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FIXED POINT THEOREMS FOR SET-VALUED GENERALIZED ASYMPTOTIC CONTRACTIONS

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Abstract. The purpose of this paper is to obtain some coincidence and fixed point theorems for a generalized hybrid pair of single-valued and set-valued non continuous maps. Our results generalize some recent results.

1. Introduction

Kirk [14] introduced a new class of maps known as *asymptotic contractions* on a metric space and obtained a fixed point theorem (see Definition 1.1 and Theorem 1.2) below.

Definition 1.1. Let (X, d) be a metric space. A self-map T of X is an *asymptotic contraction* on X if

$$d(T^n x, T^n y) \leq \varphi_n(d(x, y)) \text{ for } x,\, y \in X,$$

where φ is a continuous function, from $[0, \infty)$ into itself, $\varphi(t) < t$ for all $t > 0$ and $\{\varphi_n\}$ is a sequence of functions from $[0, \infty)$ into itself such that $\{\varphi_n\} \to \{\varphi\}$ uniformly on the range of d.

Theorem 1.2. *Let (X, d) be a complete metric space and T an asymptotic contraction on X with $\{\varphi_n\}$ and φ as in* Definition 1.1. *Assume that there exists $x \in X$ such that the orbit $\{T^n x : n \in \mathbb{N}\}$ of x is bounded, and that φ_n is continuous for $n \in \mathbb{N}$. Then there exists a unique fixed point $z \in X$. Moreover $\lim_n T^n x = z$ for all $x \in X$.*

Remark 1.3. We remark that:

(1) Theorem 1.2 is an asymptotic version of Boyd and Wong contraction [4] (see [12]).
(2) Jachymski and Jóźwic [12] showed that the continuity of the map T is essential for the conclusion of Theorem 1.2 to hold.
(3) In respect of Definition 1.1, it has been observed (cf. [1, 12, 21–23]) that $\varphi(0) = 0$.

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(4) For the equivalent formulation of Theorem 1.2 in topological spaces with the so called TCS-convergence, we refer to Tasković [24, 25].

Subsequently many extensions and generalizations of Theorem 1.2 appeared (see, for instance, [1–3, 5–13, 16, 18–23, 26, 27]). Underlying the power and importance of this new class of maps, Briseid [5, 7] has observed that a continuous self-map of a compact metric space satisfying any one of the first 50 contractive conditions listed by Rhoades [17] is an asymptotic contraction.

Recently, Fakhar [10] and Wlodarczyk *et al.* [26, 27] extended Kirk's asymptotic contraction to set-valued maps and obtained some endpoint theorems for such contractions. In [26, 27] some applications of the theory of asymptotic contractions to the analysis of set-valued dynamical systems are also discussed. On the other hand, a generalization of the well known Banach contraction principle due to Meir-Keeler [15] has been of continuing interest in fixed point theory. Recently Suzuki [21] combined the ideas of Meir-Keeler contraction and Kirk's asymptotic contraction and introduced the following notion of *asymptotic contraction of Meir-Keeler type*.

Definition 1.4. Let (X, d) be a metric space. A self-map T of X is called an *asymptotic contraction of Meir-Keeler type* if there exists a sequence φ_n of functions from $[0, \infty)$ into itself satisfying the following conditions:

(S1): $\limsup \varphi_n(\varepsilon) \leq \varepsilon$ for all $\varepsilon \geq 0$;

(S2): for each $\varepsilon > 0$, there exists $\delta > 0$ and $\nu \in \mathbb{N}$ such that $\varphi_\nu(t) \leq \varepsilon$ for all $t \in [\varepsilon, \varepsilon + \delta]$;

(S3): $d(T^n x, T^n y) < \varphi_n(d(x, y))$, for all $n \in \mathbb{N}$ and x, $y \in X$ with $x \neq y$.

In this paper first we introduce the notion of *set-valued generalized asymptotic contraction of Meir-Keeler type*, which includes the known notions of asymptotic contractions due to Kirk [14], Suzuki [21] and Fakhar [10] (see Example 2.7 for illustration). Subsequently, this notion is utilized to obtain some coincidence and fixed point theorems for such contractions which generalize, and unify several known results including [10], [26] and others.

2. Generalized asymptotic contractions

Throughout this section, Y denotes an arbitrary nonempty set, (X, d) a metric space, $CB(X)$ the collection of all nonempty closed bounded subsets of X, φ_n as in Definition 1.4 and H the Hausdorff metric induced by d, i.e.,

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B),\ \sup_{y\in B} d(y, A)\right\},$$

for all $A, B \subseteq CB(X)$, where $d(x, B) = \inf_{y\in B} d(x, y)$.

We denote by $\delta(A) = \sup\{d(x, y) : x, y \in A\}$.

Further, let

$$m(x, y) := \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\};$$

$$M(x, y) := \max\left\{d(fx, fy), d(fx, Tx), d(fy, Ty), \frac{1}{2}[d(fx, Ty) + d(fy, Tx)]\right\}.$$

Now, we introduce the notion of *set-valued generalized asymptotic contraction of Meir-Keeler type* as follows.

Definition 2.1. Let (X,d) be a metric space $f : Y \to X$ and $T : Y \to CB(X)$. The map T will be called *a generalized asymptotic contraction of Meir-Keeler type* with respect to f if the following hold:

- **(G1):** $\limsup_n \varphi_n(\varepsilon) \leq \varepsilon$ for all $\varepsilon \geq 0$;
- **(G2):** for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\varphi_k(t) < \varepsilon$ for all $t \in [\varepsilon, \varepsilon+\delta]$ and $k \in \mathbb{N}$;
- **(G3):** $H(T^n x, T^n y) < \varphi_n(M(x,y))$ for all $n \in \mathbb{N}$ and $x, y \in Y$ with $M(x,y) > 0$.

As a special case of the above definition, we have the following:

Definition 2.2. Let (X,d) be a metric space and $T : X \to CB(X)$. The map T will be called *a generalized asymptotic contraction of Meir-Keeler type* if the following hold:

- $\limsup_n \varphi_n(\varepsilon) \leq \varepsilon$ for all $\varepsilon \geq 0$;
- for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\varphi_k(t) < \varepsilon$ for all $t \in [\varepsilon, \varepsilon + \delta]$ and $k \in \mathbb{N}$;
- $H(T^n x, T^n y) < \varphi_n(m(x,y))$ for all $n \in \mathbb{N}$ and $x, y \in X$ with $m(x,y) > 0$.

The following theorem is our main result.

Theorem 2.3. *Let (X,d) be a metric space, $f : Y \to X$ and $T : Y \to CB(X)$ such that $TY \subseteq fY$. Let T be a generalized asymptotic contraction of Meir-Keeler type with respect to f.*

If $T(Y)$ or $f(Y)$ is a complete subspace of X then T and f have a coincidence point.

Further, if $Y = X$, then T and f have a common fixed point provided that $ffu = fu$ and T and f commute at a coincidence point.

Proof. Pick $x_0 \in Y$. We construct a sequence $\{x_n\}$ in the following manner. Since $TY \subseteq fY$, we may choose a point $x_1 \in Y$ such that $fx_1 \in Tx_0$. If $Tx_0 = Tx_1$ then $x_1 = z$ is a coincidence point of T and f and we are done. So assume that $Tx_0 \neq Tx_1$ and choose $x_2 \in Y$ such that $fx_2 \in Tx_1$ and

$$d(fx_1, fx_2) \leq H(Tx_0, Tx_1).$$

If $Tx_1 = Tx_2$, i.e., x_2 is a coincidence point of T and f, we are done. If not continuing in the same manner we have

$$d(fx_{n+1}, fx_{n+2}) \leq H(Tx_n, Tx_{n+1}).$$

By (G3),

$$d(fx_n, fx_{n+1}) \leq H(Tx_{n-1}, Tx_n) < \varphi_n(M(x_0, x_1)).$$

First we show that

$$\lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0. \tag{1}$$

It initially holds if $x_1 = x_2$. In the other case of $x_1 \neq x_2$, we assume that

$$\alpha := \limsup_n d(fx_{n+1}, fx_{n+2}) > 0.$$

From the condition (G2), we can choose $k \in \mathbb{N}$ satisfying $\varphi_k(d(fx_1, fx_2)) < d(fx_1, fx_2)$. By (G3) and (G1),

$$d(fx_{k+1}, fx_{k+2}) \leq H(Tx_k, Tx_{k+1}) < \varphi_k(M(x_0, x_1)) < M(x_1, x_2). \tag{2}$$

Now, we have

$$\begin{aligned}
\alpha := & \lim_{n\to\infty}\sup d(fx_{k+n+1}, fx_{k+n+2}) \leq \lim_{n\to\infty}\sup H(Tx_{k+n}, Tx_{k+n+1}) \\
\leq & \lim_{n\to\infty}\sup \varphi_n(M(x_k, x_{k+1})) \leq M(x_k, x_{k+1}) \\
= & \max\{d(fx_k, fx_{k+1}), d(fx_k, Tx_k), d(fx_{k+1}, Tx_{k+1}), \\
& \frac{1}{2}[d(fx_k, Tx_{k+1}) + d(fx_{k+1}, Tx_k)\} \\
= & \max\{d(fx_k, fx_{k+1}), d(fx_k, fx_{k+1}), d(fx_{k+1}, fx_{k+2}), \\
& \frac{1}{2}[d(Tx_k, Tx_{k+2}) + 0)\} \\
= & \max\{d(fx_k, fx_{k+1}), d(fx_{k+1}, fx_{k+2}), \\
& \frac{1}{2}[d(fx_k, fx_{k+1}) + d(fx_{k+1}, fx_{k+2})\} \\
= & \max\{d(fx_k, fx_{k+1}), d(fx_{k+1}, fx_{k+2})\}.
\end{aligned}$$

If

$$\max\{d(fx_k, fx_{k+1}), d(fx_{k+1}, fx_{k+2})\} = d(fx_{k+1}, fx_{k+2})$$

then

$$\begin{aligned}
d(fx_{k+1}, fx_{k+2}) \leq & H(Tx_k, Tx_{k+1}) \\
< & \varphi_1(M(x_k, x_{k+1})) < M(x_k, x_{k+1}) \\
= & \max\{d(fx_k, fx_{k+1}), d(fx_k, Tx_{k+1}), d(fx_{k+1}, Tx_{k+1}), \\
& \frac{1}{2}[d(fx_k, Tx_{k+1}) + d(fx_{k+1}, Tx_k)]\} \\
= & \max\{d(fx_k, fx_{k+1}), d(fx_k, fx_{k+1}), d(fx_{k+1}, fx_{k+2}), \\
& \frac{1}{2}[d(fx_k, fx_{k+1}) + 0]\} \\
= & \max\{d(fx_k, fx_{k+1}), d(fx_{k+1}, fx_{k+2})\} \\
= & d(fx_{k+1}, fx_{k+2}),
\end{aligned}$$

a contradiction. Therefore

$$\max\{d(fx_k, fx_{k+1}), d(fx_{k+1}, fx_{k+2})\} = d(fx_k, fx_{k+1})$$

and we conclude that $M(x_k, x_{k+1}) = d(fx_k, fx_{k+1})$.
By (2),

$$\begin{aligned}
d(fx_{k+2}, fx_{k+3}) \leq & H(Tx_{k+1}, Tx_{k+2}) \\
< & \varphi_k(M(x_1, x_2)) < M(x_1, x_2) \\
= & \max\{d(fx_1, fx_2), d(fx_1, Tx_2), d(fx_1, Tx_2), \\
& \frac{1}{2}[d(fx_1, Tx_2) + d(fx_1, Tx_2)]\} \\
= & d(fx_1, fx_2).
\end{aligned}$$

So $\alpha < d(fx_1, fx_2)$. By a similar argument, we obtain $\alpha < d(fx_{k+1}, fx_{k+2})$ for all $k \in \mathbb{N}$. Hence $\{d(fx_n, fx_{n+1}\}$ converges to α.

Since $0 < \alpha < d(fx_1, fx_2) < \infty$, there exists $\delta_2 > 0$ and $l \in \mathbb{N}$ such that

$$\varphi_l(t) \leq \alpha \text{ for all } t \in [\alpha, \alpha + \delta_2].$$

We choose $p \in \mathbb{N}$ with $d(fx_{p+1}, fx_{p+2}) < \alpha + \delta_2$. Then we have

$$d(fx_{l+p+1}, fx_{l+p+2}) \leq H(Tx_{l+p}, Tx_{l+p+1}) < \varphi_l d(fx_p, fx_{p+1}) \leq \alpha,$$

a contradiction. This proves that $\lim_{n\to\infty} d(fx_n, fx_{n+1}) = 0$. Now following the proof of Theorem 3.1 [20], it can be easily shown that $\{fx_n\}$ is a Cauchy sequence.

Suppose $f(Y)$ is complete. Then $\{fx_n\}$ being contained in $f(Y)$ has a limit in $f(Y)$. Call it z. Let $u \in f^{-1}z$. Then $fu = z$. Using (G2),

$$\begin{aligned} d(fu, Tu) &\leq H(Tx_n, Tu) < \varphi_1(M(u, x_n)) \\ &= \varphi_1(\max\{d(fu, fx_n), d(fu, Tu), d(fx_n, Tx_n), \\ &\quad \frac{1}{2}[d(fu, Tx_n) + d(fx_n, Tu)]\}). \end{aligned}$$

Making $n \to \infty$, $d(fu, Tu) \leq \varphi_1(d(fu, Tu)) < d(fu, Tu)$. This yields $fu \in Tu$.

Further, if $Y = X$, $ffu = fu$, and the maps f and T commute at their coincidence point u then $fu \in fTu \subseteq Tfu$ and fu is a common fixed point of f and T.

In case TY is a complete subspace of X, the condition $TY \subseteq fY$ implies that the sequence $\{fx_n\}$ converges in fY and the previous argument works. □

Remark 2.4. We remark that a set-valued *asymptotic contraction of Meir-Keeler type* is the set-valued *generalized contraction of Meir-Keeler type* when $m(x, y) = d(x, y)$. Further it includes the set-valued *asymptotic contraction* given in [10] and [26].

Now in the view of Definition 2.2 and the above remark we have the following corollaries.

Corollary 2.5. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ a generalized asymptotic contraction of Meir-Keeler type. Then T has a fixed point in X.*

Corollary 2.6. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ an asymptotic contraction of Meir-Keeler type. Then T has a fixed point in X.*

The following example shows the generality of Theorem 2.3 over [26, Th. 2.1] and [10, Th. 2.3].

Example 2.7. Let $Y = (-\infty, \infty)$ and $X = [0, \infty)$ endowed with the usual metric d. Let $f : Y \to X$ and $T : Y \to CB(X)$ be defined by

$$fx = \begin{cases} -2x & \text{if } x < 0, \\ 2x & \text{if } x \geq 0 \end{cases} \quad \text{and } Tx = \begin{cases} \{-x\} & \text{if } x < 0, \\ [0, x] & \text{if } 0 \leq x \leq 1, \\ \{x\} & \text{if } x > 1 \end{cases}$$

for all $x \in Y$. Let $\varphi_n(t) = \frac{3}{4}t$ for $t > 0$.

Then for $x > 1$ and $y > 1$,

$$H(T^n x, T^n y) = | x - y | > \frac{3}{4} | x - y | = \varphi_n(d(x, y)),$$

and the contractive condition of Theorem 2.3 [10] is not satisfied.

Further, $\delta(T^n([0, 1])) = \delta([0, 1])$ and condition (d) of Theorem 2.1 [26] is not satisfied. It can be verified that the maps f and T satisfy all the hypotheses of Theorem

2.3. Notice that $TY \subseteq fY$ and f and T commute at 0. Hence $f0 \in T0$ is a common fixed point of f and T.

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