

DEFORMATION RETRACT AND FOLDING OF THE 5D SCHWARZCHILD FIELD

(COMMUNICATED BY KRISHAN L. DUGGAL)

NASR AHMED, H. RAFAT

ABSTRACT. In this article we introduce some types of the deformation retracts of the 5D Schwarzschild space making use of Lagrangian equations. The retraction of this space into itself and into geodesics has been presented. The relation between folding and deformation retract of this space has been achieved. A relation for energy conservation similar to the one obtained in four dimensions has been obtained for the five dimensional case.

1. INTRODUCTION

The real revolution in mathematical physics in the second half of twentieth century (and in pure mathematics itself) was algebraic topology and algebraic geometry [1]. In the nineteenth century, Mathematical physics was essentially the classical theory of ordinary and partial differential equations. The variational calculus, as a basic tool for physicists in theoretical mechanics, was seen with great reservation by mathematicians until Hilbert set up its rigorous foundation by pushing forward functional analysis. This marked the transition into the first half of twentieth century, where under the influence of quantum mechanics and relativity, mathematical physics turned mainly into functional analysis, complemented by the theory of Lie groups and by tensor analysis. All branches of theoretical physics still can expect the strongest impacts from use of the unprecedented wealth of results of algebraic topology and algebraic geometry of the second half of twentieth century [1].

Today, the concepts and methods of topology and geometry have become an indispensable part of theoretical physics. They have led to a deeper understanding of many crucial aspects in condensed matter physics, cosmology, gravity, and particle physics. Moreover, several intriguing connections between only apparently disconnected phenomena have been revealed based on these mathematical tools [2].

Topology enters General Relativity through the fundamental assumption that spacetime exists and is organized as a manifold. This means that spacetime has a

2010 *Mathematics Subject Classification.* 54E35; 54C15; 58H15; 32G10; 14B12.

Key words and phrases. Space-time Geodesics, Extra-dimensions, folding, deformation retract.

©2015 Universiteti i Prishtinës, Prishtinë, Kosovë.

Submitted February 2, 2015. Published August 18, 2015.

well-defined dimension, but it also carries with it the inherent possibility of modified patterns of global connectivity, such as distinguish a sphere from a plane, or a torus from a surface of higher genus. Such modifications can be present in the spatial topology without affecting the time direction, but they can also have a genuinely spacetime character in which case the spatial topology changes with time [4]. The topology change in classical general relativity has been discussed in [7]. See [9] for some applications of differential topology in general relativity.

Most of the studies on deformation retract and folding, if not all, are pure mathematical studies. The authors believe that these two concepts should be given more attention in modern mathematical physics. A topological studies of some famous metrics of mathematical physics could be a nice topological exploration start.

1.1. Deformation Retract - Definitions. The theory of deformation retract is very interesting topic in Euclidean and non-Euclidean spaces. It has been investigated from different points of view in many branches of topology and differential geometry. A retraction is a continuous mapping from the entire space into a subspace which preserves the position of all points in that subspace [8].

(i) Let M and N be two smooth manifolds of dimensions m and n respectively. A map $f : M \rightarrow N$ is said to be an isometric folding of M into N if and only if for every piecewise geodesic path $\gamma : J \rightarrow M$, the induced path $f \circ \gamma : J \rightarrow N$ is a piecewise geodesic and of the same length as γ [3]. If f does not preserve the lengths, it is called topological folding. Many types of foldings are discussed in [11, 12, 13, 14]. Some applications are discussed in [6, 10].

(ii) A subset A of a topological space X is called a retract of X , if there exists a continuous map $r : X \rightarrow A$ such that [16]

- (a) X is open
- (b) $r(a) = a, \forall a \in A$.

(iii) A subset A of a topological space X is said to be a deformation retract if there exists a retraction $r : X \rightarrow A$, and a homotopy $f : X \times I \rightarrow X$ such that [16]

$$f(x, 0) = x, \forall x \in X,$$

$$f(x, 1) = r(x), \forall x \in X,$$

$$f(a, t) = a, \forall a \in A, t \in [0, 1].$$

The deformation retract is a particular case of homotopy equivalence, two spaces are homotopy equivalent if and only if they are both deformation retracts of a single larger space.

Deformation retracts of Stein spaces has been studied in [5]. The deformation retract of the 4D Schwarzschild metric has been discussed in [22] where it was found that the retraction of the Schwarzschild space is spacetime geodesic. The retraction of Eguchi-Hanson space has been discussed in [15] in this paper we are going to discuss the retraction for the five dimensional case.

1.2. Schwarzschild metric in five dimensions. For the Schwarzschild metric in $(n + 1)$ dimensions we can write [17]

$$ds^2 = - \left(1 - \frac{\mu}{r^{n-2}}\right) dt^2 + \left(1 - \frac{\mu}{r^{n-2}}\right)^{-1} dr^2 + r^2 d\Omega_{n-1}^2 \quad (1.1)$$

$d\Omega_{n-1}^2$ is defined as

$$d\Omega_{n-1}^2 = d\chi_2^2 + \sin^2 \chi_2 d\chi_3^2 + \dots + \prod_{m=2}^{n-1} \sin^2 \chi_m d\chi_n^2 \quad (1.2)$$

In five dimensions $n = 4$ and we have:

$$ds^2 = - \left(1 - \frac{\mu}{r^2}\right) 4\pi^2 \mu d\tau^2 + \left(1 - \frac{\mu}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 \quad (1.3)$$

Where the coordinate t has been identified with a period $2\pi\sqrt{\mu}\tau$ to remove the singularity at $r^2 = \mu$. If we set $\mu = 2m$ then $dt^2 = 8\pi^2 m d\tau^2$ (see [17]). The 5D flat metric can be written as

$$ds^2 = -dx_o^2 + \sum_{i=1}^4 dx_i^2 \quad (1.4)$$

Comparing (1.3) and (1.4) making use of the basic metric definition $ds^2 = g_{ij} dx^i dx^j$. The following relations could be obtained

$$\begin{aligned} x_o = \pm \sqrt{\left(1 - \frac{\mu}{r^2}\right) 4\pi^2 \mu \tau^2 + C_o}, \quad x_1 = \pm \sqrt{r^2 + 2r\sqrt{\mu} \tanh^{-1} \left(\frac{r}{\sqrt{\mu}}\right) - \mu \ln \left(1 - \frac{r^2}{\mu}\right) + C_1}, \\ x_2 = \pm \sqrt{r^2 \theta^2 + C_2}, \quad x_3 = \pm \sqrt{r^2 \phi^2 \sin^2 \theta + C_3}, \quad x_4 = \pm \sqrt{r^2 l^2 \sin^2 \theta \sin^2 \phi + C_4} \end{aligned} \quad (1.5)$$

where C_o, C_1, C_2, C_3 and C_4 are constants of integration. These relations will play a key role in the study of the deformation retract of the 5D Schwarzschild spacetime.

2. EULER-LAGRANGE EQUATIONS OF 5D SCHWARZCHILD FIELD

In general relativity, the geodesic equation is equivalent to the Euler-Lagrange equations

$$\frac{d}{d\lambda} \left(\frac{\partial L}{\partial \dot{x}^\alpha} \right) - \frac{\partial L}{\partial x^\alpha} = 0, \quad i = 1, 2, 3, 4 \quad (2.1)$$

associated to the Lagrangian

$$L(x^\mu, \dot{x}^\mu) = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \quad (2.2)$$

The Lagrangian of the 5D Schwarzschild field can be written as

$$L = - \left(1 - \frac{\mu}{r^2}\right) 4\pi^2 \mu \dot{\tau}^2 + \left(1 - \frac{\mu}{r^2}\right)^{-1} \dot{r}^2 + r^2 \left(\dot{\theta}^2 + (\sin^2 \theta) \dot{\phi}^2 + (\sin^2 \theta \sin^2 \phi) \dot{l}^2 \right) \quad (2.3)$$

No explicit dependence on either t or l , and thus $\frac{\partial L}{\partial \dot{\tau}}$ and $\frac{\partial L}{\partial \dot{l}}$ are constants of motion, i.e.

$$\left(1 - \frac{\mu}{r^2}\right) 2\pi\sqrt{\mu} \dot{\tau} = k, \quad r^2 (\sin^2 \theta \sin^2 \phi) \dot{l} = h \quad (2.4)$$

with k and h are constants. h can be regarded as an equivalent of angular momentum per unit mass in the fifth dimension. using the Euler-Lagrange equations we get the full set of components as

$$\frac{d}{d\lambda} \left(\frac{\dot{r}}{1 - \frac{\mu}{r^2}} \right) - \left[-\frac{4\mu\pi^2}{r^3} \dot{r}^2 - \frac{\mu\dot{r}^2}{r^3 \left(1 - \frac{\mu}{r^2}\right)^2} + r \left(\dot{\theta}^2 + (\sin^2 \theta) \dot{\phi}^2 + (\sin^2 \theta \sin^2 \phi) \dot{l}^2 \right) \right] = 0 \quad (2.5)$$

$$\frac{d}{d\lambda} (2r^2 \dot{\theta}) - r^2 \sin 2\theta (\dot{\phi}^2 + \dot{l}^2 \sin^2 \phi) = 0 \quad (2.6)$$

$$\frac{d}{d\lambda} (2r^2 \dot{\phi} \sin \theta) - r^2 \dot{l}^2 \sin \theta \sin 2\phi = 0 \quad (2.7)$$

$$\frac{d}{d\lambda} \left(1 - \frac{\mu}{r^2} \right) 2\pi\sqrt{\mu}\dot{\tau} = 0 \quad (2.8)$$

$$\frac{d}{d\lambda} r^2 \dot{l} \sin^2 \theta \sin^2 \phi = 0 \quad (2.9)$$

From $\left(1 - \frac{\mu}{r^2}\right) 2\pi\sqrt{\mu}\dot{\tau} = k$, setting $k = 0$ gives two cases: (1) $\dot{\tau} = 0$ or $\tau = A$. If $A = 0$ we get the following coordinates

$$\begin{aligned} x_o &= \pm\sqrt{C_o}, \quad x_1 = \pm\sqrt{r^2 + 2r\sqrt{\mu} \tanh^{-1} \left(\frac{r}{\sqrt{\mu}} \right) - \mu \ln \left(1 - \frac{r^2}{\mu} \right) + C_1}, \\ x_2 &= \pm\sqrt{r^2\theta^2 + C_2}, \quad x_3 = \pm\sqrt{r^2\phi^2 \sin^2 \theta + C_3}, \quad x_4 = \pm\sqrt{r^2\dot{l}^2 \sin^2 \theta \sin^2 \phi + C_4} \\ x_5 &= \pm\sqrt{r^2\eta^2 \sin^2 \theta \sin^2 \phi \sin^2 l + C_5} \end{aligned} \quad (2.10)$$

Since $x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_o^2 > 0$ which is the great circle S_1 in the Schwarzschild spacetime S . These geodesic is a retraction in Schwarzschild space. $ds^2 > 0$. $\mu = 0$ is not allowed as it leads to undefined quantities.

From $r^2(\sin^2 \theta \sin^2 \phi) \dot{l} = h$, if $h = 0$ we have three cases: (1) $\dot{l} = 0$ or $l = B$. If $B = 0$ we get the coordinates as:

$$\begin{aligned} x_o &= \pm\sqrt{\left(1 - \frac{\mu}{r^2}\right) 4\pi^2\mu\tau^2 + C_o}, \quad x_1 = \pm\sqrt{r^2 + 2r\sqrt{\mu} \tanh^{-1} \left(\frac{r}{\sqrt{\mu}} \right) - \mu \ln \left(1 - \frac{r^2}{\mu} \right) + C_1} \\ x_2 &= \pm\sqrt{r^2\theta^2 + C_2}, \quad x_3 = \pm\sqrt{r^2\phi^2 \sin^2 \theta + C_3}, \quad x_4 = \pm\sqrt{C_4} \end{aligned} \quad (2.11)$$

This is the geodesic hyperspacetime S_3 of the Schwarzschild space S . This is a retraction. $ds^2 > 0$.

(2) $\phi = 0$ and in this case we get

$$\begin{aligned} x_o &= \pm\sqrt{\left(1 - \frac{\mu}{r^2}\right) 4\pi^2\mu\tau^2 + C_o}, \quad x_1 = \pm\sqrt{r^2 + 2r\sqrt{\mu} \tanh^{-1} \left(\frac{r}{\sqrt{\mu}} \right) - \mu \ln \left(1 - \frac{r^2}{\mu} \right) + C_1}, \\ x_2 &= \pm\sqrt{r^2\theta^2 + C_2}, \quad x_3 = \pm\sqrt{C_3}, \quad x_4 = \pm\sqrt{C_4} \end{aligned} \quad (2.12)$$

This is the geodesic hyperspacetime S_4 of the Schwarzschild space S . This is a retraction. $ds^2 > 0$.

(3) $\theta = 0$ and in this case we get

$$\begin{aligned} x_o &= \pm\sqrt{\left(1 - \frac{\mu}{r^2}\right) 4\pi^2\mu\tau^2 + C_o}, \quad x_1 = \pm\sqrt{r^2 + 2r\sqrt{\mu} \tanh^{-1} \left(\frac{r}{\sqrt{\mu}} \right) - \mu \ln \left(1 - \frac{r^2}{\mu} \right) + C_1} \\ x_2 &= \pm\sqrt{C_2}, \quad x_3 = \pm\sqrt{C_3}, \quad x_4 = \pm\sqrt{C_4} \end{aligned} \quad (2.13)$$

This is the geodesic hyperspacetime S_5 of the Schwarzschild space S . This is a retraction. $ds^2 > 0$.

Theorem1:

The retraction of 5D Schwarzschild space is a 5D spacetime geodesic.

3. DEFORMATION RETRACT OF SCHWARZCHILD SPACE

The deformation retract of the 5D Schwarzschild space is defined as

$$\rho : S \times I \rightarrow S \quad (3.1)$$

where S is the 5-dimensional Schwarzschild space and I is the closed interval $[0, 1]$. The retraction of 5D Schwarzschild space S is defined as

$$R : S \rightarrow S_1, S_2, S_3, S_4 \text{ and } S_5. \quad (3.2)$$

The deformation retract of the 5D Schwarzschild space S into a geodesic $S_1 \subset S$ is defined as

$$\begin{aligned} \rho(m, t) = (1 - t) & \left\{ \pm \sqrt{\left(1 - \frac{\mu}{r^2}\right) 4\pi^2 \mu \tau^2 + C_o}, \quad (3.3) \right. \\ & \pm \sqrt{r^2 + 2r\sqrt{\mu} \tanh^{-1}\left(\frac{r}{\sqrt{\mu}}\right) - \mu \ln\left(1 - \frac{r^2}{\mu}\right) + C_1}, \pm \sqrt{r^2 \theta^2 + C_2}, \\ & \left. \pm \sqrt{r^2 \phi^2 \sin^2 \theta + C_3}, \pm \sqrt{r^2 l^2 \sin^2 \theta \sin^2 \phi + C_4} \right\} + t \left\{ \pm \sqrt{C_o} \right. \\ & , \pm \sqrt{r^2 + 2r\sqrt{\mu} \tanh^{-1}\left(\frac{r}{\sqrt{\mu}}\right) - \mu \ln\left(1 - \frac{r^2}{\mu}\right) + C_1}, \pm \sqrt{r^2 \theta^2 + C_2} \\ & \left. , \pm \sqrt{r^2 \phi^2 \sin^2 \theta + C_3}, \pm \sqrt{r^2 l^2 \sin^2 \theta \sin^2 \phi + C_4} \right\} \end{aligned}$$

where

$$\begin{aligned} \rho(m, 0) = & \left\{ \pm \sqrt{\left(1 - \frac{\mu}{r^2}\right) 4\pi^2 \mu \tau^2 + C_o}, \quad (3.4) \right. \\ & \pm \sqrt{r^2 + 2r\sqrt{\mu} \tanh^{-1}\left(\frac{r}{\sqrt{\mu}}\right) - \mu \ln\left(1 - \frac{r^2}{\mu}\right) + C_1}, \pm \sqrt{r^2 \theta^2 + C_2}, \\ & \left. \pm \sqrt{r^2 \phi^2 \sin^2 \theta + C_3}, \pm \sqrt{r^2 l^2 \sin^2 \theta \sin^2 \phi + C_4} \right\} \end{aligned}$$

$$\begin{aligned} \rho(m, 1) = & \left\{ \pm \sqrt{C_o}, \pm \sqrt{r^2 + 2r\sqrt{\mu} \tanh^{-1}\left(\frac{r}{\sqrt{\mu}}\right) - \mu \ln\left(1 - \frac{r^2}{\mu}\right) + C_1} \quad (3.5) \right. \\ & \left. , \pm \sqrt{r^2 \theta^2 + C_2}, \pm \sqrt{r^2 \phi^2 \sin^2 \theta + C_3}, \pm \sqrt{r^2 l^2 \sin^2 \theta \sin^2 \phi + C_4} \right\} \end{aligned}$$

The deformation retract of the 5D Schwarzschild space S into a geodesic $S_2 \subset S$ is defined as

$$\begin{aligned} \rho(m, t) = (1-t) & \left\{ \pm \sqrt{\left(1 - \frac{\mu}{r^2}\right) 4\pi^2 \mu \tau^2 + C_o}, \right. \\ & \pm \sqrt{r^2 + 2r\sqrt{\mu} \tanh^{-1}\left(\frac{r}{\sqrt{\mu}}\right) - \mu \ln\left(1 - \frac{r^2}{\mu}\right) + C_1}, \pm \sqrt{r^2 \theta^2 + C_2}, \\ & \left. \pm \sqrt{r^2 \phi^2 \sin^2 \theta + C_3}, \pm \sqrt{r^2 l^2 \sin^2 \theta \sin^2 \phi + C_4} \right\} + t \left\{ \pm \sqrt{\left(1 - \frac{\mu}{r^2}\right) 4\pi^2 \mu \tau^2 + C_o} \right. \\ & \left. , \pm \sqrt{r^2 + 2r\sqrt{\mu} \tanh^{-1}\left(\frac{r}{\sqrt{\mu}}\right) - \mu \ln\left(1 - \frac{r^2}{\mu}\right) + C_1}, \pm \sqrt{r^2 \theta^2 + C_2} \right. \\ & \left. , \pm \sqrt{r^2 \phi^2 \sin^2 \theta + C_3}, \pm \sqrt{C_4} \right\} \end{aligned} \quad (3.6)$$

The deformation retract of the 5D Schwarzschild space S into a geodesic $S_3 \subset S$ is defined as

$$\begin{aligned} \rho(m, t) = (1-t) & \left\{ \pm \sqrt{\left(1 - \frac{\mu}{r^2}\right) 4\pi^2 \mu \tau^2 + C_o}, \right. \\ & \pm \sqrt{r^2 + 2r\sqrt{\mu} \tanh^{-1}\left(\frac{r}{\sqrt{\mu}}\right) - \mu \ln\left(1 - \frac{r^2}{\mu}\right) + C_1}, \pm \sqrt{r^2 \theta^2 + C_2}, \\ & \left. \pm \sqrt{r^2 \phi^2 \sin^2 \theta + C_3}, \pm \sqrt{r^2 l^2 \sin^2 \theta \sin^2 \phi + C_4} \right\} + t \left\{ \pm \sqrt{\left(1 - \frac{\mu}{r^2}\right) 4\pi^2 \mu \tau^2 + C_o}, \right. \\ & \left. , \pm \sqrt{r^2 + 2r\sqrt{\mu} \tanh^{-1}\left(\frac{r}{\sqrt{\mu}}\right) - \mu \ln\left(1 - \frac{r^2}{\mu}\right) + C_1}, \pm \sqrt{r^2 \theta^2 + C_2} \right. \\ & \left. , \pm \sqrt{C_3}, \pm \sqrt{C_4} \right\} \end{aligned} \quad (3.7)$$

The deformation retract of the 5D Schwarzschild space S into a geodesic $S_4 \subset S$ is defined as

$$\begin{aligned} \rho(m, t) = (1-t) & \left\{ \pm \sqrt{\left(1 - \frac{\mu}{r^2}\right) 4\pi^2 \mu \tau^2 + C_o}, \right. \\ & \pm \sqrt{r^2 + 2r\sqrt{\mu} \tanh^{-1}\left(\frac{r}{\sqrt{\mu}}\right) - \mu \ln\left(1 - \frac{r^2}{\mu}\right) + C_1}, \pm \sqrt{r^2 \theta^2 + C_2}, \\ & \left. \pm \sqrt{r^2 \phi^2 \sin^2 \theta + C_3}, \pm \sqrt{r^2 l^2 \sin^2 \theta \sin^2 \phi + C_4} \right\} + t \left\{ \pm \sqrt{\left(1 - \frac{\mu}{r^2}\right) 4\pi^2 \mu \tau^2 + C_o} \right. \\ & \left. , \pm \sqrt{r^2 + 2r\sqrt{\mu} \tanh^{-1}\left(\frac{r}{\sqrt{\mu}}\right) - \mu \ln\left(1 - \frac{r^2}{\mu}\right) + C_1}, \pm \sqrt{C_2}, \pm \sqrt{C_3}, \pm \sqrt{C_4} \right\} \end{aligned} \quad (3.8)$$

Now we are going to discuss the folding f of the 5D Schwarzschild space S . Let $f : S \rightarrow S$ where

$$f(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, x_3, x_4, |x_5|) \quad (3.9)$$

An isometric folding of the 5D Schwarzschild space into itself may be defined as:

$$f : \left\{ \pm \sqrt{\left(1 - \frac{\mu}{r^2}\right) 4\pi^2 \mu \tau^2 + C_o}, (3.10) \right. \\ \left. \pm \sqrt{r^2 + 2r\sqrt{\mu} \tanh^{-1}\left(\frac{r}{\sqrt{\mu}}\right) - \mu \ln\left(1 - \frac{r^2}{\mu}\right) + C_1}, \pm \sqrt{r^2 \theta^2 + C_2}, \right. \\ \left. \pm \sqrt{r^2 \phi^2 \sin^2 \theta + C_3}, \pm \sqrt{r^2 l^2 \sin^2 \theta \sin^2 \phi + C_4} \right\} \rightarrow \left\{ \pm \sqrt{\left(1 - \frac{\mu}{r^2}\right) 4\pi^2 \mu \tau^2 + C_o}, \right. \\ \left. \pm \sqrt{r^2 + 2r\sqrt{\mu} \tanh^{-1}\left(\frac{r}{\sqrt{\mu}}\right) - \mu \ln\left(1 - \frac{r^2}{\mu}\right) + C_1}, \pm \sqrt{r^2 \theta^2 + C_2}, \right. \\ \left. \pm \sqrt{r^2 \phi^2 \sin^2 \theta + C_3}, \left| \pm \sqrt{r^2 l^2 \sin^2 \theta \sin^2 \phi + C_4} \right| \right\}$$

The deformation retract of the folded 5D Schwarzschild space S into the folded S_1 is

$$\rho f : \left\{ \pm \sqrt{\left(1 - \frac{\mu}{r^2}\right) 4\pi^2 \mu \tau^2 + C_o}, (3.11) \right. \\ \left. \pm \sqrt{r^2 + 2r\sqrt{\mu} \tanh^{-1}\left(\frac{r}{\sqrt{\mu}}\right) - \mu \ln\left(1 - \frac{r^2}{\mu}\right) + C_1}, \pm \sqrt{r^2 \theta^2 + C_2}, \right. \\ \left. \pm \sqrt{r^2 \phi^2 \sin^2 \theta + C_3}, \left| \pm \sqrt{r^2 l^2 \sin^2 \theta \sin^2 \phi + C_4} \right| \right\} \times I \rightarrow \left\{ \pm \sqrt{\left(1 - \frac{\mu}{r^2}\right) 4\pi^2 \mu \tau^2 + C_o}, \right. \\ \left. \pm \sqrt{r^2 + 2r\sqrt{\mu} \tanh^{-1}\left(\frac{r}{\sqrt{\mu}}\right) - \mu \ln\left(1 - \frac{r^2}{\mu}\right) + C_1}, \pm \sqrt{r^2 \theta^2 + C_2}, \right. \\ \left. \pm \sqrt{r^2 \phi^2 \sin^2 \theta + C_3}, \left| \pm \sqrt{r^2 l^2 \sin^2 \theta \sin^2 \phi + C_4} \right| \right\}$$

with

$$\rho f(m, t) = (1 - t) \{S\} + t \{S_1\} \quad (3.12) \\ = (1 - t) \left\{ \pm \sqrt{\left(1 - \frac{\mu}{r^2}\right) 4\pi^2 \mu \tau^2 + C_o}, \right. \\ \left. \pm \sqrt{r^2 + 2r\sqrt{\mu} \tanh^{-1}\left(\frac{r}{\sqrt{\mu}}\right) - \mu \ln\left(1 - \frac{r^2}{\mu}\right) + C_1}, \pm \sqrt{r^2 \theta^2 + C_2}, \right. \\ \left. \pm \sqrt{r^2 \phi^2 \sin^2 \theta + C_3}, \pm \sqrt{r^2 l^2 \sin^2 \theta \sin^2 \phi + C_4} \right\} + t \left\{ \pm \sqrt{C_o} \right. \\ \left. , \pm \sqrt{r^2 + 2r\sqrt{\mu} \tanh^{-1}\left(\frac{r}{\sqrt{\mu}}\right) - \mu \ln\left(1 - \frac{r^2}{\mu}\right) + C_1}, \pm \sqrt{r^2 \theta^2 + C_2} \right. \\ \left. , \pm \sqrt{r^2 \phi^2 \sin^2 \theta + C_3}, \pm \sqrt{r^2 l^2 \sin^2 \theta \sin^2 \phi + C_4} \right\}$$

The same for S_2, S_3, S_4 and S_5 . Then the following theorem has been proved

Theorem2:

Under the defined folding, the deformation retract of the folded 5D Schwarzschild space into the folded geodesics is the same as the deformation retract of the 5D Schwarzschild space into the geodesics.

Theorem3:

The end of the limits of the folding of n -dimensional Schwarzschild space S^n is a 0-dimensional geodesic, it is a minimal retraction.

Proof:

let $f_1 : S^n \rightarrow S^n$, $f_2 : f_1(S^n) \rightarrow f_1(S^n)$, $f_3 : f_2(f_1(S^n)) \rightarrow f_2(f_1(S^n)) \dots$

$f_n : f_{n-1}(f_{n-2}(f_{n-3}(\dots(f_1(S^n)))))) \rightarrow f_{n-1}(f_{n-2}(f_{n-3}(\dots(f_1(S^n))))))$.

$\lim_{n \rightarrow \infty} f_{n-1}(f_{n-2}(f_{n-3}(\dots(f_1(S^n)))))) = n - 1$ dimensional Schwarzschild space S^{n-1} .

Also, let

$h_1 : S^{n-1} \rightarrow S^{n-1}$, $h_2 : h_1(S^{n-1}) \rightarrow h_1(S^{n-1})$, $h_3 : h_2(h_1(S^{n-1})) \rightarrow h_2(h_1(S^{n-1})) \dots$

$h_m : h_{m-1}(h_{m-2}(h_{m-3}(\dots(h_1(S^{n-1})))))) \rightarrow h_{m-1}(h_{m-2}(h_{m-3}(\dots(h_1(S^{n-1}))))))$.

$\lim_{n \rightarrow \infty} h_{m-1}(h_{m-2}(h_{m-3}(\dots(h_1(S^{n-1})))))) = n - 2$ dimensional Schwarzschild space S^{n-2} .

Consequently,

$\lim_{S \rightarrow \infty} \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} -K_S(h_m(f_n(S^n))) = 0$ -dimensional Schwarzschild space.

4. ENERGY CONSERVATION RELATION FOR 5D SCHWARZCHILD FIELD

For the case of $(\theta = \phi = \pi/2)$, the equations to be solved are

$$\left(1 - \frac{\mu}{r^2}\right) \dot{t} = k \quad (4.1)$$

$$c^2 = -c^2 \left(1 - \frac{\mu}{r^2}\right) \dot{t}^2 + \left(1 - \frac{\mu}{r^2}\right)^{-1} \dot{r}^2 + r^2 \dot{j}^2 \quad (4.2)$$

$$r^2 \dot{j} = h \quad (4.3)$$

making use of (4.1) and (4.3) in (4.2) and after some manipulations we get

$$\frac{1}{2} \dot{r}^2 + \frac{h^2}{2r^2} \left(1 - \frac{\mu}{r^2}\right) + \frac{c^2 \mu}{2r^2} = c^2(1 - k^2) \quad (4.4)$$

For the potential in higher dimensions, we recall the familiar Newton law in $n+4$ dimensions [21]

$$V_{n+4} \simeq \frac{G_{n+4} M}{r_n^{n+1}} \quad (4.5)$$

So, for 5 dimensions the potential is inversly proportional to the square of r :

$$V_5 \simeq \frac{G_5 M}{r^2} \quad (4.6)$$

The term

$$\frac{\hbar^2}{2r^2} \left(1 - \frac{\mu}{r}\right) + \frac{c^2 \mu}{2r^2} \quad (4.7)$$

represents the 5D potential and the equation (4.4) is the Energy conservation formula in 5D. Equation (4.4) is the 5D analogy of the 4D one for the case of 4D Schwarzschild field

$$\frac{1}{2} \dot{r}^2 + \frac{\hbar^2}{2r} \left(1 - \frac{\mu}{r}\right) - \frac{c^2 \mu}{2r} = c^2 (k^2 - 1) \quad (4.8)$$

which expresses energy conservation in 4D with potential

$$V_4 = \frac{\hbar^2}{2r^2} \left(1 - \frac{\mu}{r}\right) - \frac{c^2 \mu}{2r} \quad (4.9)$$

5. CONCLUSION

The deformation retract of the five dimensional Schwarzschild space has been investigated by making use of Lagrangian equations. The retraction of this space into itself and into geodesics has been presented. The deformation retraction of the five dimensional Schwarzschild space is five dimensional space-time geodesics which found to be a great circle. The folding of this space has been discussed and it was found that this folding, and any folding homeomorphic to that folding, have the same deformation retract of the five dimensional Schwarzschild space into a geodesic. The energy conservation relation for the studied five dimensional Schwarzschild metric has been derived and compared to the ordinary four dimensional one.

REFERENCES

- [1] H. Eschrig, *Topology and Geometry for Physics (Lecture Notes in Physics, Vol. 822)*. Springer; 1st edition 2011.
- [2] E. Bick F.D. Steffen (Eds.), *Topology and Geometry in Physics, (Lecture Notes in Physics)*. Springer; 2005.
- [3] S. A. Robertson, Isometric folding of Riemannian manifolds, *Proc. Roy. Soc. Edinburgh*, 77, (1977) , 275-284.
- [4] Rafael D. Sorkin, *Consequences of spacetime topology, 1989 (SYRACUSE)*.
- [5] Helmut A. Hamm, Nicolae Mihalache, Deformation retracts of Stein spaces, *Math. Ann.* 308,(1997), 333-345.
- [6] P. DI-Francesco, Folding and coloring problem in Mathematics and Physics, *Bull. Amer. Math. Soc.* 37, (3), (2002), 251-307.
- [7] Arvind Borde, *Topology Change in Classical General Relativity*, arXiv:gr-qc/9406053.
- [8] K. Borsuk (1931). "Sur les rtractes". *Fund. Math.* 17: 220.
- [9] Andrew Chamblin, *Some Applications of Differential Topology in General Relativity*, *J.Geom.Phys.* 13 (1994) 357-377.
- [10] J.Nesetril and P.O. de Mendez, Folding, *Journal of Combinatorial Theory, Series B*, 96, (2006), 730-739.
- [11] A. E. El-Ahmady, and H. Rafat, A calculation of geodesics in chaotic flat space and its folding, *Chaos, Solitons and fractals*, 30 (2006), 836-844.
- [12] M. El-Ghoul, A. E. El-Ahmady and H. Rafat, Folding-Retraction of chaotic dynamical manifold and the VAK of vacuum fluctuation. *Chaos, Solutions and Fractals*, 20 , (2004), 209-217.

- [13] M. El-Ghoul, The deformation retract of the complex projective space and its topological folding, J.Mat. Sc., Vol.30 (1995) 4145-4148. of Pure and Applied Mathematics, Vol. 27, No. 3, (2006), 517-522.
- [14] H. Rafat, On Tiling for some types of manifolds and their foldings, App. Math. Sc. Vol 3, 26 (2009) 1275-1284.
- [15] 1- H.Rafat and Nasr Ahmed, On the deformation retract of Eguchi-Hanson space and its folding, Advances in Mathematical Physics, Vol 2015 (2015), 301928, 9 pages.
- [16] W. S. Massey, Algebraic topology, An introduction, Harcourt Brace and World, New York (1967).
- [17] M.M. Akbar and G W Gibbons, Ricci flat metrics with U(1) action and the Dirichlet boundary value problem in Riemannian quantum gravity and isoperimetric inequalities, Class.Quant.Grav. 20 (2003) 1787-1822 hep-th/0301026.
- [18] J. W. York, Black Hole Thermodynamics and the Euclidean Einstein Action, Phys. Rev. D33(1986) 2092.
- [19] S. W. Hawking, Gravitational Instantons, Phys. Lett. A 60(1977) 81.
- [20] D. N. Page, Taub-Nut Instanton with an Horizon, Phys.Lett. B 78(1978) 249.
- [21] A. Kehagias, K. Sfetsos, Deviations from the $\frac{1}{r^2}$ Newton law due to extra dimensions, Phys.Lett. B472 (2000) 39-44.
- [22] H. Rafat, A Spacetime Geodesics of the Schwarzschild Space and Its Deformation Retract, Int.J.Math.Combin. Vol.3 (2009), 23-31.

NASR AHMED

MATHEMATICS DEPARTMENT, FACULTY OF SCIENCE, TAIBAH UNIVERSITY, SAUDI ARABIA., ASTRONOMY DEPARTMENT, NATIONAL RESEARCH INSTITUTE OF ASTRONOMY AND GEOPHYSICS, HELWAN, CAIRO, EGYPT

E-mail address: abualansar@gmail.com

H. RAFAT

MATHEMATICS DEPARTMENT, FACULTY OF SCIENCE, TAIBAH UNIVERSITY, SAUDI ARABIA., DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, TANTA UNIVERSITY, TANTA, EGYPT

E-mail address: hishamrafat2000@yahoo.com