:

$$\begin{aligned}\hat{\bigtriangledown}_{\hat{T}}\hat{T} &= \hat{\kappa}\hat{N}\\ \hat{\bigtriangledown}_{\hat{T}}\hat{N} &= -\hat{\kappa}\hat{T} + \hat{\tau}\hat{B}\\ \hat{\bigtriangledown}_{\hat{T}}\hat{B} &= -\hat{\kappa}\hat{N}.\end{aligned}$$

Here $\hat{\kappa}$ and $\hat{\tau}$ are the curvature and the torsion of γ, respectively. For a unit speed curve $\gamma(s)$ in 3-dimensional Lorentzian Sasakian manifold, by virtue of (2.3) and (2.8) we get

$$\hat{\bigtriangledown}_{\hat{T}}\hat{T} = \bigtriangledown_{\hat{T}}\hat{T} + 2\eta(\hat{T})\phi\hat{T} = \bigtriangledown_{\hat{T}}\hat{T} + 2cos\theta(s)\phi\hat{T}, \tag{6.1}$$

where $\bigtriangledown$ is the Lorentzian connection of M, [4],[5]. Now, if $\gamma(s)$ is a Legendre curve in M and $\{T, N, B\}$ the Frenet frame along $\gamma(s)$, then the tangent vector field T can be defined by $T(s)$=$\acute{\gamma}$ and the curvature $\kappa(s)$ of $\gamma(s)$ is given by $\bigtriangledown_T T$=κN. Since the formula (6.1) implies that every Legendre curve $\gamma(s)$ in M satisfies

$$\hat{\bigtriangledown}_{\acute{\gamma}}\acute{\gamma} = \bigtriangledown_{\acute{\gamma}}\acute{\gamma}, \tag{6.2}$$

the mean curvature vector field $\bigtriangledown_{\acute{\gamma}}\acute{\gamma}$ coincides with the $\hat{\bigtriangledown}_{\acute{\gamma}}\acute{\gamma}$ so that we have $\hat{N} = N = \phi T$ and $\hat{\kappa} = \kappa$. Thus every Legendre curve has zero torsion with respect to the Tanaka-Webster connection and so we have that every Legendre curve in M is

$\hat{\bigtriangledown}$-geodesic if and only if it is a $\bigtriangledown$-geodesic, [1]. Now we choose an adapted local orthonormal frame field $X, \phi X, \xi$ of M such that $\eta(X) = 0$.

Let $\gamma(s)$ be a non-geodesic Frenet curve in 3-dimensional Lorentzian Sasakian manifold with Tanaka-Webster connection.
Differentiating the equation $cos\theta(s)$=$g(\hat{T}(s), \xi)$ along $\gamma(s)$, then it follows that

$$-\theta'.sin\theta = g(\hat{\bigtriangledown}_{\hat{T}}\hat{T}, \xi) + g(\hat{T}, \hat{\bigtriangledown}_{\hat{T}}\xi) = g(\hat{\kappa}\hat{N}, \xi) + g(\hat{T}, \bigtriangledown_T \xi + \phi\hat{T} + g(\phi\hat{T}, \xi)\xi). \tag{6.3}$$

If $\gamma(s)$ is a slant curve of M, then from (6.3) it follows that

$$g(\hat{\kappa}\hat{N}, \xi) + g(\hat{T}, \bigtriangledown_T \xi) + g(\hat{T}, \phi\hat{T}) + cos\theta(\eta(\phi T)) = 0.$$

This equation and from (2.1) we have $\eta(\hat{N}) = 0$. Hence we proved the following result, see [4],[9].

Proposition 6.1. *A non-geodesic curve $\gamma(s)$ in a 3-dimensional Lorentzian Sasakian manifold with Tanaka-Webster connection is a slant curve if and only if it satisfies $\eta(\hat{N}) = 0$.*

Hence the reeb vector field ξ can be written as follows $\xi = cos\theta\hat{T} \mp sin\theta\hat{B}$. This means that the reeb vector field is in the plane spanned by $\hat{T}$ and $\hat{B}$, namely $g(\xi, \hat{N}) = 0$. On the other hand, with respect to an adapted local orthonormal frame fields $X, \phi X, \xi$ of M such that $\eta(X) = 0$ we have the following equalities of the Frenet vector fields $\hat{T}$, $\hat{N}$ and $\hat{B}$ for some function $\lambda(s)$,

$$\hat{T} = sin\theta\{cos\lambda X + sin\lambda\phi X\} + cos\theta\xi,$$
$$\hat{N} = -sin\lambda X + cos\lambda\phi X.$$
$$\hat{B} = \mp cos\theta cos\lambda X \mp cos\theta sin\lambda\phi X \pm cosec\theta\xi.$$

Differentiating the equation $g(\xi, \hat{N}) = 0$ along the slant curve $\gamma(s)$ of M and using (6.1) Frenet-Serre equations and the following identities

$$\phi\hat{T} = -sin\theta sin\lambda X + sin\theta cos\lambda\phi X - g(X, \xi)sin\theta sin\lambda\xi$$
$$\phi\hat{N} = -cos\lambda X - sin\lambda\phi X - g(X, \xi)cos\lambda\xi,$$

it follows that

$$\begin{aligned}
g(\hat{\bigtriangledown}_{\hat{T}}\hat{N}, \xi) + g(\hat{N}, \hat{\bigtriangledown}_{\hat{T}}\xi) &= 0,\\
g(\bigtriangledown_{\hat{T}}\hat{N} - g(\hat{T}, \xi)\phi\hat{N} + g(\phi\hat{T}, \hat{N})\xi, \xi) + g(\hat{N}, g(\phi\hat{T}, \xi), \xi) &= 0,\\
g(-\hat{\kappa}\hat{T} + \hat{\tau}\hat{B} - cos\theta\phi\hat{N}, \xi) - g(\phi\hat{T}, \hat{N}) + g(\hat{N}, g(\phi\hat{T}, \xi)\xi) &= 0,\\
\hat{\kappa}cos\theta \pm \hat{\tau}sin\theta - cos\theta g(\phi\hat{N}, cos\hat{T} \pm sin\theta\hat{B}) - g(\phi\hat{T}, \hat{N}) &= 0,\\
\hat{\kappa}cos\theta \pm \hat{\tau}sin\theta - cos^2\theta\{-sin\theta + cos\theta cos\lambda g(X, \xi)\}&\\
\mp sin\theta cos\theta\{\pm cos\theta \pm \frac{cos\lambda}{sin\theta}g(X, \xi)\} - sin\theta &= 0,
\end{aligned}$$

from this we have that $\hat{\kappa}cos\theta + (\pm\hat{\tau} + 1)sin\theta = 0$ or $\frac{\hat{\kappa}}{\pm\hat{\tau}-1} = const.$. Thus we proved that the following result

Theorem 6.2. *If a non-geodesic curve of a 3-dimensional Lorentzian Sasakian manifold with Tanaka-Webster connection is a slant curve, then $\frac{\hat{\kappa}}{\pm\hat{\tau}-1} = const.$*

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