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APPROXIMATELY n-MULTIPLICATIVE AND APPROXIMATELY ADDITIVE FUNCTIONS IN NORMED ALGEBRAS

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Abstract. We derive some properties of (ε, δ, n)-multiplicative maps between normed algebras and establish the superstability of (δ, n)-multiplicative functionals on normed algebras. We also prove that if φ is an (ε, δ, n)-multiplicative such that in the case where n is odd, $1 \in \varphi(A)$, then $|||\varphi||| \leq (1+\delta)^{1/(n-1)}$. Moreover, under certain conditions, we prove that if $\varphi : A \to C_0(X)$ is an (ε, δ, n)-multiplicative, then $|||\varphi||| \leq (1+\delta)^{1/(n-1)}$, where $C_0(X)$ is the algebra of all real valued continuous functions which vanish at infinity defined on a locally compact Hausdorff space X.

1. Introduction

The notion of n-homomorphism between (Banach) algebras was introduced in [5]. Suppose that $n \geq 2$ is an integer. A mapping $\varphi : A \to B$ between (Banach) algebras is called n-multiplicative if $\varphi(a_1 a_2 \cdots a_n) = \varphi(a_1)\varphi(a_2)\cdots\varphi(a_n)$ for all elements $a_1, a_2, \cdots, a_n \in A$. Moreover, φ is called an n-ring if φ is n-multiplicative and additive. If φ is also linear, it is called an n-homomorphism. For further details on the above concepts and properties one can refer, for example, to [4, 5, 6, 7, 8, 10, 14].

Let A and B be normed algebras, $\varphi : A \to B$ a map and δ be a non-negative real number. The mapping φ is said to be δ-multiplicative if $\|\varphi(xy) - \varphi(x)\varphi(y)\| \leq \delta\|x\|\|y\|$ for all $x, y \in A$.

A mapping $\varphi : A \to B$ is called (δ, n)-multiplicative if $\|\varphi(x_1...x_n) - \varphi(x_1)...\varphi(x_n)\| \leq \delta\|x_1\|...\|x_n\|$ for all $x_1, ..., x_n \in A$. If further φ is linear mapping, then it is called (δ, n)-homomorphism. Also we say that φ is approximately n-homomorphism if there exists a constant $\delta > 0$ such that φ is (δ, n)-homomorphisms. The (δ, n)-homomorphisms are near to the δ-homomorphisms but they are not the same. An example of an approximately n-homomorphism which is not approximately homomorphism is given in [1, 3.6].

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Also, a mapping $\varphi : A \to B$ is called an ε-additive for some $\varepsilon > 0$ if $\|\varphi(x+y) - \varphi(x) - \varphi(y)\| \leq \varepsilon(\|x\| + \|y\|)$ for all $x, y \in A$. It is clear that $\varphi(0) = 0$. The mapping φ is said to be approximately additive if there exists a constant $\varepsilon > 0$ such that φ is ε-additive. We say that φ is an (ε, δ, n)-multiplicative if φ is ε-additive and (δ, n)-multiplicative. For further details on the above concepts and properties one can refer, for example, to [1, 2, 9, 15].

In 1980, Baker [3] proved that if φ is a complex valued function on a semigroup A such that $|\varphi(xy) - \varphi(x)\varphi(y)| \leq \delta$ for all $x, y \in A$, then either φ is multiplicative or $|\varphi(x)| \leq \frac{1+\sqrt{1+4\delta}}{2}$ for all $x \in A$.

Šemrl [13] indicated that there exists an $(\varepsilon, \delta, 2)$-multiplicative such that it is continuous only at the origin.
Let $\varphi : A \to B$ be (ε, δ, n)-multiplicative. Define $|||\varphi||| = \sup_{a \in A \setminus \{0\}} \frac{\|\varphi(a)\|}{\|a\|} \leq \infty$. The map φ is called bounded if $|||\varphi||| < \infty$.

Šemrl [12] proved that if A is a real Banach algebra and $\varphi : A \to \mathbb{R}$ is an $(\varepsilon, \delta, 2)$-multiplicative, then $|||\varphi||| \leq \frac{1+\sqrt{1+4\delta}}{2}$.

In this paper, we derive some properties of (ε, δ, n)-multiplicative maps between normed algebras and establish the superstability of (δ, n)-multiplicative functionals on normed algebras. We also prove that if φ is an (ε, δ, n)-multiplicative such that in the case where n is odd, $1 \in \varphi(A)$, then $|||\varphi||| \leq (1+\delta)^{1/(n-1)}$. Moreover, under certain conditions, we prove that if $\varphi : A \to C_0(X)$ is an (ε, δ, n)-multiplicative, then $|||\varphi||| \leq (1+\delta)^{1/(n-1)}$, where $C_0(X)$ is the algebra of all real valued continuous functions which vanish at infinity defined on a locally compact Hausdorff space X. Finally, we show that if A is a real Banach algebra and $\phi : A \to \mathbb{R}$ is (δ, δ, n)-multiplicative such that $0 < \delta < 1$ and in the case where n is odd, we have $1 \in \phi(A)$ and also, if $\psi : A \to \mathbb{R}$ satisfies $|\phi(x) - \psi(x)| \leq \varepsilon\|x\|$ for all $x \in A$ where $\varepsilon > 0$, then ψ is (γ, γ, n)-multiplicative whenever $\gamma = \varepsilon + \delta + \varepsilon[2(1+\delta)^{1/(n-1)} + \varepsilon]^{1/(n-1)}$.

2. Approximately n-multiplicative and approximately additive

For the sake of completeness we first state the following result, which appeared in [2, 2.4].

Theorem 2.1. *Let A be a normed algebra, and $p \geq 0$. If $\varphi : A \to \mathbb{C}$ satisfies $|\varphi(x_1 \cdots x_n) - \varphi(x_1) \cdots \varphi(x_n)| \leq \delta\|x_1\|^p \cdots \|x_n\|^p$ for all $x_1 \cdots x_n \in A$, then φ is n-multiplicative or there exists a constant k such that $|\varphi(x)| \leq k\|x\|^p$ for all $x \in A$.*

Proof. Suppose that φ is not n-multiplicative, that is, there exist $a_1, \cdots, a_n \in A$ such that

$$\varphi(a_1 \cdots a_n) \neq \varphi(a_1) \cdots \varphi(a_n).$$

Then for every non-zero element $x \in A$, we have

$$\begin{aligned}|\varphi(x)|^{(n-1)}|\varphi(a_1\cdots a_n)-&\varphi(a_1)\cdots\varphi(a_n)|\\
&=|\varphi(x)^{(n-1)}\varphi(a_1\cdots a_n)-\varphi(x)^{(n-1)}\varphi(a_1)\cdots\varphi(a_n)\pm\\
&\quad\varphi(x^{(n-1)}a_1\cdots a_n)\pm\varphi(x^{(n-1)}a_1)\varphi(a_2)\cdots\varphi(a_n)|\\
&\leq|\varphi(x)^{(n-1)}\varphi(a_1\cdots a_n)-\varphi(x^{(n-1)}a_1\cdots a_n)|+\\
&\quad|\varphi(x^{(n-1)}a_1\cdots a_n)-\varphi(x^{(n-1)}a_1)\varphi(a_2)\cdots\varphi(a_n)|+\\
&\quad|\varphi(x^{(n-1)}a_1)\varphi(a_2)\cdots\varphi(a_n)-\varphi(x)^{(n-1)}\varphi(a_1)\cdots\varphi(a_n)|\\
&\leq 2\delta\|x\|^{p(n-1)}\|a_1\|^p\cdots\|a_n\|^p+|\varphi(a_2)\cdots\varphi(a_n)|\delta\|x\|^{p(n-1)}\|a_1\|^p\\
&=\delta\|x\|^{p(n-1)}\|a_1\|^p[2\|a_2\|^p\cdots\|a_n\|^p+|\varphi(a_2)\cdots\varphi(a_n)|].\end{aligned}$$

Therefore, if

$$k=\left(\frac{\delta\|a_1\|^p[2\|a_2\|^p\cdots\|a_n\|^p+|\varphi(a_2)\cdots\varphi(a_n)|]}{|\varphi(a_1\cdots a_n)-\varphi(a_1)\cdots\varphi(a_n)|}\right)^{\frac{1}{(n-1)}},$$

then we have $|\varphi(x)|\leq k\|x\|^p$. □

Theorem 2.2. *Let A be a normed algebra and let φ be (δ,n)-multiplicative functional. Then either φ is n-multiplicative or $|\varphi(x)|\leq(1+\delta)\|x\|$ for each $x\in A$.*

Proof. Suppose that φ is not n-multiplicative. Then by the Theorem 2.1, there exists $k>0$ such that $|\varphi(x)|\leq k\|x\|$ for all $x\in A$, so $\varphi(0)=0$. Assume towards a contradiction that there exists $a\in A$ with $|\varphi(a)|>(1+\delta)\|a\|$, then $|\varphi(a)|=(1+\delta+p)\|a\|$ for some $p>0$. Since φ is (δ,n)-multiplicative, $|\varphi(a)^n-\varphi(a^n)|\leq\delta\|a\|^n$. Hence

$$|\varphi(a^n)|\geq|\varphi(a)^n|-|\varphi(a)^n-\varphi(a^n)|\geq(1+\delta+p)^n\|a\|^n-\delta\|a\|^n\geq(1+\delta+p)\|a\|^n.$$

Now, we assume the induction assumption $|\varphi(a^{n^m})|\geq(1+\delta+mp)\|a\|^{n^m}$. We have

$$\begin{aligned}|\varphi(a^{n^{m+1}})|&\geq|\varphi(a^{n^m})^n|-|\varphi(a^{n^m})^n-\varphi(a^{n^{m+1}})|\\
&\geq(\delta+1+mp)^n\|a\|^{n^{m+1}}-\delta\|a^{n^m}\|^n\\
&\geq(\delta+1+mp)^n\|a\|^{n^{m+1}}-\delta\|a\|^{n^{m+1}}\\
&\geq(\delta+1+(m+1)p)\|a\|^{n^{m+1}}.\end{aligned}$$

Therefore $|\varphi(a^{n^m})|\geq(1+\delta+mp)\|a\|^{n^m}$ holds for all positive integer m. Now, let $x_1,...,x_{n+1}\in A$. We have

$$|\varphi(x_1...x_n)-\varphi(x_1)...\varphi(x_n)||\varphi(x_{n+1})|\leq\delta k\|x_1\|...\|x_{n+1}\| \tag{2.1}$$

In particular, if $x_{n+1}=a^{n^m}$ by (2.1) we have

$$|\varphi(x_1...x_n)-\varphi(x_1)...\varphi(x_n)|\leq\frac{\delta k\|x_1\|...\|x_n\|\|a^{n^m}\|}{|\varphi(a^{n^m})|}\leq\frac{\delta k\|x_1\|...\|x_n\|}{1+\delta+mp}$$

Letting $m\to\infty$ shows that φ is n-multiplicative, which is a contradiction. □

Theorem 2.3. *Let A be a semigroup and φ be a complex valued function defined on A such that $|\varphi(x_1...x_n)-\varphi(x_1)...\varphi(x_n)|\leq\delta$ for all $x_1,..,x_n\in A$. Then either $|\varphi(x)|\leq 1+\delta$ for all $x\in A$ or φ is n-multiplicative.*

Proof. If φ is not n-multiplicative, then by the Theorem 2.1, there exists $k > 0$ such that $|\varphi(x)| \leq k$ for all $x \in A$. Suppose there exists $a \in A$ such that $|\varphi(a)| > 1 + \delta$. Thus, $|\varphi(a)| = 1 + \delta + p$ for some $p > 0$. By a similar argument to that in the Theorem 2.2, we obtain $|\varphi(a^{n^m})| \geq 1 + \delta + mp$ for all positive integer m, and the rest of the proof is similar to the proof of the Theorem 2.2. □

Theorem 2.4. *Let A be a Banach algebra and $\varphi : A \to \mathbb{C}$ be a nonzero (δ, n)-homomorphism. Then $\|\varphi\| \leq 1 + \delta$.*

Proof. If φ is n-homomorphism, then as in the proof of [14, Lemma 2.1] we can see that $\|\varphi\| \leq 1$. If φ is (δ, n)-homomorphism such that φ is not n-multiplicative, then by the Theorem 2.2, the result follows. □

Lemma 2.5. *Let A be a normed algebra and $0 < \delta < 1$. If $\phi : A \to \mathbb{C}$ is (ε, δ, n)-homomorphism such that $|||\phi||| < \infty$, then $|||\phi||| \leq (1 + \delta)^{1/(n-1)}$.*

Proof. Suppose that $|||\phi||| = k > 0$. By the hypothesis, we have $|\phi(x)^n - \phi(x^n)| \leq \delta\|x\|^n$ for all $x \in A$, so $|\phi(x)|^n \leq \delta\|x\|^n + |\phi(x^n)|$. Therefore for all $x \neq 0$, where $x^n \neq 0$, we have

$$\frac{|\phi(x)|^n}{\|x\|^n} \leq \delta + \frac{|\phi(x^n)|}{\|x\|^n} \leq \delta + \frac{|\phi(x^n)|}{\|x^n\|} \leq \delta + k.$$

If $x^n = 0$, since $\varphi(0) = 0$ then

$$\frac{|\phi(x)|^n}{\|x\|^n} \leq \delta + \frac{|\phi(x^n)|}{\|x\|^n} = \delta + 0 \leq \delta + k,$$

and so $k^n - k \leq \delta$. Hence, if $k > 1$, then $k < (1 + \delta)^{1/(n-1)}$, since $\delta < 1$ and if $k \leq 1$, then $k \leq 1 < (1 + \delta)^{1/(n-1)}$, so the result follows. □

Theorem 2.6. *Suppose that A be a normed algebra, $0 < \delta < 1$ and $p \geq 0$. Let $\phi : A \to \mathbb{C}$ is a functional such that*

$$|\phi(x_1 \cdots x_n) - \phi(x_1) \cdots \phi(x_n)| \leq \delta\|x_1\|^p \cdots \|x_n\|^p \ (x_1, \cdots, x_n \in A).$$

Then either ϕ is n-multiplicative or $|\phi(x)| \leq (1 + \delta)^{1/(n-1)}\|x\|^p$ for all $x \in A$.

Proof. Suppose that ϕ is not n-multiplicative. Then by the Theorem 2.1 there exists $M > 0$ such that $|\phi(x)| \leq M\|x\|^p$ for all $x \in A$. It is clear that $\phi(0) = 0$ for $p \neq 0$. If $k = \sup_{x\in A} |\phi(x)|$ for $p = 0$ and $k = \sup_{x\in A\setminus\{0\}} \frac{|\phi(x)|}{\|x\|^p}$ for $p \neq 0$, then we have

$$|\phi(x)|^n \leq \delta\|x\|^{np} + |\phi(x^n)| \leq \delta\|x\|^{np} + k\|x^n\|^p \leq (\delta + k)\|x\|^{np}$$

for all $x \in A$. Finally, by using a similar argument as in the proof of the Lemma 2.5, we have $|\phi(x)| \leq (1 + \delta)^{1/(n-1)}\|x\|^p$ for all $x \in A$, as desired. □

Theorem 2.7. [2, 2.6] *Let A be a normed algebra and ε, δ, p be non-negative real numbers. Suppose that $\varphi : A \to \mathbb{C}$ is a functional such that*

$$|\varphi(x + y) - \varphi(x) - \varphi(y)| \leq \varepsilon(\|x\|^p + \|y\|^p) \ (x, y \in A),$$

and

$$|\varphi(x_1 \cdots x_n) - \varphi(x_1) \cdots \varphi(x_n)| \leq \delta\|x_1\|^p \cdots \|x_n\|^p \ (x_1, \cdots, x_n \in A).$$

Then φ is additive and n-multiplicative or there is a constant k such that

$$|\varphi(x)| \leq k\|x\|^p \ (x \in A).$$

Proof. Suppose that φ is not additive or, is not n-multiplicative. If φ is not n-multiplicative, then by the Theorem 2.1, the result follows. Now, if φ is not additive, then there exist $a, b \in A$, such that $\varphi(a+b) - \theta(a) - \varphi(b) \neq 0$. For any $x \in A$, by the hypothesis, we have

$$\begin{aligned}
|\varphi(x)|^{(n-1)}|\varphi(a+b)-\varphi(a)-\varphi(b)| =& |\varphi(x)^{(n-1)}\varphi(a+b)-\varphi(x)^{(n-1)}\varphi(a)-\varphi(x)^{(n-1)}\varphi(b) \pm \\
& \varphi(x^{(n-1)}(a+b)) \pm \varphi(x^{(n-1)}a) \pm \varphi(x^{(n-1)}b)| \\
\leq & |\varphi(x)^{(n-1)}\varphi(a+b)-\varphi(x^{(n-1)}(a+b))| + \\
& |\varphi(x^{(n-1)}a+x^{(n-1)}b)-\varphi(x^{(n-1)}a)-\varphi(x^{(n-1)}b)| + \\
& |\varphi(x)^{(n-1)}\varphi(a)-\varphi(x^{(n-1)}a)| + \\
& |\varphi(x^{(n-1)}b)-\varphi(x)^{(n-1)}\varphi(b)| \\
\leq & \delta\|x\|^{(n-1)p}\|a+b\|^p + \varepsilon(\|x^{(n-1)}a\|^p + \|x^{(n-1)}b\|^p) + \\
& \delta\|x\|^{(n-1)p}\|a\|^p + \delta\|x\|^{(n-1)p}\|b\|^p \\
\leq & \|x\|^{(n-1)p}\Big(\delta\left(\|a+b\|^p+\|a\|^p+\|b\|^p\right)+\varepsilon\left(\|a\|^p+\|b\|^p\right)\Big).
\end{aligned}$$

Therefore, if

$$k = \left(\frac{\delta\left(\|a+b\|^p+\|a\|^p+\|b\|^p\right)+\varepsilon\left(\|a\|^p+\|b\|^p\right)}{|\varphi(a+b)-\varphi(a)-\varphi(b)|}\right)^{\frac{1}{(n-1)}},$$

then we have $|\varphi(x)| \leq k\|x\|^p$, as desired. □

Theorem 2.8. *Let A be a real Banach algebra, $0 < \delta < 1$ and n be an even number. Let $\phi : A \to \mathbb{R}$ be (ε, δ, n)-multiplicative. Then $|||\phi||| \leq (1+\delta)^{1/(n-1)}$.*

Proof. If ϕ is not additive or is not n-multiplicative, then by the Theorem 2.7 there exists $k > 0$ such that $|\phi(x)| \leq k\|x\|$ for all $x \in A$, so by the Lemma 2.5, $|||\phi||| < (1+\delta)^{1/(n-1)}$.

Now suppose ϕ is additive and n-multiplicative. Fix $x \in A$ with $\|x\| = 1$. Then the mapping $h : \mathbb{R} \to \mathbb{R}$ defined by $h(t) = \phi(tx)$ is additive. We now show that h is linear. To do this, we define two functions $f, g : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ by $f(t) = \frac{\phi(t^{\frac{n}{2}}x^n)}{t^{\frac{n}{2}}}$ and $g(t) = (\frac{\phi(tx)}{t})^{\frac{n}{2}}$. It is easy to see that $f(ts) = g(t)g(s)$ for all $s, t \in R\backslash\{0\}$. Then by [12, Theorem 3] we can assume that either g is bounded or $g(1) \neq 0$ and $g(t) = g(1)k(t)$, where k is multiplicative. First, suppose that g is bounded, then there exists $M > 0$ such that $|g(t)| \leq M$ for all $t \in \mathbb{R}\backslash\{0\}$. Then $|h(t)| \leq M^{\frac{2}{n}}|t|$ and so h is continuous at zero. Since h is additive, it is easy to see that h is linear. In the second case, define $\psi : \mathbb{R} \to \mathbb{R}$ by $\psi(t) = tk(t)^{\frac{n}{2}} = g(1)^{-\frac{2}{n}}h(t)$ for all $t \in \mathbb{R}\backslash\{0\}$ and $\psi(0) = 0$. Since k is multiplicative, then ψ is multiplicative and since h is additive, so ψ is additive. Now, because $\psi : \mathbb{R} \to \mathbb{R}$ is additive and multiplicative map and $g(1) \neq 0$, then it is easy to see that it is the identity map. Thus h is continuous, and so it is easy to see that h is linear. Therefore $\phi(tx) = t\phi(x)$ for all $t \in \mathbb{R}$ and $x \in A$ with $\|x\| = 1$. Now suppose that $y \in A\backslash\{0\}$. For all $t \in \mathbb{R}$ we have $\phi(ty) = \phi(t\|y\|\frac{y}{\|y\|}) = t\|y\|\phi(\frac{y}{\|y\|})$. Consequently ϕ is linear and then it is an n-homomorphism. Now the result follows by the Theorem 2.4 and the Lemma 2.5. □

Corollary 2.9. *Let A be a real Banach algebra, $0 < \delta < 1$, X be a locally compact Hausdorff space and let $\phi : A \to C_0(X)$ be (ε, δ, n)-multiplicative. If n is an even number, then $|||\phi||| \leq (1+\delta)^{1/(n-1)}$.*

Proof. By the hypothesis the functional $\phi_x : A \to \mathbb{R}$ by $\phi_x(a) = \phi(a)(x)$ is also (ε, δ, n)-multiplicative for every $x \in X$. Then by the Theorem 2.8, $|||\phi_x||| \leq (1+\delta)^{1/(n-1)}$ for all $x \in X$. Hence

$$|||\phi||| = \sup_{a \in A\setminus\{0\}} \frac{\|\phi(a)\|}{\|a\|} = \sup_{a \in A\setminus\{0\}} \sup_{x \in X} \frac{|\phi_x(a)|}{\|a\|} \leq (1+\delta)^{1/(n-1)}.$$

□

Theorem 2.10. *Let A be a real Banach algebra, $0 < \delta < 1$ and n an odd number. Suppose $\phi : A \to \mathbb{R}$ is (ε, δ, n)-multiplicative such that in the case where ϕ is an additive and n-multiplicative, we have $1 \in \phi(A)$. Then $|||\phi||| \leq (1+\delta)^{1/(n-1)}$.*

Proof. If ϕ is not additive or not n-multiplicative, then by the Theorem 2.7 there exists $k > 0$ such that $|\phi(x)| \leq k\|x\|$ for all $x \in A$, so by the Lemma 2.5, $|||\phi||| < (1+\delta)^{1/(n-1)}$.

If ϕ is additive and n-multiplicative, then by the hypothesis there exists $a \in A$ such that $\phi(a) = 1$. Now we define additive function $\psi : A \to \mathbb{R}$ by $\psi(x) = \phi(ax)$ for all $x \in A$. By the following proof of [14, Lemma 2.1], ψ is multiplicative and $\psi^{n-1}(x) = \phi^{n-1}(x)$ for all $x \in A$. Using the proof of the Theorem 2.8, Since ψ is additive and multiplicative so it is linear, and hence ϕ is linear. Therefore the result follows by the Theorem 2.4 and the Lemma 2.5.

□

Corollary 2.11. *Let A be a real Banach algebra, $0 < \delta < 1$, n be an odd number and let X be a locally compact Hausdorff space. Suppose $\phi : A \to C_0(X)$ is (ε, δ, n)-multiplicative and $1 \in \phi_x(A)$ for all $x \in X$. Then $|||\phi||| \leq (1+\delta)^{1/(n-1)}$.*

Proof. By adopting the proof of the Corollary 2.9, the result follows. □

Theorem 2.12. *Let A be a real Banach algebra, $0 < \delta < 1$ and $\varepsilon > 0$. Let $\phi : A \to \mathbb{R}$ be (δ, δ, n)-multiplicative such that in the case where n is odd, we have $1 \in \phi(A)$. If $\psi : A \to \mathbb{R}$ satisfies $|\phi(x) - \psi(x)| \leq \varepsilon\|x\|$ for all $x \in A$, then ψ is (γ, γ, n)-multiplicative whenever $\gamma = \varepsilon + \delta + \varepsilon[2(1+\delta)^{1/(n-1)} + \varepsilon]^{1/(n-1)}$.*

Proof. By the Theorems 2.8 and 2.10, we have $|||\phi||| \leq (1+\delta)^{1/(n-1)} = k$, so $|||\psi||| \leq \varepsilon + k$. We prove that the following inequality

$$|\phi(a_1)...\phi(a_m) - \psi(a_1)...\psi(a_m)| \leq \varepsilon\|a_1\|...\|a_m\|[k + (\varepsilon + k)]^{m-1}, \tag{2.2}$$

for all $1 \leq m \leq n$. By the hypothesis, the inequality (2.2) is certainly true if $m = 1$. Assume that (2.2) is true for $m - 1$. Therefore

$$\begin{aligned}|\phi(a_1)...\phi(a_m) - \psi(a_1)...\psi(a_m)| \leq& |\phi(a_1)...\phi(a_{m-1})||\phi(a_m) - \psi(a_m)| + \\ & |\psi(a_m)|||\phi(a_1)...\phi(a_{m-1}) - \psi(a_1)...\psi(a_{m-1})| \\ \leq& \varepsilon[k^{m-1} + (\varepsilon + k)[k + (\varepsilon + k)]^{m-2}]\|a_1\|...\|a_m\| \\ \leq& \varepsilon[k + (\varepsilon + k)]^{m-1}]\|a_1\|...\|a_m\|,\end{aligned}$$

which complete the proof of (2.2). Now by (2.2) for all $a_1, ..., a_n \in A$, we have

$$\begin{aligned}|\psi(a_1...a_n) - \psi(a_1)...\psi(a_n)| \leq & |\psi(a_1...a_n) - \phi(a_1...a_n)| + |\phi(a_1...a_n) - \phi(a_1)...\phi(a_n)| \\ & + |\phi(a_1)...\phi(a_n) - \psi(a_1)...\psi(a_n)| \\ \leq & [\varepsilon + \delta + \varepsilon[k + (\varepsilon + k)]^{n-1}]\|a_1\|...\|a_n\|.\end{aligned}$$

Moreover,

$$\begin{aligned}|\psi(x+y) - \psi(x) - \psi(y)| \leq & |\psi(x+y) - \phi(x+y)| + |\phi(x+y) - \phi(x) - \phi(y)\| \\ & + |\phi(x) - \psi(x)| + |\phi(y) - \psi(y)| \\ \leq & (2\varepsilon + \delta)(\|x\| + \|y\|) \\ \leq & [\varepsilon + \delta + \varepsilon(2k + \varepsilon)^{n-1}](\|x\| + \|y\|)\end{aligned}$$

for all $x, y \in A$, as desired. □

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