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A NOTE ON Ψ-BOUNDED SOLUTIONS FOR NON-HOMOGENEOUS MATRIX DIFFERENCE EQUATIONS

(COMMUNICATED BY AGACIK ZAFER)

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Abstract. This paper deals with obtaing necessary and sufficient conditions for the existence of at least one Ψ-bounded solution for the non-homogeneous matrix difference equation $X(n+1) = A(n)X(n)B(n)+F(n)$, where $F(n)$ is a Ψ-bounded matrix valued function on $\mathbb{Z}^+$. Finally, we prove a result relating to the asymptotic behavior of the Ψ-bounded solutions of this equation on $\mathbb{Z}^+$.

1. Introduction

The theory of difference equations is a lot richer than the corresponding theory of differential equations. Many authors have studied several problems related to difference equations, such as existence and uniqueness theorem [11], transmission of information [6], signal processing, oscillation [16], control and dynamic systems [10, 14]. The application of theory of difference equations is already extended to various fields such as numerical analysis, finite element techniques, control theory and computer science [1, 2, 8]. This paper deals with the linear matrix difference equation

$$X(n+1) = A(n)X(n)B(n) + F(n), \tag{1.1}$$

where $A(n)$, $B(n)$, and $F(n)$ are $m \times m$ matrix-valued functions on $\mathbb{Z}^+ = \{1, 2, \ldots\}$.

The Ψ-bounded solutions for system of difference equations were developed by Han and Hong [9], Diamandescu [3, 5]. The existence and uniqueness of solutions of matrix difference equation (1.1) was studied by Murty, Anand and Lakshmi [11]. Murty and Suresh Kumar [12, 13] and Dimandescu [4] obtained results on Ψ-bounded solutions for matrix Lyapunov systems. Recently in [15], we obtained a necessary and sufficient condition for the existence of Ψ-bounded solution of the matrix difference equation (1.1), provided $F(n)$ is Ψ-summable in $\mathbb{Z}$.

The aim of this paper is to provide a necessary and sufficient condition for the existence of Ψ-bounded solution of the non homogeneous matrix difference equation (1.1) via Ψ-bounded sequences. The introduction of the matrix function Ψ permits

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to obtain a mixed asymptotic behavior of the components of the solutions. Here, Ψ is a matrix-valued function. This paper include the results of Diamandescu[5] as a particular case when $B = I$, X and F are column vectors.

2. Preliminaries

In this section we present some basic definitions, notations and results which are useful for later discussion.

Let $\mathbb{R}^m$ be the Euclidean m-space. For $u = (u_1, u_2, u_3, \dots, u_m)^T \in \mathbb{R}^m$, let $\|u\| = \max\{|u_1|, |u_2|, |u_3|, \dots, |u_m|\}$ be the norm of u. Let $\mathbb{R}^{m\times m}$ be the linear space of all $m \times m$ real valued matrices. For an $m \times m$ real matrix $A = [a_{ij}]$, we use the matrix norm $|A| = \sup_{\|u\|\le 1} \|Au\|$.

Let $\Psi_k : \mathbb{Z}^+ \to \mathbb{R} - \{0\}$ ($\mathbb{R} - \{0\}$ is the set of all nonzero real numbers), $k = 1, 2, \dots m$, and let

$$\Psi = \mathrm{diag}[\Psi_1, \Psi_2, \dots, \Psi_m].$$

Then the matrix $\Psi(n)$ is an invertible square matrix of order m, for all $n \in \mathbb{Z}^+$.

Definition 2.1. *A matrix function $X(n)$ is said to be Ψ-bounded solution of* (1.1) *if $X(n)$ satisfies the equation* (1.1) *and also $\Psi(n)X(n)$ is bounded for all $n \in \mathbb{Z}^+$.*

Definition 2.2. [7] *Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, then the Kronecker product of A and B is written as $A \otimes B$ and is defined to be the partitioned matrix*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & . & . & . & a_{1n}B \\ a_{21}B & a_{22}B & . & . & . & a_{2n}B \\ . & . & . & . & . & . \\ a_{m1}B & a_{m2}B & . & . & . & a_{mn}B \end{bmatrix}$$

which is an $mp \times nq$ matrix and in $\mathbb{R}^{mp\times nq}$.

Definition 2.3. [7] *Let $A = [a_{ij}] \in \mathbb{R}^{m\times n}$, then the vectorization operator $Vec : \mathbb{R}^{m\times n} \to \mathbb{R}^{mn}$ is defined as*

$$\hat{A} = VecA = \begin{bmatrix} A_{.1} \\ A_{.2} \\ . \\ . \\ A_{.n} \end{bmatrix}, \textit{ where } A_{.j} = \begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ a_{mj} \end{bmatrix}, (1 \le \ j \le \ n).$$

Lemma 2.1. *The vectorization operator $Vec : \mathbb{R}^{m\times m} \to \mathbb{R}^{m^2}$, is a linear and one-to-one operator. In addition, Vec and Vec^{-1} are continuous operators.*

Proof. The fact that the vectorization operator is linear and one-to-one is immediate. Now, for$A = [a_{ij}] \in \mathbb{R}^{m\times m}$, we have

$$\|Vec(A)\| = \max_{1\le i,j\le m} \{|a_{ij}|\} \le \max_{1\le i\le m} \left\{ \sum_{j=1}^{m} |a_{ij}| \right\} = |A| .$$

Thus, the vectorization operator is continuous and $\|Vec\| \le 1$.

In addition, for $A = I_m$ (identity $m \times m$ matrix) we have$\|Vec(I_m)\| = 1 = |I_m|$ and then $\|Vec\| = 1$.

Obviously, the inverse of the vectorization operator, $Vec^{-1}:\mathbb{R}^{m^2}\to\mathbb{R}^{m\times m}$, is defined by

$$Vec^{-1}(u)=\begin{bmatrix} u_1 & u_{m+1} & . & . & . & u_{m^2-m+1}\\ u_2 & u_{m+2} & . & . & . & u_{m^2-m+2}\\ . & . & . & . & . & .\\ . & . & . & . & . & .\\ . & . & . & . & . & .\\ u_m & u_{2m} & . & . & . & u_{m^2}\end{bmatrix},$$

where $u=(u_1,u_2,u_3,.....,u_{m^2})^T\in\mathbb{R}^{m^2}$. We have

$$\left|Vec^{-1}(u)\right|=\max_{1\le i\le m}\left\{\sum_{j=0}^{m-1}|u_{mj+i}|\right\}\le m\max_{1\le i\le m}\{|u_i|\}=m\,\|u\|.$$

Thus, Vec^{-1} is a continuous operator. Also, if we take $u=VecA$ in the above inequality, then the following inequality holds

$$|A|\le m\|VecA\|,$$

for every $A\in\mathbb{R}^{m\times m}$. □

Regarding properties and rules for Kronecker product of matrices we refer to [7].

Now by applying the Vec operator to the linear nonhomogeneous matrix difference equation (1.1) and using Kronecker product properties, we have

$$\hat{X}(n+1)=G(n)\hat{X}(n)+\hat{F}(n), \tag{2.1}$$

where $G(n)=B^T(n)\otimes A(n)$ is a $m^2\times m^2$ matrix and $\hat{F}(n)=VecF(n)$ is a column matrix of order m^2. The equation (2.1) is called the Kronecker product difference equation associated with (1.1).
The corresponding homogeneous difference equation of (2.1) is

$$\hat{X}(n+1)=G(n)\hat{X}(n). \tag{2.2}$$

Definition 2.4. [3] *A function $\phi:\mathbb{Z}^+\to\mathbb{R}^m$ is said to be Ψ- bounded on $\mathbb{Z}^+$ if $\Psi(n)\phi(n)$ is bounded on $\mathbb{Z}^+$ (i.e., there exists $L>0$ such that $\|\Psi(n)\phi(n)\|\le L$, for all $n\in\mathbb{Z}^+$).*

Extend this definition for matrix functions.

Definition 2.5. *A matrix function $F:\mathbb{Z}^+\to\mathbb{R}^{m\times m}$ is said to be Ψ-bounded on $\mathbb{Z}^+$ if the matrix function ΨF is bounded on $\mathbb{Z}^+$ (i.e., there exists $L>0$ such that $|\Psi(n)F(n)|\le L$, for all $n\in\mathbb{Z}^+$).*

Now we shall assume that $A(n)$ and $B(n)$ are invertable $m\times m$ matrices on $\mathbb{Z}^+$ and $F(n)$ is a Ψ-bounded matrix function on $\mathbb{Z}^+$.
The following lemmas play a vital role in the proof of main result.

Lemma 2.2. *The matrix function $F:\mathbb{Z}^+\to\mathbb{R}^{m\times m}$ is Ψ-bounded on $\mathbb{Z}^+$ if and only if the vector function $VecF(n)$ is $I_m\otimes\Psi$-bounded on $\mathbb{Z}^+$.*

Proof. From the proof of Lemma 2.1, it follows that

$$\frac{1}{m}|A|\le\|VecA\|_{\mathbb{R}^{m^2}}\le|A|$$

for every $A\in\mathbb{R}^{m\times m}$.

Put $A = \Psi(n)F(n)$ in the above inequality, we have

$$\frac{1}{m}\left|\Psi(n)F(n)\right| \leq \left\|(I_m \otimes \Psi(n)).VecF(n)\right\|_{\mathbb{R}^{m^2}} \leq \left|\Psi(n)F(n)\right|, \tag{2.3}$$

$n \in \mathbb{Z}^+$, for all matrix functions $F(n)$.

Suppose that $F(n)$ is Ψ-bounded on $\mathbb{Z}^+$. From (2.3)

$$\left\|(I_m \otimes \Psi(n)).VecF(n)\right\|_{\mathbb{R}^{m^2}} \leq \left|\Psi(n)F(n)\right|,$$

From Definitions 2.4 and 2.5, $\hat{F}(n)$ is $I_m \otimes \Psi$-bounded on $\mathbb{Z}^+$.

Conversely, suppose that $\hat{F}(n)$ is $I_m \otimes \Psi$-bounded on $\mathbb{Z}^+$. Again from (2.3), we have

$$\left|\Psi(n)F(n)\right| \leq m\left\|(I_m \otimes \Psi(n)).VecF(n)\right\|_{\mathbb{R}^{m^2}}.$$

From, Definitions 2.4 and 2.5, $F(n)$ is Ψ-bounded on $\mathbb{Z}^+$. Now the proof is complete. □

Lemma 2.3. *Let $Y(n)$ and $Z(n)$ be the fundamental matrices for the matrix difference equations*

$$X(n+1) = A(n)X(n), \quad n \in \mathbb{Z}^+ \tag{2.4}$$

and

$$X(n+1) = B^T(n)X(n), \quad n \in \mathbb{Z}^+ \tag{2.5}$$

respectively. Then the matrix $Z(n) \otimes Y(n)$ is a fundamental matrix of (2.2).

Proof. Consider

$$\begin{aligned} Z(n+1) \otimes Y(n+1) &= B^T(n)Z(n) \otimes A(n)Y(n) \\ &= (B^T(n) \otimes A(n))(Z(n) \otimes Y(n)) \\ &= G(n)(Z(n) \otimes Y(n)), \end{aligned}$$

for all $n \in \mathbb{Z}^+$.

On the other hand, the matrix $Z(n)\otimes Y(n)$ is an invertible matrix for all $n \in \mathbb{Z}^+$ (because $Z(n)$ and $Y(n)$ are invertible matrices for all $n \in \mathbb{Z}^+$). □

Let $\mathbf{X}_1$ denote the subspace of $\mathbb{R}^{n\times n}$ consisting of all matrices which are values of Ψ-bounded solution of $X(n+1) = A(n)X(n)B(n)$ on $\mathbb{Z}^+$ at $n = 1$ and let $\mathbf{X}_2$ an arbitrary fixed subspace of $\mathbb{R}^{n\times n}$, supplementary to $\mathbf{X}_1$. Let P_1, P_2 denote the corresponding projections of $\mathbb{R}^{n\times n}$ onto $\mathbf{X}_1$, $\mathbf{X}_2$ respectively.

Then $\overline{\mathbf{X}}_1$ denote the subspace of $\mathbb{R}^{n^2}$ consisting of all vectors which are values of $I_n \otimes \Psi$-bounded solution of (2.2) on $\mathbb{Z}^+$ at $n = 1$ and $\overline{\mathbf{X}}_2$ a fixed subspace of $\mathbb{R}^{n^2}$, supplementary to $\overline{\mathbf{X}}_1$. Let Q_1, Q_2 denote the corresponding projections of $\mathbb{R}^{n^2}$ onto $\overline{\mathbf{X}}_1$, $\overline{\mathbf{X}}_2$ respectively.

Theorem 2.1. *Let $Y(n)$ and $Z(n)$ be the fundamental matrices for the systems* (2.4) *and* (2.5). *If*

$$\begin{aligned} \hat{X}(n) = &\sum_{k=1}^{n-1}(Z(n) \otimes Y(n))Q_1(Z^{-1}(k+1) \otimes Y^{-1}(k+1))\hat{F}(k) \\ &- \sum_{k=1}^{\infty}(Z(n) \otimes Y(n))Q_2(Z^{-1}(k+1) \otimes Y^{-1}(k+1))\hat{F}(k) \end{aligned} \tag{2.6}$$

is convergent, then it is a solution of (2.1) *on* $\mathbb{Z}^+$.

Proof. It is easily seen that $\hat{X}(n)$ is the solution of (2.1) on $\mathbb{Z}^+$. □

The following theorems are useful in the proofs of our main results.

Theorem 2.2. [5] *The equation*

$$x(n+1) = A(n)x(n) + f(n) \tag{2.7}$$

has at least one Ψ*-bounded solution on* $N = \{1, 2, \ldots\}$ *for every* Ψ*-bounded sequence* f *on* N *if and only if there is a positive constant* K *such that, for all* $n \in N$,

$$\sum_{k=1}^{n-1} |\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)| + \sum_{k=n}^{\infty} |\Psi(n)Y(n)P_2Y^{-1}(k+1)\Psi^{-1}(k)| \leq K. \tag{2.8}$$

Theorem 2.3. [5] *Suppose that:*

(1) *The fundamental matrix* $Y(n)$ *of* $x(n+1) = A(n)x(n)$ *satisfies the inequality* (2.8)*, for all* $n \geq 1$*, where* K *is positive constant.*
(2) *The matrix* Ψ *satisfies the condition* $|\Psi(n)\Psi^{-1}(n+1)| \leq T$*, for all* $n \in N$*, where* T *is positive constant.*
(3) *The* Ψ*-bounded function* $f : N \to \mathbb{R}^m$ *is such that* $\lim_{n\to\infty} \|\Psi(n)f(n)\| = 0$.

Then, every Ψ*-bounded solution* $x(n)$ *of* (2.7) *satisfies*

$$\lim_{n\to\infty} \|\Psi(n)x(n)\| = 0.$$

3. Main results

Our first theorem is as follows.

Theorem 3.1. *Let* $A(n)$ *and* $B(n)$ *be bounded matrices on* $\mathbb{Z}^+$*, then* (1.1) *has at least one* Ψ*-bounded solution on* $\mathbb{Z}^+$ *for every* Ψ*-bounded matrix function* $F : \mathbb{Z}^+ \to \mathbb{R}^{m\times m}$ *on* $\mathbb{Z}^+$ *if and only if there exists a positive constant* K *such that*

$$\begin{aligned}
&\sum_{k=1}^{n-1} |(Z(n) \otimes \Psi(n)Y(n))Q_1(Z^{-1}(k+1) \otimes Y^{-1}(k+1)\Psi^{-1}(k))| \\
&\quad + \sum_{k=n}^{\infty} |(Z(n) \otimes \Psi(n)Y(n))Q_2(Z^{-1}(k+1) \otimes Y^{-1}(k+1)\Psi^{-1}(k))| \leq K.
\end{aligned} \tag{3.1}$$

Proof. Suppose that the equation (1.1) has at least one Ψ-bounded solution on $\mathbb{Z}^+$ for every Ψ-bounded matrix function $F : \mathbb{Z}^+ \to \mathbb{R}^{m\times m}$. Let $\hat{F} : \mathbb{Z}^+ \to \mathbb{R}^{m^2}$ be $I_m \otimes \Psi$-bounded function on $\mathbb{Z}^+$. From Lemma 2.2, it follows that the matrix function $F(n) = Vec^{-1}\hat{F}(n)$ is Ψ - bounded matrix function on $\mathbb{Z}^+$. From the hypothesis, the system (1.1) has at least one Ψ - bounded solution $X(n)$ on $\mathbb{Z}^+$. From Lemma 2.2, it follows that the vector valued function $\hat{X}(n) = VecX(n)$ is a $I_m \otimes \Psi$-bounded solution of (2.1) on $\mathbb{Z}^+$.

Thus, equation (2.1) has at least one $I_m \otimes \Psi$-bounded solution on $\mathbb{Z}^+$ for every $I_m \otimes \Psi$-bounded function $\hat{F}$ on $\mathbb{Z}^+$. From Theorem 2.2, there exists a positive number K, the fundamental matrix $U(n)$ of (2.2) satisfies

$$\sum_{k=1}^{n-1} |(I_m \otimes \Psi(n))U(n)Q_1T^{-1}(k+1)(I_m \otimes \Psi(k))| + \sum_{k=n}^{\infty} |(I_m \otimes \Psi(n))U(n)Q_2T^{-1}(k+1)(I_m \otimes \Psi(k))| \leq K$$

From Lemma 2.3, $U(n) = Z(n) \otimes Y(n)$ and using Kronecker product properties, (3.1) holds. Conversely suppose that (3.1) holds for some $K > 0$.

Let $F : \mathbb{Z}^+ \to \mathbb{R}^{n\times n}$ be a Ψ-bounded matrix function on $\mathbb{Z}^+$. From Lemma 2.2, it follows that the vector valued function $\hat{F}(n) = VecF(n)$ is a $I_m \otimes \Psi$-bounded function on $\mathbb{Z}^+$.

Since $A(n)$, $B(n)$ are invertible, then $G(n) = B^T(n) \otimes A(n)$ is also invertible. Now from Theorem 2.2, the difference equation (2.1) has at least one $I_m \otimes \Psi$ -bounded solution on $\mathbb{Z}^+$. Let $x(n)$ be this solution.

From Lemma 2.2, it follows that the matrix function $X(n) = Vec^{-1}x(n)$ is a Ψ-bounded solution of the equation (1.1) on $\mathbb{Z}^+$ (because $F(n) = Vec^{-1}\hat{F}(n)$).

Thus, the matrix difference equation (1.1) has at least one Ψ-bounded solution on $\mathbb{Z}^+$ for every Ψ-bounded matrix function F on $\mathbb{Z}^+$. □

Finally, we give a result in which we will see that the asymptotic behavior of solution of (1.1) is completely determined by the asymptotic behavior of F.

Theorem 3.2. *Suppose that:*
(1) The fundamental matrices $Y(n)$ and $Z(n)$ of (2.4) and (2.5) satisfies:
(a) $|\Psi(n)\Psi^{-1}(n+1)| \leq M$, *where M is a positive constant*
(b) *condition (3.1), for some $K > 0$.*
(2) The matrix function $F : \mathbb{Z}^+ \to \mathbb{R}^{m\times m}$ is Ψ-bounded on $\mathbb{Z}^+$ such that $\lim_{n\to\infty} |\Psi(n)F(n)| = 0$.
Then, every Ψ-bounded solution X of (1.1) is such that

$$\lim_{n\to\infty} |\Psi(n)X(n)| = 0.$$

Proof. Let $X(n)$ be a Ψ-bounded solution of (1.1). From Lemma 2.2, the function $\hat{X}(n) = VecX(n)$ is a $I_m \otimes \Psi$- bounded solution of the difference equation (2.1) on $\mathbb{Z}^+$. Also from hypothis (2), Lemma 2.2, the function $\hat{F}(n)$ is $I_m \otimes \Psi$-bounded on $\mathbb{Z}^+$ and $\lim_{n\to\infty} \|(I_m \otimes \Psi(n))\hat{F}(n)| = 0$. From the Theorem 2.3, it follows that

$$\lim_{n\to\infty} \left\| (I_m \otimes \Psi(n))\, \hat{X}(n) \right\| = 0.$$

Now, from the inequality (2.3) we have

$$|\Psi(n)X(n)| \leq m \left\| (I_m \otimes \Psi(n))\, \hat{X}(n) \right\|, n \in \mathbb{Z}^+$$

and, then

$$\lim_{n\to\infty} |\Psi(n)X(n)| = 0.$$

□

The following examples illustrate the above theorems.

Example 3.1. Consider the matrix difference equation (1.1) with

$$A(n) = \begin{bmatrix} \frac{n+1}{n} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}, \quad B(n) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \text{ and } F(n) = \begin{bmatrix} \frac{n(n+1)}{6^n} & 0 \\ 0 & \frac{n^2}{2^n} \end{bmatrix}.$$

Then,

$$Y(n) = \begin{bmatrix} n & 0 \\ 0 & 3^{1-n} \end{bmatrix} \text{ and } Z(n) = \begin{bmatrix} 2^{1-n} & 0 \\ 0 & 1 \end{bmatrix}$$

are the fundamental matrices for (2.4) and (2.5) respectively. Consider

$$\Psi(n) = \begin{bmatrix} \frac{3^n}{n+1} & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{for all } n \in \mathbb{Z}^+.$$

If we take projections

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } Q_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

then condition (1) is satisfied with $M = 1$ and $K = 7.5$.

In addition, the hypothesis (2) of Theorem 3.2 is satisfied. Because

$$|\Psi(n)F(n)| = \frac{n}{2^n} \leq \frac{1}{2}$$

and

$$\lim_{n\to\infty} |\Psi(n)F(n)| = \lim_{n\to\infty} \frac{n}{2^n} = 0.$$

From Theorems 3.1 and 3.2, the difference equation has at least one Ψ-bounded solution and every Ψ-bounded solution X of (1.1) is such that $\lim_{n\to\infty} |\Psi(n)X(n)| = 0$.

Remark 3.1. In Theorem 3.2, if we do not have $\lim_{n\to\infty} |\Psi(n)F(n)| = 0$, then the solution $X(n)$ of (1.1) may be such that $\lim_{n\to\infty} |\Psi(n)X(n)| \neq 0$.

The following example illustrates Remark 3.1, that the Theorem 3.2 fail if the matrix function F is Ψ-bounded and $\lim_{n\to\infty} |\Psi(n)F(n)| \neq 0$.

Example 3.2. Consider the matrix difference equation (1.1) with

$$A(n) = \begin{bmatrix} \frac{n^3}{(n+1)^3} & 0 \\ 0 & \frac{n}{n+1} \end{bmatrix}, \quad B(n) = \begin{bmatrix} \frac{(n+1)^2}{n^2} & 0 \\ 0 & \frac{n+1}{n} \end{bmatrix}$$

and

$$F(n) = \begin{bmatrix} \frac{2^n}{n+1} & \frac{3^{-n}}{(n+1)^2} \\ 6^{-n}(n+1) & 3^{-n} \end{bmatrix}.$$

Then,

$$Y(n) = \begin{bmatrix} \frac{1}{n^3} & 0 \\ 0 & \frac{1}{n} \end{bmatrix} \text{ and } Z(n) = \begin{bmatrix} n^2 & 0 \\ 0 & n \end{bmatrix}$$

are the fundamental matrices for (2.4) and (2.5) respectively. Consider

$$\Psi(n) = \begin{bmatrix} (n+1)2^{-n} & 0 \\ 0 & \frac{3^n}{n+1} \end{bmatrix}, \quad \text{for all } n \in \mathbb{Z}^+.$$

If we take projections

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } Q_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

then condition (1) is satisfied with $M = 2$ and $K = 2.5$. Also $|\Psi(n)F(n)| = 1$, for $n \in \mathbb{Z}^+$. Therefore, F is Ψ-bounded on $\mathbb{Z}^+$ and $\lim_{n\to\infty} |\Psi(n)F(n)| = 1 \neq 0$.

The solutions of the equation (1.1) are

$$X(n) = \begin{bmatrix} \frac{1}{n}(2^n - 2 + c_1) & \frac{1}{2n^2}(1 - 3^{1-n} + 2c_2) \\ \frac{n}{5}(1 - 6^{1-n} + 5c_3) & \frac{1}{2}(1 - 3^{1-n} + 2c_4) \end{bmatrix},$$

where c_1, c_2, c_3 and c_4 are arbitrary constants and

$$\Psi(n)X(n) = \begin{bmatrix} \frac{n+1}{n}[1 + 2^{-n}(c_1 - 2)] & \frac{n+1}{2n^2}[(2^{-n}(1 + 2c_2) - 3(6^{-n})] \\ \frac{n}{5(n+1)}[3^n(1 + 5c_3) - 6(2^{-n})] & \frac{1}{2(n+1)}[3^n(1 + 2c_4) - 3] \end{bmatrix}.$$

It is easily seen that, there exist Ψ-bounded solutions of (1.1) for $c_3 = -\frac{1}{5}$ and $c_4 = -\frac{1}{2}$. But $\lim_{n\to\infty} |\Psi(n)X(n)| \neq 0$.

References

[1] Agarwal, R.P. : *Difference Equations and Inequalities Theory Methods and Applications*, Second edition, Marcel Dekker, New York, 2000.

[2] Agarwal, R.P. and Wong, P.J.Y. : *Advanced Topics in Difference Equations*, Kluwer, Dordrecht, 1997.

[3] Diamandescu, A. : *Existence of Ψ - bounded solutions for linear difference equations on $\mathbb{Z}$*, Electronic Journal of Qualitative Theory of Differential Equations, Vol.2008(2008), No.26, 1-13.

[4] Diamandescu, A. : *On Ψ - bounded solutions of a Lyapunov matrix differential equation*, Electronic Journal of Qualitative Theory of Differential Equations, Vol.2009(2009), No.17, 1-11.

[5] Diamandescu, A. : *Existence of Ψ - bounded solutions for nonhomogeneous linear difference equations*, Applied Mathematics E-Notes, Vol.10 (2010), 94-102.

[6] Elaydi Saber, N. : *An introduction to difference equations*, Springer-Verlag, New York Inc., 2005.

[7] Graham, A. : *Kronecker Products and Matrix Calculus ; With Applications*, Ellis Horwood Ltd. England (1981).

[8] Hong, J. and Núñez, C. : *The almost periodic type difference equations*, Math. Comput.Modeling, Vol.28, (1998) 21-31.

[9] Han, Y. and Hong, J. : *Existence of Ψ - bounded solutions for linear difference equations*, Applied Mathematics Letters, Vol.20, (2007) 301-305.

[10] Kolla, S.R., Farison, J.B. : *Techniques in reduced order dynamic compensator design for stability robustness of linear discrete time systems*, Control and Dynamic systems, Vol.63, (1994), 77-128.

[11] Murty, K.N., Anand, P.V.S. and Lakshmi Prasannam, V. : *First order difference systems - existence and uniqueness*, Proceedings of the American Mathematical Society, Vol.125, No.12 (1997), 3533-3539.

[12] Murty, M.S.N. and Suresh Kumar, G. : *On Ψ-Boundedness and Ψ-stability of matrix Lyapunov Systems*, Journal of Applied Mathematics and Computing, Vol.26, (2008) 67-84.

[13] Murty, M.S.N. and Suresh Kumar, G. : *On Ψ-Bounded solutions for non-homogeneous matrix Lyapunov systems on $\mathbb{R}$*, Electronic Journal of Qualitative Theory of Differential Equations, Vol.2009(2009), No.62, 1-12.

[14] Ogata, K. : *Discrete time control systems*, Printice Hall, Englewood Cliffs, NJ, 1987.

[15] Suresh Kumar, G., Appa Rao, B.V. and Murty, M.S.N. : *Ψ-bounded solutions for non-homogeneous matrix difference equations on* $\mathbb{Z}$, Journal of Advanced Research in Applied Mathematics, Vol.3, No.3 (2011), 21-31.

[16] Szafranski, Z. and Szmanda, B. : *Oscilatory behavior of difference equations of second order*, Journal of Mathematical Analysis and Applications, Vol.150, (1990), 414-424.

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