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A NEW APPLICATION OF GENERALIZED ALMOST INCREASING SEQUENCES

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Abstract. In the present paper, a general theorem dealing with $|A, p_n; \delta|_k$ summability factors of infinite series has been proved by using almost increasing sequence. This theorem also includes some known and new results.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n), and let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ... \tag{1.1}$$

The series $\sum a_n$ is said to be summable $|A|_k$, $k \geq 1$, if (see [10])

$$\sum_{n=1}^{\infty} n^{k-1} \left|\bar{\Delta} A_n(s)\right|^k < \infty, \tag{1.2}$$

where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s).$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, i \geq 1). \tag{1.3}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{1.4}$$

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defines the sequence (σ_n) of the Riesz mean or simply the $(\bar{N}, p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [5]). The series $\sum a_n$ is said to be summable $\mid \bar{N}, p_n \mid_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} \mid \Delta\sigma_{n-1} \mid^k < \infty, \tag{1.5}$$

and it is said to be summable $|A, p_n; \delta|_k$, $k \geq 1$ and $\delta \geq 0$, if (see [7])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k+k-1} |\bar{\Delta} A_n(s)|^k < \infty. \tag{1.6}$$

If we take $a_{nv} = \frac{p_v}{P_n}$ and $\delta = 0$, then $|A, p_n; \delta|_k$ summability reduces to $|\bar{N}, p_n|_k$ summability. Also, if we take $\delta = 0$, then $|A, p_n; \delta|_k$ summability reduces to $|A, p_n|_k$ summability (see [9]). In the special case $\delta = 0$ and $p_n = 1$ for all n, $|A, p_n; \delta|_k$ summability is the same as $|A|_k$ summability. Furthermore, if we take $a_{nv} = \frac{p_v}{P_n}$, then $|A, p_n; \delta|_k$ summability is the same as $|\bar{N}, p_n; \delta|_k$ summability.
Before stating the main theorem we must first introduce some further notations.
Given a normal matrix $A = (a_{nv})$, we associate two lover semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{1.7}$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{1.8}$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively. Then, we have

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v = \sum_{v=0}^{n} \bar{a}_{nv} a_v \tag{1.9}$$

and

$$\bar{\Delta} A_n(s) = \sum_{v=0}^{n} \hat{a}_{nv} a_v. \tag{1.10}$$

2. Known Result

In [3], Bor has proved the following theorem for $\left|\bar{N}, p_n\right|_k$ summability factors of infinite series.

Theorem 2.1. *Let (X_n) be a positive non-decreasing sequence and let there be sequences (β_n) and (λ_n) such that*

$$\mid \Delta\lambda_n \mid \leq \beta_n, \tag{2.1}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{2.2}$$

$$\sum_{n=1}^{\infty} n \mid \Delta\beta_n \mid X_n < \infty, \tag{2.3}$$

$$\mid \lambda_n \mid X_n = O(1). \tag{2.4}$$

If

$$\sum_{v=1}^{n} \frac{\mid t_v \mid^k}{v} = O(X_n) \quad as \quad n \to \infty, \tag{2.5}$$

where (t_n) *is the* n*th* $(C,1)$ *mean of the sequence* (na_n)*, and* (p_n) *is a sequence such that*

$$P_n = O(np_n), \tag{2.6}$$

$$P_n \Delta p_n = O(p_n p_{n+1}), \tag{2.7}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ *is summable* $\mid \bar{N}, p_n \mid_k$, $k \geq 1$.

3. Main Result

The aim of this paper is to generalize Theorem 2.1 for $|A, p_n; \delta|_k$ summability by using almost increasing sequence. For this we need the concept of an almost increasing sequence. A positive sequence (b_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). Obviously every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking the example, say $b_n = ne^{(-1)^n}$.
Now, we shall prove the following theorem.

Theorem 3.1. *Let* (X_n) *be an almost increasing sequence. The conditions (2.1)-(2.4) and (2.6)-(2.7) of Theorem 2.1 and the conditions*

$$\sum_{n=1}^{m} \left(\frac{P_n}{p_n}\right)^{\delta k-1} |t_n|^k = O(X_m) \quad as \quad m \to \infty, \tag{3.1}$$

$$\sum_{n=v+1}^{m+1} \left(\frac{P_n}{p_n}\right)^{\delta k} |\Delta_v \hat{a}_{nv}| = O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k-1}\right\} \quad as \quad m \to \infty, \tag{3.2}$$

where (t_n) *as is in Theorem 2.1, are satisfied. If* $A = (a_{nv})$ *is a positive normal matrix such that*

$$\overline{a}_{n0} = 1, \; n = 0, 1, ..., \tag{3.3}$$

$$a_{n-1,v} \geq a_{nv}, \; for \; n \geq v+1, \tag{3.4}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{3.5}$$

$$\mid \widehat{a}_{n,v+1} \mid = O(v \mid \Delta_v(\widehat{a}_{nv}) \mid), \tag{3.6}$$

then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{np_n}$ *is summable* $|A, p_n; \delta|_k$, $k \geq 1$ *and* $0 \leq \delta < 1/k$.

We need the following lemmas for the proof of Theorem 3.1.

Lemma 3.2. *([6]) If (X_n) is an almost increasing sequence, then under the conditions (2.2)-(2.3) we have that*

$$nX_n\beta_n = O(1), \tag{3.7}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{3.8}$$

Lemma 3.3. *([3]) If conditions (2.6) and (2.7) are satisfied, then we have*

$$\Delta\left(\frac{P_n}{n^2 p_n}\right) = O\left(\frac{1}{n^2}\right). \tag{3.9}$$

Lemma 3.4. *([3]) If conditions (2.1)-(2.4) are satisfied, then we have that*

$$\lambda_n = O(1), \tag{3.10}$$

$$\Delta\lambda_n = O\left(\frac{1}{n}\right). \tag{3.11}$$

4. Proof of Theorem 3.1

Let (I_n) denotes A-transform of the series $\sum_{n=1}^{\infty} \frac{a_n P_n \lambda_n}{n p_n}$. Then, we have by (1.9) and (1.10)

$$\bar{\Delta} I_n = \sum_{v=1}^{n} \hat{a}_{nv} \frac{a_v P_v \lambda_v}{v p_v}.$$

Applying Abel's transformation to this sum, we get that

$$\begin{aligned}
\bar{\Delta} I_n &= \sum_{v=1}^{n} \hat{a}_{nv} \frac{v a_v P_v \lambda_v}{v^2 p_v} \\
&= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} P_v \lambda_v}{v^2 p_v}\right) \sum_{r=1}^{v} r a_r + \frac{\hat{a}_{nn} P_n \lambda_n}{n^2 p_n} \sum_{r=1}^{n} r a_r \\
&= \sum_{v=1}^{n-1} \Delta_v \left(\frac{\hat{a}_{nv} P_v \lambda_v}{v^2 p_v}\right) (v+1) t_v + \frac{a_{nn} P_n \lambda_n}{n^2 p_n} (n+1) t_n \\
&= \frac{a_{nn} P_n \lambda_n}{n^2 p_n} (n+1) t_n + \sum_{v=1}^{n-1} \Delta_v (\hat{a}_{nv}) \frac{(v+1)}{v^2} \frac{P_v \lambda_v}{p_v} t_v \\
&\quad + \sum_{v=1}^{n-1} \frac{\hat{a}_{n,v+1} P_v}{p_v} \Delta\lambda_v t_v \frac{(v+1)}{v^2} + \sum_{v=1}^{n-1} \hat{a}_{n,v+1} \lambda_{v+1} \Delta\left(\frac{P_v}{v^2 p_v}\right) t_v (v+1) \\
&= I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}.
\end{aligned}$$

Since

$$|I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}|^k \leq 4^k (|I_{n,1}|^k + |I_{n,2}|^k + |I_{n,3}|^k + |I_{n,4}|^k)$$

to complete the proof of Theorem 3.1, it is sufficient to show that

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{\delta k + k - 1} | I_{n,r} |^k < \infty, \quad for \quad r = 1, 2, 3, 4. \tag{4.1}$$

First, by using Abel's transformation, we have that

$$\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid I_{n,1}\mid^k &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} a_{nn}^k\left(\frac{P_n}{p_n}\right)^k |\lambda_n|^k\frac{|t_n|^k}{n^k}\\
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k-1}|\lambda_n|^{k-1}|\lambda_n||t_n|^k\\
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k-1}|\lambda_n||t_n|^k\\
&= O(1)\sum_{n=1}^{m-1}\Delta|\lambda_n|\sum_{r=1}^{n}\left(\frac{P_r}{p_r}\right)^{\delta k-1}|t_r|^k+O(1)|\lambda_m|\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k-1}|t_n|^k\\
&= O(1)\sum_{n=1}^{m-1}|\Delta\lambda_n|X_n+O(1)|\lambda_m|X_m\\
&= O(1)\sum_{n=1}^{m-1}\beta_nX_n+O(1)|\lambda_m|X_m\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by (2.1), (2.4), (2.6), (3.1), (3.5), (3.8), (3.10) and (3.11).
Now, using the fact that $P_v=O(vp_v)$ by (2.6), we have that

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid I_{n,2}\mid^k = O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\,|\lambda_v|\,|t_v|\right)^k.$$

Now, applying Hölder's inequality with indices k and k', where $k>1$ and $\frac{1}{k}+\frac{1}{k'}=1$, as in $I_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|I_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v||t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\right)\times\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\right)\\
&= O(1)\sum_{v=1}^{m}|\lambda_v|^{k-1}|\lambda_v||t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v(\hat{a}_{nv})|\\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k-1}|\lambda_v||t_v|^k\\
&= O(1) \quad as \quad m\to\infty,
\end{aligned}$$

by (2.1), (2.4), (3.1), (3.2), (3.3), (3.4), (3.5), (3.8) and (3.11).
Now, using Hölder's inequality, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|I_{n,3}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}v|\Delta_v(\hat{a}_{nv})|\beta_v|t_v|^k\right)\times\left(\sum_{v=1}^{n-1}v|\Delta_v(\hat{a}_{nv})|\beta_v\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\left(\sum_{v=1}^{n-1}v|\Delta_v(\hat{a}_{nv})|\beta_v|t_v|^k\right) \\
&= O(1)\sum_{v=1}^{m}v\beta_v|t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v(\hat{a}_{nv})| \\
&= O(1)\sum_{v=1}^{m}v\beta_v|t_v|^k\left(\frac{P_v}{p_v}\right)^{\delta k-1} \\
&= O(1)\sum_{v=1}^{m-1}\Delta\left(v\beta_v\right)\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\delta k-1}|t_r|^k+O(1)m\beta_m\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k-1}|t_v|^k \\
&= O(1)\sum_{v=1}^{m-1}\Delta\left(v\beta_v\right)X_v+O(1)m\beta_mX_m \\
&= O(1)\sum_{v=1}^{m-1}(v+1)|\Delta\beta_v|X_v+O(1)\sum_{v=1}^{m-1}\beta_vX_v+O(1)m\beta_mX_m \\
&= O(1)\quad as\quad m\to\infty,
\end{aligned}$$

by (2.1), (2.3), (2.6), (3.1), (3.2), (3.5), (3.6), (3.7) and (3.8).
Finally, since $\Delta\left(\frac{P_v}{v^2p_v}\right)=O\left(\frac{1}{v^2}\right)$ by Lemma 3.3, as in $I_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}|I_{n,4}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\lambda_{v+1}|\frac{|t_v|}{v}\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}||\lambda_{v+1}||t_v|\right)^k \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1}\left(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}||\lambda_{v+1}|^k|t_v|^k\right)\times\left(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}|\right)^{k-1} \\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}\left(\sum_{v=1}^{n-1}|\Delta_v\hat{a}_{nv}||\lambda_{v+1}|^k|t_v|^k\right) \\
&= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|^{k-1}|\lambda_{v+1}||t_v|^k\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\delta k}|\Delta_v(\hat{a}_{nv})| \\
&= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\delta k-1}|\lambda_{v+1}||t_v|^k=O(1)\quad as\quad m\to\infty,
\end{aligned}$$

by (2.1), (2.4), (2.6), (3.1), (3.2), (3.5), (3.6) and (3.11).
Therefore we get

$$\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\delta k+k-1} \mid I_{n,r} \mid^k = O(1) \quad as \quad m\to\infty, \quad for \quad r=1,2,3,4.$$

This completes the proof of Theorem 3.1.

If we take $a_{nv}=\frac{p_v}{P_n}$ and $\delta=0$, then we get a result of Bor [4] for $|\bar{N},p_n|_k$ summability factors. Also, if we take $\delta=0$, then we get a result of Özarslan [8] for $|A,p_n|_k$ summability factors. Furthermore, if we take (X_n) as a positive non-decreasing sequence, then we get a new result dealing with $|A,p_n;\delta|_k$ summability factors.

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