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POWER OF m-PARTIAL ISOMETRIES ON HILBERT SPACES

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Abstract. A bounded linear operator T on a Hilbert space $\mathcal{H}$ is called an m-partial isometry for a positive integer m, if

$$T\left(T^{*m}T^m - \binom{m}{1}T^{*m-1}T^{m-1} + \binom{m}{2}T^{*m-2}T^{m-2} - .... + (-1)^m I\right) = 0.$$

In this article we will give some further properties of m-partial isometries in Hilbert spaces. In particular, we study some cases in which a power of an m-partial isometry is again an m-partial isometry.

1. Introduction and Terminologies

Let $\mathcal{H}$ denotes a complex separable infinite dimensional Hilbert space and $\mathcal{L}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$ into itself. For $T \in \mathcal{L}(\mathcal{H})$, T^* denotes the adjoint of T, $\mathcal{R}(T)$ and $\mathcal{N}(T)$ denote the range and the null-space of T, respectively, $I = I_{\mathcal{H}}$ is the identity operator.

Partial isometries provide an extensively studied extension of isometries which played significant role in the study of Hilbert space operators. The operator theory of partial isometries has been studied by several authors ([5],[16],[18],[19],[26]..).

Every bounded operator T on a Hilbert space $\mathcal{H}$ can be written as $T = U|T|$, where U is a partial isometry ,$|T| = \sqrt{T^*T}$, $\mathcal{R}(U) = \mathcal{R}(|T|)$, and $U^*U|T| = |T|$. Moreover U and $|T|$ are unique if $\mathcal{N}(U) = \mathcal{N}(|T|)$. An additional fact about partial isometries is the following result,[11].

Theorem 1.1. *If T is an operator on a Hilbert space $\mathcal{H}$, then the following statements are mutually equivalent:*

(1) *T is a partial isometry.*
(2) *T^* is a partial isometry.*
(3) $TT^*T = T$

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J. Agler and M.Stankus studied another extension called m-isometries on Hilbert space. An operator $T \in \mathcal{L}(\mathcal{H})$ is an m-isometry for some integer $m \geq 1$ if

$$\sum_{k=0}^{m}(-1)^k \binom{m}{k} T^{*m-k}T^{m-k} = 0, \tag{1}$$

where $\binom{m}{k}$ is the binomial coefficient. Simple manipulation proves that (1.1) is equivalent to

$$\sum_{k=0}^{m}(-1)^k \binom{m}{k} \|T^{m-k}x\|^2 = 0, \quad \text{for all } x \in \mathcal{H} \tag{2}$$

Evidently, an isometric operator (i.e., a 1-isometric operator) is m-isometric for all integers $m \geq 1$. Indeed the class of m-isometric operators is a generalization of the class of isometric operators. A detailed study of this class, in particular 2-isometric operators on a Hilbert space has been the object of some intensive study, especially by J.Agler and M. Stankus in [1], [2] and [3], but also by S.Richter [23], Shimorin [27] ,S.M. Patel [22] and C.Hillings [17], J.Gleasum and S.Richter [15]. Also we refer the reader to [10, 12, 13, 14] for more information about $m-$ isometric operators on Hilbert space. m-Isometries are not only a natural extension of an isometry, but they are also important in the study of Dirichlet operators and some other classes of operators.

The following characterization of 3-isometry is given by M. Scott in [25]. An operator $T \in \mathcal{L}(\mathcal{H})$ is a 3-isometry if

$$T^{*3}T^3 - 3T^{*2}T^2 + 3T^*T - I = 0.$$

Equivalently T is a 3-isometry if and only if there exist operators $B_1(T^*, T)$ and $B_2(T^*, T)$ such that for all natural numbers n,

$$T^{*n}T^n = I + nB_1(T^*, T) + n^2 B_2(T^*, T). \tag{3}$$

In this case it is straightforward to verify that

$$2B_2(T^*, T) = T^{*2}T^2 - 2T^*T + I \tag{4}$$

and

$$2B_1(T^*, T) = -T^{*2}T^2 + 4T^*T - 3I. \tag{5}$$

Evidently, each $B_j(T^*, T)$ is self-adjoint.

The notion of m-isometric operators on Hilbert spaces has been generalized to operators on general Banach spaces in papers of Botelho [8], Sid Ahmed [20], [21] and Bayart [4]. An operator $T \in \mathcal{L}(X)$ on a Banach space X is called an (m, p)-isometry if there exists an integer $m \geq 1$ and a $p \in [1, \infty)$, with

$$\forall\, x \in X, \ \sum_{k=0}^{m}(-1)^k \binom{m}{k} \|T^{m-k}x\|^p = 0 \tag{6}$$

It is easy to see that, if $X = \mathcal{H}$ is a Hilbert space and $p = 2$, this definition coincides with the original definition (1.1) of m-isometries.

Recall that $T \in \mathcal{L}(\mathcal{H})$ is normal if it satisfies the following condition: $T^*T = TT^*$, quasi-normal if: $T(T^*T) = (T^*T)T$ and quasi-isometry if $T^{*2}T^2 = T^*T$.

In [24] Adel and Sid Ahmed considered an extension of the notion of partial isometries to m-partial isometries. We say that $T \in \mathcal{L}(\mathcal{H})$ is an m-partial isometry if T satisfies

$$TB_m(T) = T\sum_{k=0}^{m}\binom{m}{k}(-1)^k T^{*m-k}T^{m-k} = 0, \tag{7}$$

where $B_m(T)$ is obtained formally from the binomial expansion of $B_m(T) = (T^*T - I)^m$ by understanding $(T^*T)^{m-k} = T^{*m-k}T^{m-k}$. The case when $m = 1$ is the partial isometries class. The class of m-partial isometries properly contains class of m-isometries. Contrary to the case of quasi-normal operators, in general, the kernel of an m-partial isometry is not reducing. Adel and Sid Ahmed [24] have proved the following interesting results: if T is an m-partial isometry such that the null space $\mathcal{N}(T)$ of T is a reducing subspace for T, and S_T is defined by $S_T := T^*B_{m-1}(T)T$, then S_T is positive , $\mathcal{N}(S_T)$ is invariant subspace for T and $T_{/\mathcal{N}(S_T)}$ is an $(m-1)$-partial isometry. Furthermore, if $\mathcal{M}$ is an invariant subspace for T and $T_{/\mathcal{M}}$ is $(m-1)$-partial isometry, then $\mathcal{M} \subset \mathcal{N}(S_T)$. In addition if $m = 2$ and T is a finitely cyclic, then T is compact.
A subspace $\mathcal{M}$ of $\mathcal{H}$ is called a reducing subspace for T if both $\mathcal{M}$ and $\mathcal{M}^{\perp}$ are $T-$invariant or equivalently if $\mathcal{M}$ is invariant for both T and T^*.

Theorem 1.2. *([24]) If $T \in \mathcal{L}(\mathcal{H})$ and $\mathcal{N}(T)$ is a reducing subspace for T, then the following properties are equivalent*

(1) *T is an m-partial isometry.*
(2) *$T|_{\mathcal{N}(T)^{\perp}}$ is an m-isometry.*

Remark 1.3. The assertion (2) in Theorem 1.2 is equivalent to

$$\|T^mT^*x\|^2 - \binom{m}{1}\|T^{m-1}T^*x\|^2 + ... + (-1)^{m-1}\binom{m}{m-1}\|TT^*x\|^2 + (-1)^m\|T^*x\|^2 = 0, \ \ \forall\, x \in \mathcal{H}.$$

The following proposition gives a more detailed description of several spectral properties of some m- partial isometries concerning the approximate spectrum σ_{ap}, the spectrum $\sigma(T)$ and the point spectrum σ_p.

Proposition 1.4. *([24]) If $T \in \mathcal{L}(\mathcal{H})$ is an m-partial isometry such that the null space $\mathcal{N}(T)$ of T is a reducing subspace for T, then*

(1) *$\sigma_{ap}(T) \subset \partial\mathbf{D} \cup \{0\}$, where $\mathbf{D}$ is the unit disc of the complex plane $\mathbb{C}$.*
(2) *$\sigma(T) \subset \partial\mathbf{D}$ or $\sigma(T) = \overline{\mathbf{D}}$.*
(3) *$\lambda \in \sigma_{ap}(T)\backslash\{0\}$ implies that $\overline{\lambda} \in \sigma_{ap}(T^*)$, i.e., if $(T-\lambda)x_n \longrightarrow 0$ for some bounded sequence $\{x_n\} \subset \mathcal{H}:\ \|x_n\| = 1$, then $(T-\lambda)^*x_n \longrightarrow 0$.*
(4) *$\lambda \in \sigma_p(T)\backslash\{0\}$ implies that $\overline{\lambda} \in \sigma_p(T^*)$.*
(5) *Eigenvectors of T corresponding to distinct eigenvalues are orthogonal, i.e.*

$$\mathcal{N}(T-\lambda)\perp\mathcal{N}(T-\mu) \ \text{ if } \ \lambda, \mu \in \sigma_p(T) \ \text{ and } \ \lambda \neq \mu.$$

2. Main Results

Theorem 2.1. *Let $T,\ N \in \mathcal{L}(\mathcal{H})$ such that T is an invertible m-isometry, $TN = NT$, $TN^* = N^*T$ and $N^2 = 0$. Then $T + N$ is an $(m+2)$- isometry.*

Proof. To prove that $T+N$ is an $(m+2)$-isometry, we need to prove that

$$\sum_{k=0}^{m+2}(-1)^k\binom{m+2}{k}(T^*+N^*)^{m+2-k}(T+N)^{m+2-k}=0.$$

From [20] , we have T^{-1} is an m-isometry and T is an $(m+1)$-isometry. It follows that

$$\begin{aligned}
&\sum_{k=0}^{m+2}(-1)^k\binom{m+2}{k}(T^*+N^*)^{m+2-k}(T+N)^{m+2-k}\\
=\ &\sum_{k=0}^{m+2}(-1)^k\binom{m+2}{k}\Bigg(\sum_{j=0}^{1}\binom{m+2-k}{j}T^{*m+2-k-j}N^{*j}\sum_{j=0}^{1}\binom{m+2-k}{j}T^{m+2-k-j}N^j\Bigg)\\
=\ &\sum_{k=0}^{m+2}(-1)^k\binom{m+2}{k}\Bigg(T^{*m+2-k}+(m+2-k)T^{*m+1-k}N^*\Bigg)\Bigg(T^{m+2-k}+(m+2-k)T^{m+1-k}N\Bigg).
\end{aligned}$$

Simple computation gives

$$\sum_{k=0}^{m+2}(-1)^k\binom{m+2}{k}kT^{*m+2-k}T^{m+1-k}N=-(m+2)\sum_{k=0}^{m+1}(-1)^k\binom{m+1}{k}T^{*m+1-k}T^{m+1-k}T^{-1}N=0,$$

and

$$\begin{aligned}
&\sum_{k=0}^{m+2}(-1)^k\binom{m+2}{k}k^2T^{*m+1-k}N^*T^{m+1-k}N\\
=\ &\sum_{k=1}^{m+2}(-1)^k\binom{m+2}{k}(k(k-1)+k)T^{*m+1-k}N^*T^{m+1-k}N\\
=\ &(m+2)(m+1)(T^*)^{-1}\sum_{k=0}^{m}(-1)^k\binom{m}{k}T^{*m-k}T^{m-k}N^*T^{-1}N\\
=\ &0.
\end{aligned}$$

We deduce that

$$\sum_{k=0}^{m+2}(-1)^k\binom{m+2}{k}(T^*+N^*)^{m+2-k}(T+N)^{m+2-k}=0.$$

Hence, the result.

Proposition 2.2. *If T is an m-partial isometry such that T^k is an partial isometry for $k=1,2,3,...m-1$, then T^m is a partial isometry for $m\geq 2$.*

Proof. Since T is an m-partial isometry, we have

$$T\Bigg(T^{*m}T^m-\tbinom{m}{1}T^{*m-1}T^{m-1}+\tbinom{m}{2}T^{*m-2}T^{m-2}-....+(-1)^mI\Bigg)=0.$$

Multiplying the above equation from the left by T^{m-1} we get

$$T^mT^{*m}T^m-\tbinom{m}{1}T^mT^{*m-1}T^{m-1}+\tbinom{m}{2}T^mT^{*m-2}T^{m-2}-....+(-1)^mT^m=0.$$

Since T^k is a partial isometry for $k = 1, ..., m-1$, we deduce that

$$T^mT^{*m}T^m - \binom{m}{1}T^m + \binom{m}{2}T^m - .... + (-1)^mT^m = 0.$$

Thus,

$$T^mT^{*m}T^m + T^m\Bigg(- \binom{m}{1} + \binom{m}{2} .... + (-1)^m\Bigg) = 0.$$

Hence

$$T^mT^{*m}T^m = T^m.$$

Remark 2.3. The conditions on T^k in Proposition 2.2 are necessary as shown in the following example.

Example 2.4. Consider the operator $S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{\frac{1+\sqrt{5}}{2}} & 0 \\ 0 & 1 & 0 \end{pmatrix}$,

acting on $\mathbb{C}^3$. Simple calculation shows that S is a 2-partial isometry but S is not a 1-partial isometry. Therefore S^2 is not a partial isometry (see [24]).

Proposition 2.5. *Let $T \in \mathcal{L}(\mathcal{H})$ be an m-partial isometry such that T^k is a partial isometry for $k = 2, 3, ..., m$. Then*

$$T^m = T^mT^*T.$$

Proof. We have

$$T\Bigg(T^{*m}T^m - \binom{m}{1}T^{*m-1}T^{m-1} + \binom{m}{2}T^{*m-2}T^{m-2} - .... + (-1)^mI\Bigg) = 0.$$

Multiplying the above equation from the left by T^{m-1}, we obtain

$$T^mT^{*m}T^m - \binom{m}{1}T^mT^{*m-1}T^{m-1} + \binom{m}{2}T^mT^{*m-2}T^{m-2} - .... + (-1)^{m-1}\binom{m}{m-1}T^mT^*T + (-1)^mT^m = 0.$$

By the assumption we get

$$T^m - \binom{m}{1}T^m + \binom{m}{2}T^m - .... + (-1)^{m-1}\binom{m}{m-1}T^mT^*T + (-1)^mT^m = 0.$$

A simple computation shows that $T^m(I - T^*T) = 0$. Hence, the result.

Remark 2.6. It is known that if T is an m-partial isometry such that T is a quasi-isometry or a quasi-normal , then T is a partial isometry ([24]).

The general case is given in the following corollary.

Corollary 2.7. *Let $T \in \mathcal{L}(\mathcal{H})$ be an m-partial isometry such that T^k is a partial isometry for $k = 2, 3, ..., m$. If $\mathcal{N}(T) = \mathcal{N}(T^2)$, then T is a partial isometry.*

Proof. It is a consequence of Proposition 2.2 and the fact that $\mathcal{N}(T) = \mathcal{N}(T^n)$ for all positive integer n.

Proposition 2.8. *Let $T \in \mathcal{L}(\mathcal{H})$ be an 2-partial isometry and an 3-partial isometry. Then the power of T satisfies the following relation*

$$\|T^kx\|^2 = (k-1)\|T^2x\|^2 - (k-2)\|Tx\|^2, \quad k = 1, 2, 3, ... \tag{8}$$

Proof. Since T is an 2-partial isometry and an 3-partial isometry, we obtain

$$T\Big(T^{*2}T^2 - 2T^*T + I\Big) = 0, \tag{9}$$

and

$$T\Big(T^{*3}T^3 - 3T^{*2}T^2 + 3T^*T - I\Big) = 0. \tag{10}$$

From (2.2) and (2.3), we deduce that

$$TT^*(T^{*2}T^2 - 2T^*T + I)T = 0,$$

or

$$T^*(T^{*2}T^2 - 2T^*T + I)T = 0, \big(\text{ as } \mathcal{N}(TT^*) = \mathcal{N}(T^*)\big).$$

Then

$$T^{*3}T^3 - 2T^{*2}T^2 + T^*T = 0 \Longleftrightarrow T^{*2}(T^*T - I)T^2 = T^*(T^*T - I)T.$$

Therefore

$$T^{*3}(T^*T - I)T^3 = T^{*2}(T^*T - I)T^2 = T^*(T^*T - I)T.$$

We deduce that

$$T^{*p}(T^*T - I)T^p = T^*(T^*T - I)T, \;\; p = 1, 2, ... \tag{11}$$

which implies that

$$\begin{aligned} & \langle T^{*p}(T^*T - I)T^p x|\, x\rangle = \langle T^*(T^*T - I)Tx|\, x\rangle \\ \Longleftrightarrow \;\; & \langle (T^*T - I)T^p x|\, T^p x\rangle = \langle (T^*T - I)Tx|\, Tx\rangle \\ \Longleftrightarrow \;\; & \|T^{p+1}x\|^2 - \|T^p x\|^2 = \|T^2 x\|^2 - \|Tx\|^2. \end{aligned}$$

Then

$$\sum_{p=1}^{k-1}\Big(\|T^{p+1}x\|^2 - \|T^p x\|^2\Big) = (k-1)\Big(\|T^2x\|^2 - \|Tx\|^2\Big), \;\; k = 2, 3, ...$$

Hence

$$\|T^k x\|^2 = (k-1)\|T^2 x\|^2 - (k-2)\|Tx\|^2, \;\; k = 1, 2, 3, ...$$

The following examples inspired from [24] and [9] which justify the next theorems.

Example 2.9. Let $T = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{pmatrix} \in \mathcal{L}(\mathbb{C}^3)$. It is easy to show that T is a 1-partial isometry but T is not a 2-partial isometry.

Example 2.10. Let $\mathcal{H} = \displaystyle\bigoplus_{i=-\infty}^{\infty} \mathbb{C}e_i$ be a Hilbert space generated by orthonormal vectors $\big(e_i\big)_{i=-\infty}^{\infty}$. Define on $\mathcal{H}$ the operator S by

$$S(e_i) = \begin{cases} 0\,, & i \leq -1 \\ e_{i+1}, & i \geq 0 \end{cases}$$

A computation shows that S is a 1-partial isometry and 2-partial isomerty.

Theorem 2.11. *([24], Proposition 3.5) Let $T \in \mathcal{L}(\mathcal{H})$ be an m-partial isometry such that $\mathcal{N}(T)$ is a reducing subspace for T. Then T is an $(m+n)$-partial isometry for $n = 0, 1, 2, ...$*

Theorem 2.12. *Let $T \in \mathcal{L}(\mathcal{H})$ be an m-partial isometry such that T is an m-isometry on $\mathcal{R}(T)$. Then T is an $(m+1)$-partial isometry.*

Proof. To show that T is an $(m+1)$-isometry using the Definition (1.7)

$$\begin{aligned}
& T\sum_{k=0}^{m+1}(-1)^k\tbinom{m+1}{k}T^{*m+1-k}T^{m+1-k} \\
= \; & T\Big(T^{*m+1}T^{m+1}+\sum_{k=1}^{m}(-1)^k\tbinom{m+1}{k}T^{*m+1-k}T^{m+1-k}-(-1)^m I\Big) \\
= \; & T\Big(T^{*m+1}T^{m+1}+\sum_{k=1}^{m}(-1)^k\Big(\tbinom{m}{k}+\tbinom{m}{k-1}\Big)T^{*m+1-k}T^{m+1-k}-(-1)^m I\Big) \\
= \; & T\Big(T^{*m+1}T^{m+1}+\sum_{k=1}^{m}(-1)^k\tbinom{m}{k}T^{*m+1-k}T^{m+1-k}+\sum_{k=1}^{m}(-1)^k\tbinom{m}{k-1}T^{*m+1-k}T^{m+1-k}-(-1)^m I\Big) \\
= \; & T\Big(T^{*m+1}T^{m+1}+\sum_{k=1}^{m}(-1)^k\tbinom{m}{k}T^{*m+1-k}T^{m+1-k}-\sum_{j=0}^{m-1}(-1)^j\tbinom{m}{j}T^{*m-j}T^{m-j}-(-1)^m I\Big) \\
= \; & T\Big(T^{*m+1}T^{m+1}+\sum_{k=1}^{m}(-1)^k\tbinom{m}{k}T^{*m+1-k}T^{m+1-k}\Big) \\
= \; & TT^*\Big(T^{*m}T^{m}+\sum_{k=1}^{m}(-1)^k\tbinom{m}{k}T^{*m-k}T^{m-k}\Big)T \\
= \; & TT^*\Big(T^{*m}T^{m}-\tbinom{m}{1}T^{*m-1}T^{m-1}+\tbinom{m}{2}T^{*m-2}T^{m-2}-....+(-1)^m I\Big)T \\
= \; & 0.
\end{aligned}$$

Example 2.1 justifies the following proposition.

Proposition 2.13. *Let $T \in \mathcal{L}(\mathcal{H})$ be an m-partial isometry such that T is an $(m-1)$-partial isometry on $\mathcal{R}(T)$. Then T is an $(m-1)$-partial isometry.*

Proof. This follows from equation (1.7) and the following identity

$$\binom{m}{k} = \binom{m-1}{k} + \binom{m-1}{k-1}, \; k = 1, 2, ..., m-1.$$

In the following theorem, we generalize Proposition 3.5 given in [24].

Theorem 2.14. *Let T, $S \in \mathcal{L}(\mathcal{H})$ such that T is an m-partial isometry . If $\mathcal{N}(T)$ is a reducing subspace for T (or T is an m-isometry on $\mathcal{R}(T)$) and S is an 3-isometry for which $TS = ST$ and $TS^* = S^*T$, then TS is an $(m+2)$-partial isometry.*

Proof. Since S is a 3-isometry and $TS = ST$,$TS^* = S^*T$, we have from equation (1.3)

$$\begin{aligned}
& ST\sum_{k=0}^{m+2}(-1)^k\tbinom{m+2}{k}(ST)^{*m+2-k}(ST)^{m+2-k} \\
= \; & ST\sum_{k=0}^{m+2}(-1)^k\tbinom{m+2}{k}(T)^{*m+2-k}(T)^{m+2-k}\big(S^{*m+2-k}S^{m+2-k}\big) \\
= \; & ST\sum_{k=0}^{m+2}(-1)^k\tbinom{m+2}{k}(T)^{*m+2-k}(T)^{m+2-k}\big(I+(m+2-k)B_1(S^*,S)+(m+2-k)^2B_2(S^*,S)\big) \\
= \; & \underbrace{ST\sum_{k=1}^{m+2}(-1)^k\tbinom{m+2}{k}k(T)^{*m+2-k}(T)^{m+2-k}\big(-B_1(S^*,S)+(-2(m+2))B_2(S^*,S)\big)}_{I} \\
+ \; & \underbrace{ST\sum_{k=1}^{m+2}(-1)^k\tbinom{m+2}{k}k^2(T)^{*m+2-k}(T)^{m+2-k}\big(B_2(S^*,S)\big)}_{J}
\end{aligned}$$

$$\begin{aligned}
I &= ST\sum_{k=1}^{m+2}(-1)^k(m+2)\tbinom{m+1}{k-1}(T)^{*m+2-k}(T)^{m+2-k}\big(-B_1(S^*,S)+(-2(m+2))B_2(S^*,S)\big) \\
&= -(m+2)ST\sum_{k=0}^{m+1}(-1)^k\tbinom{m+1}{k}(T)^{*m+1-k}(T)^{m+1-k}\big(-B_1(S^*,S)+(-2(m+2))B_2(S^*,S)\big) \\
&= 0 \quad \Big(\text{since } T \text{ is an } (m+1)-\text{partial isometry}\Big).
\end{aligned}$$

$$\begin{aligned}
J &= ST\sum_{k=1}^{m+2}(-1)^k\tbinom{m+2}{k}k^2(T)^{*m+2-k}(T)^{m+2-k}\big(B_2(S^*,S)\big) \\
&= ST\sum_{k=1}^{m+2}(-1)^k\tbinom{m+2}{k}(k(k-1)+k)(T)^{*m+2-k}(T)^{m+2-k}\big(B_2(S^*,S)\big) \\
&= ST\sum_{k=1}^{m+2}(-1)^k\tbinom{m+2}{k}k(k-1)(T)^{*m+2-k}(T)^{m+2-k}\big(B_2(S^*,S)\big)\Big(\text{by a similar argument as in } I\Big) \\
&= (m+2)(m+1)ST\sum_{k=2}^{m+2}(-1)^k\tbinom{m}{k-2}(T)^{*m+2-k}(T)^{m+2-k}\big(B_2(S^*,S)\big) \\
&= (m+2)(m+1)ST\sum_{k=0}^{m}(-1)^k\tbinom{m}{k}(T)^{*m-k}(T)^{m-k}\big(B_2(S^*,S)\big) \\
&= 0 \quad \Big(\text{since } T \text{ is an } m-\text{partial isometry}\Big).
\end{aligned}$$

Hence $I+J=0$. Thus TS is a $(m+2)$-partial isometry.

Theorem 2.15. *([6]) Let X be a Banach space, $T \in \mathcal{L}(X)$, m be a positive integer and $p \geq 1$ is a real number. If T is a (m,p)-isometry, so is any power T^r.*

Adel and Sid Ahmed ([24]) proved that any power of some 2-partial isometries is again a 2 -partial isometry.

Theorem 2.16. *Let $T \in \mathcal{L}(\mathcal{H})$ be a 2-partial isometry such that $\mathcal{N}(T)$ is a reducing subspace for T. Then any power of T is also a 2-partial isometry.*

In the following theorem, we generalize this result for m-partial isometries.

Theorem 2.17. *Let $T \in \mathcal{L}(\mathcal{H})$ be an m-partial isometry such that $\mathcal{N}(T)$ is a reducing subspace for T. Then any power of T is also an m-partial isometry.*

Proof. Let r be a positive integer. If T is an m-partial isometry then $T|_{\mathcal{N}(T)^\perp}$ is an m-isometry by Theorem 1.2. It follows from Theorem 2.13 that $T^r|_{\mathcal{N}(T)^\perp}$ is a m-isometry. Hence T^r is a m-partial isometry.

Corollary 2.18. *Let $T,\ S \in \mathcal{L}(\mathcal{H})$ such that T is an m-partial isometry . If $\mathcal{N}(T)$ is a reducing subspace for T and S is an 3- isometry for which $TS = ST$ and $TS^* = S^*T$, then T^kS^p is an $(m+2)$-partial isometry for $k = 1, 2, 3, ...$ and $p = 1, 2, ...$*

Proof. It is a consequence of Theorem 2.12 and Theorem 2.15.

Remark 2.19. It is trivial that the m-partial isometry is invariant under unitary equivalence. But the similarity does not preserve the m-partial isometry.

Example 2.20. Let $\mathcal{H}$ be a 3-dimensional Hilbert space and suppose S and T be defined on $\mathcal{H}$ as

$$S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } T = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}.$$

Then S is a 2-partial isometry, and if we take $X = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$ then $T = XSX^{-1}$. Hence T is similar to S and T in not 2-partial isometry.

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References

[1] J. Agler and M. Stankus, *m-Isometric transformations of Hilbert space I,* Integral Equations and Operator Theory,21 (1995), 383-429.

[2] J.Agler, M. Stankus, *m-Isometric transformations of Hilbert space II,* Integral Equations Operator Theory 23 (1) (1995) 1–48.

[3] J. Agler, M. Stankus, *m-Isometric transformations of Hilbert space III,* Integral Equations Operator Theory 24 (4) (1996) 379?-421.

[4] F. Bayart, *m-isometries on Banach spaces,* Math. Nachr. 284 (2011), 2141–2147.

[5] C. Badea and M.Mbekhta. *Operators similar to partial isometries.* Acta. Sci. Math (Szeged) 71(2005), 663–680.

[6] T. Bermúdez, C. Díaz-Mendoza, A. Martinón, *Powers of m-isometries,* Studia Math. 208 (2012) 249?-255.

[7] T. Bermúdez, A. Martinón, J. A. Noda,*Products of m-isometries*, Linear Algebra and its Applications 438 (2013) 80?-86.

[8] F. Botelho and J. Jamison, *Isometric properties of elementary operators,* Linear Algebra Appl. 432 (2010), 357–365.

[9] Z. Burdak , *On the decomposition of families of quasinormal operators* ,Opuscula Math. 33, no. 3 (2013), 419?-438.

[10] M. Chõ , S. Ôta, K. Tanahashic, A. Uchiyama, *Spectral properties of m-isometric*. operators. Functional Analysis, Approximation and Computation.4:2 (2012), 33?-39

[11] J. B. Conway, *A course in Functional analysis Second Edition.* Springer-Verlag 1990.

[12] B. P. Duggal, *Tensor product of n-isometries,* Linear Alg. Appl. 437(2012), 307-318.

[13] B. P. Duggal, *Tensor product of n-isometries II,* Functional Analysis, Approximation and Computation 4:1(2012), 27?-32.

[14] B. P. Duggal, *Tensor product of n-isometries III* , Functional Analysis, Approximation and Computation 4:2(2012),61-?67.

[15] J. Gleason and S. Richter, *m-Isometries commuting Tuples of Operators on a Hilbert Space*, Integral Equations and Operator Theory, 56(2)(2006), 181-196.

[16] P. R. Halmos and L. J. Wallen, *Powers of partial isometries* , J. Math. Mech, 19 (1970), 657-663.

[17] C. Hellings, *Two-Isometries on Pontryagin Spaces* ,Integr. equ. oper. theory 61 (2008), 211–239.

[18] T. Kato, *Peturbation Theory for Linear Operators* , Springer-Verlag, Berlin,1966.

[19] M. Mbekhta, *Partial isometries and generalised inverses* , Acta Sci. Math.(Szeged), 70, (2004), 767–781.

[20] Ould Ahmed Mahmoud Sid Ahmed, *m-isometric operators on Banach spaces*, Asian-European J. Math. 3(2010), 1?-19.

[21] Ould Ahmed Mahmoud Sid Ahmed, *Some properties of m-isometries and m-invertible operators on Banach spaces* , Acta Math. Scientia 32(B)(2)(2012), 520– 530.

[22] S. M. Patel, 2-*Isometric Operators,* Glasnik Matematicki. Vol. 37(57)(2002), 143-147.

[23] S. Richter, *A Representation Theorem for Cyclic Analytic Two-Isometries* , Transactions of the American Mathematical Society, Vol. 328, No. 1 (Nov., 1991), pp. 325-349.

[24] A.Saddi and Ould Ahmed Mahmoud Sid Ahmed, *m-partial isometries on Hilbert spaces Intern* . J. Funct. Anal., Operators Theory Appl. 2 (2010), No 1, 67–83.

[25] M.Scott, 3-*Isometries, Thesis* . University of California, San Diego (1987).

[26] Ch. Schmoeger, *Partial isometries on Banach spaces,* Seminar LV, No.20, 13 pp 28.12.2005.

[27] S. M. Shimorin, *Wold-type decompositions and wandering subspaces for operators close to isometries*, J. reine angew. Math. 531 (2001), 147–189.

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