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# ON ABSOLUTE FACTORABLE MATRIX SUMMABILITY METHODS

## (COMMUNICATED BY HUSEYIN BOR)

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Abstract. In this paper we give necessary and sufficient conditions for $|C,0|_k \Rightarrow |A_f|_s$ and $|A_f|_k \Rightarrow |C,0|_s$ for the case $1 < k \leq s < \infty$, where $|A_f|_k$ is absolute factorable summability. So we obtain some known results.

## 1. Introduction

Let $\sum x_v$ be a given infinite series with partial sums $(s_n)$. By $\sigma_n^\alpha$ we denote $n$.th Cesàro mean of order $\alpha$, $\alpha > -1$ , of the sequence $(s_n)$. The series $\sum x_v$ is said to be absolutely summable $(C,\alpha)$ with index $k$, or simply summable $|C,\alpha|_k$ , $k \geq 1$, if (see [6])

$$\sum_{n=1}^{\infty} n^{k-1} \left|\sigma_n^\alpha - \sigma_{n-1}^\alpha\right|^k < \infty. \tag{1.1}$$

Since $\sigma_n^0 = s_n$, the summability $|C,0|_k$ is equivalent to

$$\sum_{n=1}^{\infty} n^{k-1} |x_n|^k < \infty. \tag{1.2}$$

Let $(p_n)$ be a sequence of positive real constants with $P_n = p_0 + p_1 + ... + p_n \to \infty$ as $n \to \infty$. The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v$$

defines the sequence $(t_n)$ of the $(R,p_n)$ Riesz mean of the sequence $(s_n)$, generated by the sequence of coefficients $(p_n)$. The series $\Sigma x_v$ is then said to be summable $|R,p_n|_k$, $k \geq 1$, if (see [13])

$$\sum_{n=1}^{\infty} n^{k-1} |t_n - t_{n-1}|^k < \infty. \tag{1.3}$$

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Now, by $|R_p|_k$ let us denote the set of series summable by the summability method $|R,p_n|_k$. Then it is easily seen that

$$|R_p|_k = \left\{ a = (a_n) : \sum_{n=1}^{\infty} n^{k-1} \left| \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} x_v \right|^k < \infty \right\}, \ k \geq 1,$$

and so it means that the series $\Sigma x_v$ is summable $|R,p_n|_k$ if and only if the sequence $x = (x_v) \in |R_p|_k$.

Here, we extend the summability $|R,p_n|_k$ with factorable matrix as follows: The series $\Sigma x_v$ is said to be summable $|A_f|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1} \left| \widehat{a}_n \sum_{v=1}^{n} a_v x_v \right|^k < \infty. \tag{1.4}$$

A factorable matrix $A_f = (a_{nv})$ is one in which each entry

$$a_{nv} = \begin{cases} \widehat{a}_n a_v, & 0 \leq v \leq n \\ 0, & v > n \end{cases} \tag{1.5}$$

where $(\widehat{a}_n)$ and $(a_n)$ are any sequences of real numbers. Note that it is possible to get from it some known notations. For example, if one takes $\widehat{a}_n = p_n/P_n P_{n-1}$, $a_v = P_{v-1}$ and $\widehat{a}_n = 1/n(n+1)$, $a_v = v$, then $|A_f|_k$ are reduced to the summabilities $|R,p_n|_k$ and $|C,1|_k$, respectively.

If A and B are methods of summability, B is said to include A (written $A \Rightarrow B$) if every series summable by the method A is also summable by the method B. A and B said to be equivalent (written $A \Leftrightarrow B$) if each methods includes the other.

Problems on inclusion dealing absolute Cesàro and absolute weighted mean summabilities have been examined by many authors ([2-14]). On this topic, Bor [2] proved sufficient conditions for equivalence of the summabilities $|R,p_n|_k$ and $|C,0|_k$ as follows.

**Theorem 1.1.** Let $k > 1$ and

$$\sum_{n=v}^{\infty} \frac{n^{k-1} p_n^k}{P_n^k P_{n-1}} = O\left( \frac{v^{k-1} p_{v-1}^k}{P_{v-1}^k} \right). \tag{1.6}$$

If

$$P_{n+1} \geq d P_n. \tag{1.7}$$

where d is a constant such that $d > 1$, then $|R,p_n|_k \Leftrightarrow |C,0|_k$.

It has been more recently shown by Sarıgöl [10] that the condition (1.6) is omitted, and the condition (1.7) is not only sufficient but also necessary for Theorem 1.1 to hold, and also been completed in the following way.

**Theorem 1.2.** Let $1 < k \leq s < \infty$.Then, $|R,p_n|_k \Rightarrow |C,0|_s$ if and only if

$$\left( \sum_{v=m-1}^{m} \frac{1}{v} \left( \frac{P_v P_{v-1}}{p_v} \right)^{k^*} \right)^{1/k^*} \left( \sum_{n=m}^{m+1} \frac{n^{s-1}}{P_{n-1}^s} \right)^{1/s} = O(1), \tag{1.8}$$

where k* denotes the conjugate index of $k$, i.e., $\frac{1}{k} + \frac{1}{k^*} = 1$

**Theorem 1.3.** Let $1 < k \leq s < \infty$.Then, $|C,0|_k \Rightarrow |R,p_n|_s$ if and only if

$$\left(\sum_{m=1}^{v} \frac{P_{m-1}^{k^*}}{m}\right)^{1/k^*} \left(\sum_{n=v}^{\infty} \left(\frac{n^{1-1/s} p_n}{P_n P_{n-1}}\right)^s\right)^{1/s} = O(1), \tag{1.9}$$

where k* denotes the conjugate index of $k$.

**Corollary 1.4.** Let $k \geq 1$. Then, $|C,0|_k \Leftrightarrow |R,p_n|_k$ if and only if condition (1.6) is satisfied.

## 2. Main Results

The aim of this paper is to generalize the above theorems for summability $|A_f|_k$. Now we prove the following theorems.

**Theorem 2.1.** Let $1 < k \leq s < \infty$ and A be a factorable matrix given by (1.5) such that $\widehat{a}_n.a_n \neq 0$ for all n. Then, $|A_f|_k \,|\!\Rightarrow |C,0|_s$ if and only if

$$\left(\sum_{v=m-1}^{m} \frac{1}{v\,|\widehat{a}_v|^{k^*}}\right)^{1/k^*} \left(\sum_{n=m}^{m+1} \frac{n^{s-1}}{|a_n|^s}\right)^{1/s} = O(1), \tag{2.1}$$

where k* denotes the conjugate index of k, i.e., $\frac{1}{k} + \frac{1}{k^*} = 1$

**Theorem 2.2.** Let $1 < k \leq s < \infty$ and A be a factorable matrix given by (1.5). Then, $|C,0|_k \Rightarrow |A_f|_s$ if and only if

$$\left(\sum_{v=1}^{m} \frac{1}{v} |a_v|^{k^*}\right)^{1/k^*} \left(\sum_{n=m}^{\infty} n^{s-1} |\widehat{a}_n|^s\right)^{1/s} = O(1), \tag{2.2}$$

where k* denotes the conjugate index of $k$.

Now Theorem 2.1 and Theorem 2.2 immediately give the following result.

**Corollary 2.3.** Let $1 < k < \infty$ and A be a factorable matrix given by (1.5) such that $\widehat{a}_n.a_n \neq 0$ for all n. Then, $|C,0|_k \Leftrightarrow |A_f|_s$ if and only if the conditions (2.1) and (2.2) with $k = s$ are satisfied.

Before proving theorems we recall a result of Bennett [1] that $T : \ell^k \to \ell^s$ if and only if

$$\left(\sum_{v=1}^{m} c_v^{k^*}\right)^{1/k^*} \left(\sum_{n=m}^{\infty} b_n^s\right)^{1/s} = O(1), \tag{2.3}$$

where $T = (t_{nv}) = b_n c_v$ is a factorable matrix with nonegative entrice $b_n c_v$.

**Proof of Theorem 2.1.** Let $x_n^* = n^{1/s^*} x_n$ and $A_n^*(x) = n^{1/k^*} A_n(x)$, where

$$A_n(x) = \widehat{a}_n \sum_{v=1}^{n} a_v x_v, \; n \geq 1. \tag{2.4}$$

Then, $\Sigma x_n$ is summable $|A_f|_k$ and $|C,0|_s$ iff $A^*(x) \in l_k$ and $x^* \in l_s$, respectively. On the other hand, it can be written from (2.4) that

$$x_n = \frac{1}{a_n}\left(\frac{A_n(x)}{\widehat{a}_n} - \frac{A_{n-1}(x)}{\widehat{a}_{n-1}}\right) \tag{2.5}$$

and so

$$x_n^* = \frac{n^{1/s^*}}{a_n}\left(\frac{n^{-1/k^*} A_n^*(x)}{\widehat{a}_n} - \frac{(n-1)^{-1/k^*} A_{n-1}^*(x)}{\widehat{a}_{n-1}}\right)$$

which gives us

$$x_n^* = \sum_{v=1}^{\infty} t_{nv} A_v^*(x),$$

where

$$t_{nv} = \begin{cases} \frac{n^{1/s^*}}{a_n}\left(-\frac{(n-1)^{-1/k^*}}{\widehat{a}_{n-1}}\right), & v = n-1 \\ \frac{n^{1/s^*}}{a_n}\left(\frac{n^{-1/k^*}}{\widehat{a}_n}\right), & v = n. \\ 0, & v \neq n-1, n \end{cases} \tag{2.6}$$

Then, $|A_f|_k \mid\Rightarrow |C,0|_s$ if and only if

$$\sum_{n=1}^{\infty} |A_n^*(x)|^k < \infty \Longrightarrow \sum_{n=1}^{\infty} |x_n^*|^s < \infty, \ i.e., \ T : \ell^k \to \ell^s,$$

where $T$ is the matrix whose entries are defined by (2.6). Therefore, applying (2.3) to the matrix $T$, we have that $|A_f|_k \mid\Rightarrow |C,0|_s$ iff the condition (2.1) holds, which completes the proof.

**Proof of Theorem 2.2.** Let $n \geq 1$ and $x_n^* = n^{1/k^*} x_n$ and $A_n^*(x) = n^{1/s^*} A_n(x)$, where $A_n(x)$ is given by (2.4). Then,

$$A_n^*(x) = n^{1/s^*} \widehat{a}_n \sum_{v=1}^{n} v^{-1/k^*} a_v x_v^* = \sum_{v=1}^{n} h_{nv} x_v^*$$

where

$$h_{nv} = \begin{cases} n^{1/s^*} \widehat{a}_n v^{-1/k^*} a_v, & 1 \leq v \leq n \\ 0, & v > n. \end{cases}$$

Since the reminder of the proof is similar to the above, so it can be omitted.

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