

**WEIGHTED COMPOSITION OPERATORS BETWEEN  
WEIGHTED BERGMAN AND  $S^p$  SPACES**

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**ABSTRACT.** Let  $\varphi$  be an analytic self-map of open unit disk  $\mathbb{D}$  and  $\psi$  is analytic on  $\mathbb{D}$ . Then a weighted composition operator induced by  $\varphi$  with weight  $\psi$  is given by  $(W_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z))$  for  $z$  in  $\mathbb{D}$  and  $f$  analytic on  $\mathbb{D}$ . For each  $p \geq 1$ , let  $S^p$  be the space of analytic functions on  $\mathbb{D}$  whose derivatives belong to the Hardy space  $H^p$ . Given  $W_{\psi,\varphi} : A_{\alpha}^p(\mathbb{D}) \rightarrow S^q(\mathbb{D})$ , using Carleson measure we characterize boundedness and compactness of  $W_{\psi,\varphi}$  for  $1 \leq p, q \leq \infty$ .

1. INTRODUCTION

Let  $\mathbb{D}$  be the open unit disk in the complex plane  $\mathbb{C}$ . The space  $H^{\infty}(\mathbb{D})$  is the set of bounded analytic functions on  $\mathbb{D}$ , with

$$\|f\|_{\infty} = \sup_{z \in \mathbb{D}} |f(z)|.$$

For  $0 < p < \infty$  and  $-1 < \alpha < \infty$ , the weighted Bergman space  $A_{\alpha}^p(\mathbb{D})$  consists of those functions  $f$  analytic on  $\mathbb{D}$  that satisfy

$$\|f\|_{A_{\alpha}^p}^p = \int_{\mathbb{D}} |f(z)|^p dA_{\alpha}(z) < \infty,$$

where  $dA_{\alpha}(z) = \frac{(1+\alpha)}{\pi} (1 - |z|^2)^{\alpha} dA(z)$  is a weighted area measure. For  $0 < p < \infty$ , the Hardy space  $H^p(\mathbb{D})$  consists of functions  $f$  analytic on  $\mathbb{D}$  that satisfy

$$\|f\|_{H^p}^p = \sup_{0 < r < 1} \int_{\partial\mathbb{D}} |f(r\zeta)|^p d\sigma(\zeta) < \infty,$$

where  $\sigma$  is the normalized Lebesgue measure on the boundary of the unit disk. For  $f$  belongs to  $H^p(\mathbb{D})$ , it is well known from Fatou's theorem that the radial limit

$$f^*(\zeta) = \lim_{r \rightarrow 1^-} f(r\zeta)$$

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exists for almost all  $\zeta$  on  $\partial\mathbb{D}$ . Moreover,

$$\|f\|_{H^p}^p = \int_{\partial\mathbb{D}} |f^*(\zeta)|^p d\sigma(\zeta),$$

for all finite values of  $p$ . We often use the standard abuse of notation of writing  $f$  instead of  $f^*$ . We will restrict ourselves to those  $p$  with  $1 \leq p < \infty$ . For each  $p$  we denote by  $S^p$  the space of all analytic functions  $f$  on the unit disk  $\mathbb{D}$  whose derivative  $f'$  lies in  $H^p$ , endowed with the norm

$$\|f\|_{S^p} = |f(0)| + \|f'\|_{H^p}.$$

One can easily show that  $S^p$  is a Banach space with respect to this norm. It is well known that  $S^p$  is a Banach algebra when the norm of  $f \in S^p$  is defined by  $\|f\|_\infty + \|f'\|_{H^p}$ , we will not use this norm in this paper.

Suppose  $\varphi$  is an analytic function mapping  $\mathbb{D}$  into itself and  $\psi$  is an analytic function on  $\mathbb{D}$ , the weighted composition operator  $W_{\psi,\varphi}$  is defined on the space  $H(\mathbb{D})$  of all analytic functions on  $\mathbb{D}$  by

$$(W_{\psi,\varphi}f)(z) = \psi(z)C_\varphi f(z) = \psi(z)f(\varphi(z)),$$

for all  $f \in H(\mathbb{D})$  and  $z \in \mathbb{D}$ . It is well known that the weighted composition operator  $W_{\varphi,\psi}f = \psi f \circ \varphi$  defines a linear operator  $W_{\varphi,\psi}$  which acts boundedly on various spaces of analytic or harmonic functions on  $\mathbb{D}$ .

These operators have been studied on many spaces of analytic functions. During the past few decades much effort has been devoted to the study of these operators with the goal of explaining the operator-theoretic properties of  $W_{\psi,\varphi}$  in terms of the function-theoretic properties of the induced maps  $\varphi$  and  $\psi$ . We refer to the monographs by Cowen and MacCluer [2], Duren and Schuster [4], Hedemalm, Korenblum, and Zhu [7], Shapiro [13], and Zhu ([14], [15]) for the overview of the field as of the early 1990s.

Composition operators on the space  $S^p$  are studied first by Roan ([11], [12]). A few years later, MacCluer [10] characterized boundedness and compactness of these operators in terms of Carleson measures showing, among other things, that for  $1 \leq p < \infty$  the composition operator  $C_\varphi$  is compact on  $S^q$  if and only if  $\|\varphi\|_\infty < 1$ . Arora, Mukherjee and Panigrahi [1] found sufficient conditions in terms of Carleson measures under which the weighted composition operators are bounded and compact on the spaces  $S^p$ . Recently, Contreras and Hernández-Díaz [3] characterized boundedness, compactness, weak compactness and complete continuity of weighted composition operators from  $S^p$  into  $S^q$ ,  $1 \leq p, q \leq \infty$ , in terms of weighted composition operators in Hardy spaces. In this paper, by using Carleson measure, we characterize boundedness and compactness of weighted composition operators act between weighted Bergman  $A_\alpha^p$  and  $S^q$  spaces,  $1 \leq p, q \leq \infty$ .

The results we obtain about weighted composition operators will be given in terms of certain measures, which we define next. For  $1 \leq p < \infty$ , let  $\varphi$  be an analytic self-map of  $\mathbb{D}$  and  $\psi \in S^p$  such that  $\psi\varphi' \in H^p$ , we define measures  $\mu_{\varphi,\psi\varphi',p}$ ,  $\mu_{\varphi,\psi',p}$  and  $\mu_{\varphi,\varphi,p}$  on  $\overline{\mathbb{D}}$  as:

$$\begin{aligned} \mu_{\varphi,\psi',p}(E) &= \int_{\varphi^{-1}(E) \cap \partial\mathbb{D}} |\psi'|^p d\sigma, \\ \mu_{\varphi,\psi\varphi',p}(E) &= \int_{\varphi^{-1}(E) \cap \partial\mathbb{D}} |\psi\varphi'|^p d\sigma, \end{aligned}$$

and for  $\psi \in H^p$ ,

$$\mu_{\varphi,\psi,p}(E) = \int_{\varphi^{-1}(E) \cap \partial\mathbb{D}} |\psi|^p d\sigma,$$

where  $E$  is a Borel subset of the closed unit disk  $\overline{\mathbb{D}}$ , and where  $\sigma$  is the normalized Lebesgue measure on the boundary of the unit disk  $\partial\mathbb{D}$ . Thus by using ([6], Theorem III.10.4) we get the following change of variable formula,

$$\int_{\mathbb{D}} g d\mu_{\varphi,\psi,p} = \int_{\varphi^{-1}(\mathbb{D})} g(\varphi) |\psi|^p d\sigma,$$

where  $g$  is an arbitrary measurable positive function on  $\overline{\mathbb{D}}$ . So for  $f \in H^p$ , by taking  $g = |f|^p$ , we get

$$\int_{\mathbb{D}} |f(z)|^p d\mu_{\varphi,\psi,p}(z) = \int_{\partial\mathbb{D}} |\psi(z)|^p f(\varphi(z)) d\sigma.$$

We will use the change of variable formula for the other measures.

The pseudo-hyperbolic distance in the unit disk  $\mathbb{D}$  is defined as

$$\rho(z, w) = |\varphi_w(z)| \quad \text{where} \quad \varphi_w(z) = \frac{w - z}{1 - \overline{w}z},$$

for  $z, w \in \mathbb{D}$ . This represents a metric, moreover the triangle inequality takes a stronger form

$$\rho(z, w) \leq \frac{\rho(z, \alpha) + \rho(\alpha, w)}{1 + \rho(z, \alpha)\rho(\alpha, w)}.$$

Using Schwarz-Pick theorem ([2], Theorem 2.39), this metric is Möbius-invariant in the sense that

$$\rho(\varphi_\beta(z), \varphi_\beta(w)) = \rho(z, w),$$

for any  $\beta \in \mathbb{D}$ . For further details on pseudo-hyperbolic distance see [5]. For  $w$  in the disk,  $D(w, r)$  denotes the disk whose pseudo-hyperbolic center is  $w$  and whose pseudo-hyperbolic radius is  $r$ :  $D(w, r) = \{z : \rho(z, w) < r\}$ . Throughout this paper the notation  $A \approx B$  means that there is a positive constant  $C$  independent of  $A$  and  $B$  such that  $C^{-1}B \leq A \leq CB$ .

**Lemma 1.1.** Suppose  $r > 0$ ,  $p > 0$ , and  $\alpha > -1$ . Then there exists a constant  $C > 0$  such that

$$|f(z)|^p \leq \frac{C}{(1 - |z|^2)^{2+\alpha}} \int_{D(z,r)} |f(w)|^p dv_\alpha(w),$$

for all  $f \in H(\mathbb{D})$  and all  $z \in \mathbb{D}$ .

The proof of Lemma 1.1 can be easily seen by using Lemma 2.12 and Lemma 2.14 in [7], so the proof is not given here. In what follows, for  $a \in \mathbb{D}$ ,  $D(a)$  denotes the disk whose pseudo-hyperbolic center is  $a$  and whose pseudo-hyperbolic radius is  $r$ .

The following lemma 1.2 is a standard estimate in Bergman space, which is a result of Luecking ([9], Lemma 2.1).

**Lemma 1.2.** Let  $a \in \mathbb{D}$  and let  $r$  be fixed,  $0 < r < 1$ . there exists a constant  $C$  such that if  $f$  is analytic in  $\mathbb{D}$  and  $p \geq 1$  then

$$|f^{(n)}(a)|^p \leq \left( C 2^n \frac{(n+2)n!}{(1-r)^{n+2}} \right)^p \frac{\int_{D(a)} |f|^p dA}{(r(1-|a|^2))^{2+nq}}.$$

Luecking characterized positive measures  $\mu$  with the property  $\|f^{(n)}\|_{L^q(\mu)} \leq \|f\|_{A_\alpha^p}$ . The following result is a special case of Luecking's result ([9], Theorem 2.2) for  $n = 1$ .

**Theorem 1.1.** *Let  $1 \leq p \leq q$  and let  $\alpha > -1$ . Let  $\mu \geq 0$  be a finite measure on  $\mathbb{D}$ . The following are equivalent.*

- (1) *There exists a constant  $C$  such that  $\|f'\|_{L^q(\mu)} \leq C\|f\|_{A_\alpha^p}$  for all  $f \in A_\alpha^p$ .*
- (2)  *$\mu(D(a)) = O\left(\left(1 - |a|^2\right)^{q(\alpha+2+p)/p}\right)$  as  $|a| \rightarrow 1$ .*

For the case  $1 \leq q < p$ , Luecking used Khinchine's inequality and other estimates to obtain a version of Theorem 1.1 for  $f^{(n)}$ , where  $f \in A_\alpha^p$  ([8], theorem 1). We are interested in the case  $n = 1$  and  $f \in A_\alpha^p$ . Theorem 1.2 is a slight modification of Luecking's result, so the proof's details are omitted.

**Theorem 1.2.** *Let  $1 \leq q < p$  and let  $\alpha > -1$ . Let  $\mu \geq 0$  be a finite measure on  $\mathbb{D}$ . Let  $L(z) = (1 - |z|^2)^{-(\alpha+2+q)} \mu(D(z))$ . The following are equivalent.*

- (1) *There exists a constant  $C$  such that  $\|f'\|_{L^q(\mu)} \leq C\|f\|_{A_\alpha^p}$  for all  $f \in A_\alpha^p$ .*
- (2)  *$L \in L^{p/(p-q)}(A_\alpha^p)$ .*

Finally before stating our results, we present the following lemma whose proof can be obtained by adapting the proof of ([2], Proposition 3.11).

**Lemma 1.3.** *For  $1 \leq p, q \leq \infty$  and  $\alpha > -1$ . Let  $\varphi$  be an analytic self-map of  $\mathbb{D}$   $\psi \in S^q$  such that  $W_{\psi, \varphi}$  is bounded operator from  $A_\alpha^p(\mathbb{D})$  into  $S^q$ . Then  $W_{\psi, \varphi}$  is compact if and only if whenever  $\{f_n\}$  is bounded sequence in  $A_\alpha^p(\mathbb{D})$  and  $f_n \rightarrow 0$  uniformly on compact subsets of  $\mathbb{D}$ , then  $\|W_{\psi, \varphi}(f_n)\|_{S^q} \rightarrow 0$  as  $n \rightarrow \infty$ .*

## 2. BOUNDEDNESS AND COMPACTNESS WHEN $1 \leq p, q < \infty$

The results of this section concern boundedness and compactness of weighted composition operators mapping  $A_\alpha^p$  into  $S^q$  for  $1 \leq p \leq q < \infty$ . An important tool in the study of weighted composition operators on analytic functions spaces is the notion of Carleson measure. We consider Carleson measure on weighted Bergman spaces.

**Definition 2.1.** Let  $\mu$  be a positive Borel measure on  $\mathbb{D}$  and let  $X$  be a Banach space of analytic functions on  $\mathbb{D}$ . Then for  $q > 0$ ,  $\mu$  is an  $(X, q)$ -Carleson measure if there is a constant  $C > 0$  such that for any  $f \in X$ ,

$$\int_{\mathbb{D}} |f(z)|^q d\mu(z) \leq C\|f\|_X^q.$$

The following Lemma 2.1 is a well-known result.

**Lemma 2.1.** Let  $0 < p \leq q < \infty$  and  $\alpha > -1$ . Let  $\mu$  be a finite positive Borel measure on  $\mathbb{D}$ . Then the measure  $\mu$  is  $(A_\alpha^p, q)$ -Carleson measure if and only if for any  $a \in \mathbb{D}$ , there exists a constant  $C > 0$  such that

$$\int_{\mathbb{D}} |\varphi'_a(z)|^{(2+\alpha)q/p} d\mu(z) \leq C.$$

In the first result Theorem 2.3, we Characterize the boundedness of weighted composition operators acting from  $A_\alpha^p$  into  $S^q$ .

**Theorem 2.2.** *Let  $1 \leq p \leq q$  and  $\alpha > -1$ . Let  $\varphi$  be an analytic self-map of  $\mathbb{D}$  and  $\psi$  be an analytic map of  $\mathbb{D}$  such that  $\psi \in S^q$  and  $\psi\varphi' \in H^q$ . Let  $\mu_{\varphi, \psi\varphi', q}(D(a)) = O(1 - |a|^2)^{q(\alpha+2+p)/p}$  as  $|a| \rightarrow 1$ . Then  $W_{\psi, \varphi} : A_{\alpha}^p \rightarrow S^q$  is bounded if and only if  $\mu_{\varphi, \psi', q}$  is  $(A_{\alpha}^p, q)$ -Carleson measure.*

*Proof.* Suppose that  $\mu_{\varphi, \psi', q}$  is Carleson measure. Then for  $f \in A_{\alpha}^p$ ,

$$\begin{aligned} \|W_{\psi, \varphi}(f)\|_{S^q}^q &= |\psi(0)f'(0)|^q + \int_{\partial\mathbb{D}} |(\psi f(\varphi))'|^q d\sigma \\ &\leq \int_{\partial\mathbb{D}} |(\psi(z)\varphi'(z))^q| |f'(z)|^q d\sigma(z) + \int_{\partial\mathbb{D}} |\psi'(z)|^q |f(\varphi(z))|^q d\sigma(z) \\ &= \int_{\mathbb{D}} |f'(w)|^q d\mu_{\varphi, \psi\varphi', q}(w) + \int_{\mathbb{D}} |f(w)|^q d\mu_{\varphi, \psi', q}(w). \end{aligned}$$

By Theorem 1.1, the first term bounded by a constant times  $\|f\|_{A_{\alpha}^p}^p$ . Since  $\mu_{\varphi, \psi', q}$  is Carleson measure, the second term bounded by a constant times  $\|f\|_{A_{\alpha}^p}^p$ . Therefore  $W_{\psi, \varphi}$  is bounded.

For the converse, suppose  $W_{\psi, \varphi}$  is bounded. Then there exists a constant  $M > 0$  such that for all  $f \in A_{\alpha}^p$ ,  $\|W_{\psi, \varphi}(f)\|_{S^q}^q \leq M\|f\|_{A_{\alpha}^p}^q$ . On the other hand, there exists a constant  $C > 0$  such that

$$\begin{aligned} \|W_{\psi, \varphi}(f)\|_{S^q}^q &\geq \int_{\partial\mathbb{D}} |(\psi f(\varphi))'|^q d\sigma \\ &\geq C \int_{\partial\mathbb{D}} |\psi'|^q |f(\varphi)|^q d\sigma - C \int_{\partial\mathbb{D}} |\psi\varphi'|^q |f'(\varphi)|^q d\sigma. \end{aligned}$$

Then by using Theorem 1.1, there exists a constant  $K > 0$

$$\begin{aligned} C \int_{\mathbb{D}} |f(z)|^q d\mu_{\varphi, \psi', q}(z) &\leq \|W_{\psi, \varphi}(f)\|_{S^q}^q + C \int_{\mathbb{D}} |f'(z)|^q d\mu_{\varphi, \psi\varphi', q}(z) \\ &\leq M\|f\|_{A_{\alpha}^p}^q + CK\|f\|_{A_{\alpha}^p}^q. \end{aligned}$$

Hence, from Definition 2.1, we get that  $\mu_{\varphi, \psi', q}$  is  $(A_{\alpha}^p, q)$ -Carleson measure.  $\square$

As a consequence of Theorem 2.2 and Lemma 2.1 we get the next Corollary.

**Corollary 2.3.** *Let  $1 \leq p \leq q$  and  $\alpha > -1$ . Let  $\varphi$  be an analytic self-map of  $\mathbb{D}$  and  $\psi$  be an analytic map of  $\mathbb{D}$  such that  $\psi \in S^q$  and  $\psi\varphi' \in H^q$ . Let  $\mu_{\varphi, \psi\varphi', q}(D(a)) = O(1 - |a|^2)^{q(\alpha+2+p)/p}$  as  $|a| \rightarrow 1$ . Then  $W_{\psi, \varphi}$  is bounded from  $A_{\alpha}^p$  into  $S^q$  if and only if*

$$\sup_{a \in \mathbb{D}} \int_{\partial\mathbb{D}} \frac{|\psi'(z)|^q (1 - |a|^2)^{q(\alpha+2)/p}}{(1 - \bar{a}\varphi(z))^{2(\alpha+2)q/p}} d\sigma(z) < \infty.$$

When a result concerning Carleson measure is established, it is then relatively straight forward to formulate and prove the corresponding ‘‘little-oh’’ version. In order to do that, we introduce vanishing Carleson measure.

**Definition 2.4.** Let  $\mu$  be a positive Borel measure on  $\mathbb{D}$ , and let  $X$  be a Banach space of analytic functions on  $\mathbb{D}$ . Then for  $q > 0$ ,  $\mu$  is called vanishing  $(X, q)$ -Carleson measure if

$$\lim_{k \rightarrow \infty} \int_{\mathbb{D}} |f_k|^q d\mu = 0,$$

whenever  $\{f_k\}$  is a bounded sequence in  $X$  that converges to 0 uniformly on compact subsets of  $\mathbb{D}$ .

The following lemma 2.2 is a well-know result.

**Lemma 2.2.** Let  $0 < p \leq q < \infty$  and  $\alpha > -1$ . Let  $\mu$  be a finite positive Borel measure on  $\mathbb{D}$ . Then the measure  $\mu$  is a vanishing  $(A_\alpha^p, q)$ -Carleson measure if and only if for any  $a \in \mathbb{D}$ ,

$$\lim_{|a| \rightarrow 1} \int_{\mathbb{D}} |\varphi'_a(z)|^{(2+\alpha)q/p} d\mu(z) = 0$$

In the next Theorem we use the notion of Vanishing Carleson measure to characterize the compactness of weighted composition operators.

**Theorem 2.5.** Let  $1 \leq p \leq q$  and  $\alpha > -1$ . Let  $\varphi$  be an analytic self-map of  $\mathbb{D}$  and  $\psi$  be an analytic map of  $\mathbb{D}$  such that  $\psi \in S^q$  and  $\psi\varphi' \in H^q$ . Suppose  $\mu_{\varphi, \psi\varphi', q}(D(w)) = o(1 - |w|^2)^{q(\alpha+2+p)/p}$  as  $|w| \rightarrow 1$ . Then  $W_{\psi, \varphi} : A_\alpha^p \rightarrow S^q$  is compact if and only if  $\mu_{\varphi, \psi', q}$  is vanishing  $(A_\alpha^p, q)$ -Carleson measure.

*Proof.* Suppose that  $W_{\psi, \varphi}$  is compact. Let  $\{f_n\}$  be a bounded sequence in  $A_\alpha^p$  such that  $f_n \rightarrow 0$  uniformly on compact subsets of  $\mathbb{D}$ . Then

$$\begin{aligned} \|W_{\psi, \varphi}(f_n)\|_{S^q}^q &= |\psi(0)f_n(\varphi(0))|^q + \int_{\partial\mathbb{D}} |(\psi f(\varphi))'|^q d\sigma \\ &\geq |\psi(0)f_n(\varphi(0))|^q + \int_{\partial\mathbb{D}} |\psi'|^q |f_n(\varphi)|^q d\sigma - \int_{\partial\mathbb{D}} |\psi\varphi'|^q |f'_n(\varphi)|^q d\sigma \end{aligned}$$

Therefore,

$$\int_{\mathbb{D}} |f_n(w)|^q d\mu_{\varphi, \psi', q}(w) \leq \|W_{\psi, \varphi}(f_n)\|_{S^q}^q - |\psi(0)f_n(\varphi(0))|^q + \int_{\mathbb{D}} |f'_n(w)|^q d\mu_{\varphi, \psi\varphi', q}(w)$$

Since  $W_{\psi, \varphi}$  is compact and  $f_n \rightarrow 0$  uniformly on compact subsets of  $\mathbb{D}$ , the first and second terms in the last inequality's right hand side each tends to zero as  $n \rightarrow 0$ . Thus to show  $\mu_{\varphi, \psi', q}$  is Carleson measure, it suffices to show  $\|f'_n\|_{L^q(\mu_{\varphi, \psi\varphi', q})}^q \rightarrow 0$  as  $n \rightarrow 0$ . By using Lemma 1.2, we have

$$|f'_n(z)|^q \leq \frac{C_1}{(1 - |z|^2)^{2+q}} \int_{D(z)} |f_n(w)|^q dA(w).$$

Integrate the last inequality with respect to  $\mu_{\varphi, \psi\varphi', q}$  and then use Fubini's Theorem to get

$$\|f'_n\|_{L^q(\mu_{\varphi, \psi\varphi', q})}^q \leq C_1 \int_{\mathbb{D}} \frac{\mu_{\varphi, \psi\varphi', q}(D(w))}{(1 - |w|^2)^{2+q}} |f_n(w)|^q dA(w).$$

From Lemma 1.1, we have for  $f \in A_\alpha^p$  and  $w \in \mathbb{D}$

$$|f_n(w)| \leq \frac{C_2 \|f_n\|_{A_\alpha^p}}{(1 - |w|^2)^{(2+\alpha)/p}}.$$

Therefore, we get

$$\|f'_n\|_{L^q(\mu_{\varphi, \psi\varphi', q})}^q \leq C_1 C_2 \|f_n\|_{A_\alpha^p}^{q-p} \int_{\mathbb{D}} |f_n(w)|^p \frac{\mu_{\varphi, \psi\varphi', q}(D(w))}{(1 - |w|^2)^{(\alpha q + 2q + pq - \alpha p)/p}} dA(w).$$

The hypothesis implies that for a given  $\epsilon > 0$ , there exists  $r \in (0, 1)$  such that

$$\int_{|w| > r} |f_n(w)|^p \frac{\mu_{\varphi, \psi\varphi', q}(D(w))}{(1 - |w|^2)^{(\alpha+2+p)q/p}} (1 - |w|^2)^\alpha dA(w) \leq \epsilon \|f_n\|_{A_\alpha^p}^p. \quad (1)$$

On the other hand, since  $f_n \rightarrow 0$  uniformly on compact subsets of  $\mathbb{D}$ , namely  $|w| \leq r$ , we can find constants  $C_3$  and  $C_4$  such that for large  $n$

$$\begin{aligned}
& \int_{\mathbb{D}} |f_n(w)|^p \frac{\mu_{\varphi, \psi\varphi', q}(D(w))}{(1-|w|^2)^{(\alpha q + 2q + pq - \alpha p)/p}} dA(w) \\
& \leq \epsilon C_3 \int_{|w| \leq r} \mu_{\varphi, \psi\varphi', q}(D(w)) dA(w) \\
& \leq \epsilon C_3 \int_{\mathbb{D}} \mu_{\varphi, \psi\varphi', q}(\mathbb{D}) dA(w) \\
& = \epsilon C_3 C_4. \tag{2}
\end{aligned}$$

Hence from (1) and (2), we have

$$\|f_n'\|_{L^q(\mu_{\varphi, \psi\varphi', q})}^q \leq \epsilon C_1 C_2 M^{q-p} (M^p + C_3 C_4),$$

which gives the desired result.

For the converse, let  $\{f_n\}$  be a bounded sequence in  $A_\alpha^p$  with  $f_n \rightarrow 0$  uniformly on compact subsets of  $\mathbb{D}$ . Then,

$$\begin{aligned}
\|W_{\psi, \varphi}(f_n)\|_{S^q}^q & \leq \int_{\partial\mathbb{D}} |\psi\varphi'|^q |f_n'(\varphi)|^q d\sigma + \int_{\partial\mathbb{D}} |\psi'|^q |f_n(\varphi)|^q d\sigma \\
& = \int_{\mathbb{D}} |f_n'(w)|^q d\mu_{\varphi, \psi\varphi', q}(w) + \int_{\mathbb{D}} |f_n(w)|^q d\mu_{\varphi, \psi', q}(w).
\end{aligned}$$

Since  $\mu_{\varphi, \psi', q}$  is vanishing Carleson measure, the second integral tends to zero as  $n \rightarrow \infty$ . By similar argument as above, the first integral tends to zero as  $n \rightarrow \infty$ . Which completes the proof.  $\square$

The following corollary can be seen by using Theorem 2.5 and Lemma 2.2, so we omit the proof's details.

**Corollary 2.6.** *Let  $1 \leq p \leq q$  and  $\alpha > -1$ . Let  $\varphi$  be an analytic self-map of  $\mathbb{D}$  and  $\psi$  be an analytic map of  $\mathbb{D}$  such that  $\psi \in S^q$  and  $\psi\varphi' \in H^q$ . Suppose  $\mu_{\varphi, \psi\varphi', q}(D(w)) = o(1-|w|^2)^{q(\alpha+2+p)/p}$  as  $|w| \rightarrow 1$ . Then  $W_{\psi, \varphi}$  is compact from  $A_\alpha^p$  into  $S^q$  if and only if*

$$\limsup_{|a| \rightarrow 1} \int_{\partial\mathbb{D}} \frac{|\psi'(z)|^q (1-|a|^2)^{q(\alpha+2)/p}}{(1-\bar{a}\varphi(z))^{2(\alpha+2)q/p}} d\sigma(z) = 0.$$

For  $1 \leq q < p$ , by using Theorem 1.2 and arguments similar to those in Theorem 2.2 and Theorem 2.5, we get the following Theorem.

**Theorem 2.7.** *Let  $1 \leq q < p$  and  $\alpha > -1$ . Let  $\varphi$  be an analytic self-map of  $\mathbb{D}$  and  $\psi$  be an analytic map of  $\mathbb{D}$  such that  $\psi \in S^q$  and  $\psi\varphi' \in H^q$ . Suppose that  $L(z) = (1-|z|^2)^{-(\alpha+q+2)} \mu_{\varphi, \psi\varphi', q}(D(z)) \in L^{p/(p-q)}(\mathbb{D}, dA_\alpha)$ . Then  $W_{\psi, \varphi}$  is bounded (respectively, compact) from  $A_\alpha^p$  into  $S^q$  if and only if  $\mu_{\varphi, \psi', q}$  is  $(A_\alpha^p, q)$ -Carleson measure (respectively, vanishing  $(A_\alpha^p, q)$ -Carleson measure).*

*Proof.* We prove compactness only. First, suppose that  $\mu_{\varphi, \psi', q}$  is vanishing Carleson measure. Let  $\{f_n\}$  be a bounded sequence in  $A_\alpha^p$  with  $f_n \rightarrow 0$  uniformly on compact

subsets of  $\mathbb{D}$ . Then,

$$\begin{aligned} \|W_{\psi,\varphi}(f_n)\|_{S^q}^q &\leq \int_{\partial\mathbb{D}} |(\psi f_n(\varphi))'|^q d\sigma \\ &\leq \int_{\partial\mathbb{D}} |\psi'|^q |f_n(\varphi)|^q d\sigma + \int_{\partial\mathbb{D}} |\psi\varphi'|^q |f_n'(\varphi)|^q d\sigma \\ &= \int_{\mathbb{D}} |f_n(w)|^q d\mu_{\varphi,\psi',q}(w) + \int_{\mathbb{D}} |f_n'(w)|^q d\mu_{\varphi,\psi\varphi',q}(w). \end{aligned}$$

Since  $\mu_{\varphi,\psi',q}$  is vanishing Carleson measure, the first integral tends to zero as  $n \rightarrow \infty$ . Thus it is enough to show that  $\|f_n'\|_{L^q(\mu_{\varphi,\psi\varphi',q})}^q \rightarrow 0$  as  $n \rightarrow \infty$ . By using similar argument as in Theorem 2.5, for  $\epsilon > 0$  there exists a constant  $C > 0$  such that  $\|f_n'\|_{L^q(\mu_{\varphi,\psi\varphi',q})}^q \leq \epsilon C$ , which gives the desired result.

For the converse, use argument similar to that in Theorem 2.5 and above argument.  $\square$

### 3. BOUNDEDNESS AND COMPACTNESS WHEN $q = \infty$

In this section we are interested in the the boundedness and compactness of weighted composition operators act between  $A_\alpha^p$  and  $S^q$  when  $p > 0$  and  $q = \infty$ .

**Theorem 3.1.** *Let  $0 < p < \infty$  and  $\alpha > -1$ . Let  $\varphi$  be an analytic self-map of  $\mathbb{D}$  and  $\psi$  be an analytic map of  $\mathbb{D}$  such that  $\psi \in S^\infty$  and  $\psi\varphi' \in H^\infty$ . Then  $W_{\psi,\varphi} : A_\alpha^p \rightarrow S^\infty$  is bounded if and only if*

$$\sup_{z \in \mathbb{D}} \frac{|\psi(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}} < \infty, \quad (3)$$

$$\sup_{z \in \mathbb{D}} \frac{|\psi'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha)/p}} < \infty. \quad (4)$$

*Proof.* Suppose  $W_{\psi,\varphi}$  is bounded. For any  $w \in \mathbb{D}$ , define

$$f_w(z) = \left( \frac{w-z}{1-\bar{w}z} \right) \frac{(1-|w|^2)^{(\alpha+2)/p}}{(1-\bar{w}z)^{2(2+\alpha)/p}}.$$

It is easy to see  $\|f_w\|_{A_\alpha^p} \approx 1$ . Setting  $w = \varphi(z)$ , we get

$$f_w(\varphi(z)) = 0 \quad \text{and} \quad f_w'(\varphi(z)) = \frac{-1}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}}.$$

From this it follows that,

$$\begin{aligned} \|W_{\psi,\varphi}(f_w)\|_{S^\infty} &\geq |\psi(z)\varphi'(z)f_w'(\varphi(z)) + \psi'(z)f_w(\varphi(z))| \\ &= \frac{|\psi(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}}, \end{aligned}$$

which proves the first condition (3), by taking supremum over all  $z \in \mathbb{D}$ .

Now, to show the second condition (4), consider for any  $w \in \mathbb{D}$  the function

$$g_w(z) = \frac{(1-|w|^2)^{(\alpha+2)/p}}{(1-\bar{w}z)^{2(2+\alpha)/p}}.$$

It is clear  $\|g_w\|_{A_\alpha^p} \approx 1$ . Again setting  $w = \varphi(z)$ , we get

$$g_w(\varphi(z)) = \frac{1}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}}, \quad \text{and} \quad g_w'(\varphi(z)) = \frac{4+2\alpha}{p} \frac{\overline{\varphi(z)}}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}}.$$



From this it follows that,

$$\begin{aligned} \|W_{\psi,\varphi}(g_w)\|_{S^\infty} &= \sup_{z \in \mathbb{D}} |\psi(z)\varphi'(z)g'_w(\varphi(z)) + \psi'(z)g_w(\varphi(z))| \\ &\geq \sup_{z \in \mathbb{D}} \frac{|\psi'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha)/p}} - \left(\frac{4 + 2\alpha}{p}\right) |\varphi(z)| \sup_{z \in \mathbb{D}} \frac{|\psi(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}}. \end{aligned}$$

Since  $\varphi(\mathbb{D}) \subseteq \mathbb{D}$  and the first condition (3) holds, we get the second condition (4).

For the converse, assume condition (3) and condition (4) are hold. By using Lemma 1.1 and Lemma 1.2, we get for each  $f \in A_\alpha^p$

$$\begin{aligned} \|W_{\psi,\varphi}(f(z))\|_{S^\infty} &= |\psi(z)\varphi'(z)f'(\varphi(z)) + \psi'(z)f(\varphi(z))| \\ &\leq C_1 \|f\|_{A_\alpha^p} \sup_{z \in \mathbb{D}} \frac{|\psi(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}} + C_2 \|f\|_{A_\alpha^p} \sup_{z \in \mathbb{D}} \frac{|\psi'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha)/p}}, \end{aligned}$$

which gives the boundedness of  $W_{\psi,\varphi}$ .  $\square$

MacCluer ([10], Theorem 2.3) showed that if  $\varphi \in S^p$ ,  $p \geq 1$ , then  $C_\varphi$  is compact on  $S^p$  if and only if  $\|\varphi\|_\infty < 1$ . Contreras and Hernández-Díaz ([3], Theorem 2.2) proved that if  $\varphi$  is an analytic self-map of  $\mathbb{D}$  and  $\psi \in S^\infty$ , then  $W_{\psi,\varphi}$  is compact from  $S^1$  into  $S^\infty$  if and only if  $\|\varphi\|_\infty < 1$ , or  $\lim_{|\varphi(z)| \rightarrow 1} |\psi'(z)| = \lim_{|\varphi(z)| \rightarrow 1} \frac{|\psi(z)\varphi'(z)|}{1 - |\varphi(z)|^2} = 0$ . In the next Theorem, we Characterize the compactness of the weighted composition operator  $W_{\psi,\varphi}$  acting from  $A_\alpha^p$  into  $S^\infty$ .

**Theorem 3.2.** *Let  $p > 0$  and  $\alpha > -1$ . Let  $\varphi$  be an analytic self-map of  $\mathbb{D}$  and  $\psi$  be an analytic map of  $\mathbb{D}$  such that  $\psi \in S^\infty$  and  $\psi\varphi' \in H^\infty$ . Then  $W_{\psi,\varphi} : A_\alpha^p \rightarrow S^\infty$  is compact if and only if*

$$\|\varphi\|_\infty < 1, \text{ or } \lim_{|\varphi(z)| \rightarrow 1} \frac{|\psi(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}} = \lim_{|\varphi(z)| \rightarrow 1} \frac{|\psi'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha)/p}} = 0.$$

*Proof.* Suppose that  $W_{\psi,\varphi}$  is compact,  $\|\varphi\|_\infty = 1$ , and  $\lim_{|\varphi(z)| \rightarrow 1} \frac{|\psi(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}} \neq 0$ . Then there exists a sequence  $\{z_n\}$  in  $\mathbb{D}$  such that  $|\varphi(z_n)| \rightarrow 1$  and  $\epsilon > 0$  such that  $\frac{|\psi(z)\varphi'(z)|}{(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}} \geq \epsilon$  for all natural  $n$ . Let us consider the function  $f_n$  in  $A_\alpha^p$  given by

$$f_n(z) = \left( \frac{\varphi(z_n) - z}{1 - \overline{\varphi(z_n)}z} \right) \frac{(1 - |\varphi(z_n)|^2)^{(\alpha+2)/p}}{(1 - \overline{\varphi(z_n)}z)^{2(2+\alpha)/p}}.$$

From the proof of Theorem 3.1, we know  $\{f_n\}$  is bounded in  $A_\alpha^p$  and goes to zero uniformly on compact subsets of  $\mathbb{D}$ . So, by Lemma 1.3,  $\|W_{\psi,\varphi}(f_n)\|_{S^\infty} \rightarrow 0$ . On the other hand, we have

$$\begin{aligned} \|W_{\psi,\varphi}(f_n)\|_{S^\infty} &\geq |\psi(z_n)\varphi'(z_n)f'_n(\varphi(z_n)) + \psi'(z_n)f_n(\varphi(z_n))| \\ &= \frac{|\psi(z_n)\varphi'(z_n)|}{(1 - |\varphi(z_n)|^2)^{(2+\alpha+p)/p}} \\ &\geq \epsilon, \end{aligned}$$

for all  $n$ , which is a contradiction. Thus, the condition holds. To prove the other condition, consider the function  $g_n$  in  $A_\alpha^p$  given by

$$g_n(z) = \frac{(1 - |\varphi(z_n)|^2)^{(2+\alpha)/p}}{(1 - \overline{\varphi(z_n)}z)^{(2+\alpha)/p}}.$$

Again from the proof of Theorem 3.1,  $\{g_n\}$  is bounded in  $A_\alpha^p$  and goes to zero uniformly on compact subsets of  $\mathbb{D}$ . So by Lemma 1.3,  $\|W_{\psi,\varphi}(g_n)\|_{S^\infty} \rightarrow 0$ . Therefore,

$$\begin{aligned} \|W_{\psi,\varphi}(g_n)\|_{S^\infty} &\geq |\psi(z_n)\varphi'(z_n)g_n'(\varphi(z_n)) + \psi'(z_n)g_n(\varphi(z_n))| \\ &\geq \frac{|\psi'(z_n)|}{(1 - |\varphi(z_n)|^2)^{(2+\alpha)/p}} - \frac{|\psi(z_n)\varphi'(z_n)|}{(1 - |\varphi(z_n)|^2)^{(2+\alpha+p)/p}}. \end{aligned}$$

By taking limit as  $|\varphi(z_n)| \rightarrow 1$  to both sides we get,

$$\lim_{|\varphi(z_n)| \rightarrow 1} \frac{|\psi'(z_n)|}{(1 - |\varphi(z_n)|^2)^{(2+\alpha)/p}} \leq \lim_{|\varphi(z_n)| \rightarrow 1} \frac{|\psi(z_n)\varphi'(z_n)|}{(1 - |\varphi(z_n)|^2)^{(2+\alpha+p)/p}},$$

which gives the desired result.

For the converse, suppose that the conditions are hold. Again we are going to apply Lemma 1.3. Let  $\{f_n\}$  be a bounded sequence in  $A_\alpha^p$  that converges to zero on compact subsets of  $\mathbb{D}$ . Take  $\epsilon > 0$ , by the hypothesis, there is  $r < 1$  such that if  $|\varphi(z)| > r$ , then

$$\begin{aligned} |\psi(z)\varphi'(z)| &< \epsilon(1 - |\varphi(z)|^2)^{(2+\alpha+p)/p}, \quad \text{and} \\ |\psi'(z)| &< \epsilon(1 - |\varphi(z)|^2)^{(2+\alpha)/p}. \end{aligned}$$

On the other hand, since  $f_n \rightarrow 0$  uniformly on compact subsets of  $\mathbb{D}$ , Cauchy's estimate gives  $f_n' \rightarrow 0$  uniformly on compact subsets of  $\mathbb{D}$ . Thus, there is a natural  $n_0$  such that if  $n \geq n_0$ , then

$$\sup_{|z| \leq r} |f_n(z)| \leq \frac{\epsilon}{2\|\psi\varphi'\|_\infty} \quad \text{and} \quad \sup_{|z| \leq r} |f_n'(z)| \leq \frac{\epsilon}{2\|\psi'\|_\infty}.$$

Then, for  $n \geq n_0$ , we have

$$\begin{aligned} \|W_{\psi,\varphi}(f_n)\|_{S^\infty} &\geq \sup_{|\varphi(z)| \leq r} |\psi(z)\varphi'(z)f_n'(\varphi(z)) + \psi'(z)f_n(\varphi(z))| \\ &\quad + \sup_{|\varphi(z)| > r} |\psi(z)\varphi'(z)f_n'(\varphi(z)) + \psi'(z)f_n(\varphi(z))| \\ &= I + II, \end{aligned}$$

where,

$$\begin{aligned} I &\leq \|\psi\varphi'\|_\infty \sup_{|w| \leq r} |f_n'(w)| + \|\psi'\|_\infty \sup_{|w| \leq r} |f_n(w)| \\ &\leq \|\psi\varphi'\|_\infty \frac{\epsilon}{2\|\psi\varphi'\|_\infty} + \|\psi'\|_\infty \frac{\epsilon}{2\|\psi'\|_\infty} \\ &= \epsilon, \end{aligned}$$

$$\begin{aligned} II &\leq \epsilon \sup_{|\varphi(z)| > r} |f_n'(\varphi(z))| (1 - |\varphi(z)|^2)^{(2+\alpha+p)/p} \\ &\quad + \epsilon \sup_{|\varphi(z)| > r} |f_n(\varphi(z))| (1 - |\varphi(z)|^2)^{(2+\alpha)/p} \\ &\leq \epsilon C \|f\|_{A_\alpha^p}, \end{aligned}$$

where the last inequality follows by consulting Lemma 1.1 and Lemma 1.2. Relations *I* and *II* give  $\|W_{\psi,\varphi}(f_n)\|_{S^\infty} \rightarrow 0$ .

If we assume that  $\|\varphi\|_\infty < 1$ , the compactness of  $W_{\psi,\varphi}$  follows from the fact that  $\overline{\varphi(\mathbb{D})}$  is compact subset of  $\mathbb{D}$  and  $\psi', \psi\varphi' \in H^\infty$ .  $\square$

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