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GENERALIZED ABSOLUTE CESÀRO SUMMABILITY FACTORS

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Abstract. In this paper, we generalize a known result dealing with an application of almost increasing sequences by using a quasi-f-power increasing sequence. Some new results are obtained.

1. Introduction

A positive sequence (b_n) is said to be an almost increasing sequence if there exists a positive increasing sequence (c_n) and two positive constants M and N such that $Mc_n \leq b_n \leq Nc_n$ (see [1]). A positive sequence $X = (X_n)$ is said to be a quasi-f-power increasing sequence if there exists a constant $K = K(X, f) \geq 1$ such that $Kf_nX_n \geq f_mX_m$ for all $n \geq m \geq 1$, where $f = \{f_n(\gamma, \eta)\} = \{n^\eta (\log n)^\gamma$, $\gamma \geq 0$, $0 < \eta < 1\}$(see [11]). If we take γ=0, then we get a quasi-η-power increasing sequence. Every almost increasing sequence is a quasi-η-power increasing for any nonnegative η, but the converse is not true (see [10]). For any sequence (λ_n) we write that $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. The sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in BV$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| < \infty$. Let $\sum a_n$ be a given infinite series. We denote by $t_n^{\alpha,\beta}$ the nth Cesàro mean of order (α, β), with $\alpha + \beta > -1$, of the sequence (na_n), that is (see [6])

$$t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v, \tag{1.1}$$

where

$$A_n^{\alpha+\beta} = O(n^{\alpha+\beta}), \quad A_0^{\alpha+\beta} = 1, \ \texttt{and} \quad A_{-n}^{\alpha+\beta} = 0 \ \texttt{for} \ n > 0. \tag{1.2}$$

Let $(\theta_n^{\alpha,\beta})$ be a sequence defined by

$$\theta_n^{\alpha,\beta} = \begin{cases} \left|t_n^{\alpha,\beta}\right|, & \alpha = 1, \beta > -1, \\ \max_{1 \leq v \leq n} \left|t_v^{\alpha,\beta}\right|, & 0 < \alpha < 1, \beta > -1. \end{cases} \tag{1.3}$$

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The series $\sum a_n$ is said to be summable $\mid C, \alpha, \beta; \delta \mid_k$, $k \geq 1$ and $\delta \geq 0$, if (see [3])

$$\sum_{n=1}^{\infty} n^{\delta k-1} \mid t_n^{\alpha,\beta} \mid^k < \infty. \tag{1.4}$$

If we take $\delta = 0$, then $\mid C, \alpha, \beta; \delta \mid_k$ summability reduces to $\mid C, \alpha, \beta \mid_k$ summability (see [7]). Also, if we take $\beta = 0$ and $\delta = 0$, then $\mid C, \alpha, \beta; \delta \mid_k$ summability reduces to $\mid C, \alpha \mid_k$ summability (see [8]). Furthermore, if we take $\beta = 0$, then we get $\mid C, \alpha; \delta \mid_k$ summability (see [9]).

2. Known result

The following theorem is known dealing with $\mid C, \alpha, \beta; \delta \mid_k$ summability factors of infinite series.

Theorem A ([3]). Let $(\theta_n^{\alpha,\beta})$ be a sequence defined as in (1.3). Let (X_n) be an almost increasing sequence. Suppose also that there exist sequences (σ_n) and (λ_n) such that

$$\mid \Delta\lambda_n \mid \leq \sigma_n \tag{2.1}$$

$$\sigma_n \to 0 \quad as \quad n \to \infty \tag{2.2}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\sigma_n \mid X_n < \infty \tag{2.3}$$

$$\mid \lambda_n \mid X_n = O(1) \quad as \quad n \to \infty. \tag{2.4}$$

If the condition

$$\sum_{n=1}^{m} n^{\delta k-1} (\theta_n^{\alpha,\beta})^k = O(X_m) \quad as \quad m \to \infty \tag{2.5}$$

satisfies, then the series $\sum a_n\lambda_n$ is summable $\mid C, \alpha, \beta; \delta \mid_k$, $0 < \alpha \leq 1$, $\beta > -1$, $k \geq 1$, $\delta \geq 0$, and $(\alpha + \beta - \delta) > 0$.

3. Main result

The aim of this paper is to generalize Theorem A by using a quasi-f-power increasing sequence instead of an almost increasing sequence. Now we shall prove the following main theorem.

Theorem. Let $(\theta_n^{\alpha,\beta})$ be a sequence defined as in (1.3). Let $(\lambda_n) \in BV$ and let (X_n) be a quasi-f-power increasing sequence. Suppose also that there exist sequences (σ_n) and (λ_n) such that conditions (2.1)- (2.4) of Theorem A are satisfied. If the condition (2.5) is satisfied, then the series $\sum a_n\lambda_n$ is summable $\mid C, \alpha, \beta; \delta \mid_k$, $0 < \alpha \leq 1$, $\beta > -1$, $k \geq 1$, $\delta \geq 0$, and $(\alpha + \beta - \delta) > 0$.

Remark. It should be noted that if we take $\beta = 0$ and $\delta = 0$, then we obtain a known result dealing with the $\mid C, \alpha \mid_k$ summability (see [5]). If we take $\gamma = 0$ and (X_n) as an almost increasing sequence, then we get Theorem A. In this case the condition "$(\lambda_n) \in BV$" is not needed. Also, if we set $\delta = 0$, then we get a result concerning the $\mid C, \alpha, \beta \mid_k$ summability factors of infinite series. Furthermore, if we take $\beta = 0$, $\delta = 0$ and $\alpha = 1$, then we get a result for $\mid C, 1 \mid_k$ summability factors of infinite series. Finally, if we take (X_n) as a quasi-η-power increasing sequence, then we obtain a new result.

We need the following lemmas for the proof of our theorem.
Lemma 1 ([2]). If $0 < \alpha \leq 1$, $\beta > -1$, and $1 \leq v \leq n$, then

$$\mid \sum_{p=0}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} a_p \mid \leq \max_{1 \leq m \leq v} \mid \sum_{p=0}^{m} A_{m-p}^{\alpha-1} A_p^{\beta} a_p \mid . \tag{3.1}$$

Lemma 2 ([4]). Except for the condition $(\lambda_n) \in BV$, under the conditions on (X_n), (σ_n) and (λ_n) as expressed in the statement of the theorem, we have the following;

$$nX_n\sigma_n = O(1), \tag{3.2}$$

$$\sum_{n=1}^{\infty} \sigma_n X_n < \infty. \tag{3.3}$$

4. Proof of the theorem

Let $(T_n^{\alpha,\beta})$ be the nth (C, α, β) mean of the sequence $(na_n\lambda_n)$. Then, by (1.1), we have that

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v.$$

Applying Abel's transformation first and then using Lemma 1, we obtain that

$$T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v,$$

$$\begin{aligned} \mid T_n^{\alpha,\beta} \mid &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \mid \Delta\lambda_v \mid \mid \sum_{p=1}^{v} A_{n-p}^{\alpha-1} A_p^{\beta} p a_p \mid + \frac{\mid \lambda_n \mid}{A_n^{\alpha+\beta}} \mid \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \mid \\ &\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{(\alpha+\beta)} \theta_v^{\alpha,\beta} \mid \Delta\lambda_v \mid + \mid \lambda_n \mid \theta_n^{\alpha,\beta} = T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}. \end{aligned}$$

To prove the theorem, by Minkowski's inequality, it is sufficient to show that for $k \geq 1$

$$\sum_{n=1}^{\infty} n^{\delta k-1} \mid T_{n,r}^{\alpha,\beta} \mid^k < \infty, \quad \texttt{for} \quad r = 1, 2.$$

When $k>1$, we can apply Hölder's inequality with indices k and k', where $\frac{1}{k}+\frac{1}{k'}=1$, and so we get that

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{\delta k-1} \mid T_{n,1}^{\alpha,\beta} \mid^k &\leq \sum_{n=2}^{m+1} n^{\delta k-1} \mid \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{(\alpha+\beta)} \theta_v^{\alpha,\beta} \Delta\lambda_v \mid^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{1+(\alpha+\beta-\delta)k}} \{\sum_{v=1}^{n-1} v^{(\alpha+\beta)k} \left|\Delta\lambda_v\right| (\theta_v^{\alpha,\beta})^k\} \times \{\sum_{v=1}^{n-1} \left|\Delta\lambda_v\right|\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} \sigma_v (\theta_v^{\alpha,\beta})^k \sum_{n=v+1}^{m+1} \frac{1}{n^{1+(\alpha+\beta-\delta)k}} \\
&= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} \sigma_v (\theta_v^{\alpha,\beta})^k \int_v^\infty \frac{dx}{x^{1+(\alpha+\beta-\delta)k}} \\
&= O(1) \sum_{v=1}^{m} \sigma_v v^{\delta k} (\theta_v^{\alpha,\beta})^k = O(1) \sum_{v=1}^{m} v \sigma_v v^{\delta k-1} (\theta_v^{\alpha,\beta})^k \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\sigma_v) \sum_{p=1}^{v} p^{\delta k-1} (\theta_p^{\alpha,\beta})^k + O(1) m\sigma_m \sum_{v=1}^{m} v^{\delta k-1} (\theta_v^{\alpha,\beta})^k \\
&= O(1) \sum_{v=1}^{m-1} \left|\Delta\left(v\sigma_v\right)\right| X_v + O(1) m\sigma_m X_m \\
&= O(1) \sum_{v=1}^{m-1} \mid (v+1)\Delta\sigma_v - \sigma_v \mid X_v + O(1) m\sigma_m X_m \\
&= O(1) \sum_{v=1}^{m-1} v \left|\Delta\sigma_v\right| X_v + O(1) \sum_{v=1}^{m-1} \sigma_v X_v + O(1) m\sigma_m X_m \\
&= O(1) \quad as \quad m \to \infty,
\end{aligned}$$

in view of hypotheses of the theorem and Lemma 2. Similarly, we have that

$$\begin{aligned}
\sum_{n=1}^{m} n^{\delta k-1} \mid \ T_{n,2}^{\alpha,\beta} \mid^k &= O(1) \sum_{n=1}^{m} \left|\lambda_n\right| n^{\delta k-1} (\theta_n^{\alpha,\beta})^k = O(1) \sum_{n=1}^{m-1} \Delta\left(\left|\lambda_n\right|\right) \sum_{v=1}^{n} v^{\delta k-1} (\theta_v^{\alpha,\beta})^k \\
&+ O(1) \left|\lambda_m\right| \sum_{v=1}^{m} v^{\delta k-1} (\theta_v^{\alpha,\beta})^k = O(1) \sum_{n=1}^{m-1} \left|\Delta\lambda_n\right| X_n + O(1) \left|\lambda_m\right| X_m \\
&= O(1) \sum_{n=1}^{m-1} \sigma_n X_n + O(1) \left|\lambda_m\right| X_m = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by virtue of hypotheses of the theorem and Lemma 2. This completes the proof of the theorem.

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