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COEFFICIENT BOUNDS FOR INVERSE OF CERTAIN UNIVALENT FUNCTIONS

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Abstract. This article provides a continuation of paper by Libera and Złotkiewicz [Proc. Amer. Math. Soc. 87(2) (1983), 251–257], in which they investigated upper bounds on initial coefficients of inverse of a function defined by integration of Carathéodory functions. We obtain upper bounds on Fekete-Szegö functional and third Hankel determinant of such functions.

1. Introduction and Preliminaries

Let $\mathcal{H}$ denote the family of analytic functions in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ denote the class of functions $f \in \mathcal{H}$, having the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots, \qquad z \in \mathbb{D}. \tag{1.1}$$

We denote by $\mathcal{S}$, the subclass of $\mathcal{A}$ consisting of functions which are also univalent in $\mathbb{D}$.

It is well-known that the function $f \in \mathcal{S}$ of the form (1.1) has an inverse f^{-1}, which is analytic in $|w| < r_0(f)$ $(r_0(f) \geq 1/4)$. If $f \in \mathcal{S}$ given by (1.1), then

$$f^{-1}(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \cdots, \qquad |w| < r_0(f). \tag{1.2}$$

Löwner [19] proved that, if $f \in \mathcal{S}$ and its inverse is given by (1.2), then the sharp estimate

$$|\gamma_n| \leq \frac{\Gamma(2n+1)}{\Gamma(n+1)\Gamma(n+2)} \tag{1.3}$$

holds. It has been shown that the inverse of the Koebe function $k(z) = z/(1-z)^2$ provides the best bounds for all $|\gamma_k|$ $(k = 2, 3, \cdots)$ in (1.3) over all members of $\mathcal{S}$.

Libera *et al.* [16] obtained a relationship between the coefficients of f and f^{-1} for all f of the form (1.1) when $f(\mathbb{D})$ is a convex region. On the other hand, Krzyż *et al.* [14] investigated bounds on initial coefficients of inverse of starlike functions of order α and their results were extended by Kapoor and Mishra [11]. Further, Ali [1] studied sharp bounds on early coefficients of inverse functions when function belongs to the class of strongly starlike functions.

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Libera *et al.* [17] investigated the bounds on early coefficients of inverse of functions, which are defined by

$$f(z) = \int_0^z p(\zeta)\, d\zeta, \quad z \in \mathbb{D}. \tag{1.4}$$

Here p is a member of the class $\mathcal{P}$ of Carathéodory functions (see [5, p. 40]) that consists of functions $p \in \mathcal{H}$ with $\Re(p(z)) > 0$, having the form

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots, \qquad z \in \mathbb{D}. \tag{1.5}$$

The class of all functions $f(z)$ satisfying (1.4) is denoted by $\mathcal{I}$.

The Hankel determinant of Taylor coefficients of functions $f \in \mathcal{A}$ of the form (1.1), is denoted by $H_{q,n}(f)$, which is defined by

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix} \quad (a_1 = 1;\, n, q \in \mathbb{N} = \{1, 2, \cdots\}).$$

Several researchers including Cantor [4], Noonan and Thomas [21], Pommerenke [22], Hayman [9], Ehrenborg [6] and many more have studied the Hankel determinant and given some remarkable results, which are useful, for example, in showing that a function of bounded characteristic in the unit disk $\mathbb{D}$.

Indeed, $H_{2,1}(f) = \Lambda_1(f)$ is the Fekete-Szegö functional, which have been studied for various subclasses of $\mathcal{S}$ [12, 13, 18]. Recently many authors have studied the problem of calculating $\max_{f \in \mathbb{F}} |H_{2,2}(f)|$ for various subclasses of $\mathcal{A}$ [2, 10, 15]. The third Hankel determinant $H_{3,1}(f)$ is given by

$$\begin{aligned} H_{3,1}(f) \quad &= \quad \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} \\ &= \quad a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2). \end{aligned}$$

Recently, authors have obtained bounds on $|H_{3,1}(f)|$ for certain classes of analytic functions [3, 20]. Also, Raza and Malik [23] have obtained the bounds on $|H_{3,1}(f)|$ for a subclasses of analytic functions associated with right half of the lemniscate of Bernoulli $(x^2 + y^2)^2 - 2(x^2 - y^2) = 0$.

In this paper, we examine the upper bounds on $|H_{2,1}(f)|$ and $|H_{3,1}(f)|$ for the coefficients of inverse functions f^{-1} of the form (1.2) when f belongs to the class $\mathcal{I}$. To obtain our main results, we shall need the following results:

Lemma 1.1. ([5, 12, 17]) *Let the function* $p \in \mathcal{P}$ *be given by the power series* (1.5). *Then*

(a) $|c_n| \leq 2,\ n \in \mathbb{N} = \{1, 2, \ldots\}$. *This inequality is sharp and equality holds for every function* $p_\epsilon(z) = \dfrac{1 + \epsilon z}{1 - \epsilon z}$ $(z \in \mathbb{D},\ |\epsilon| = 1)$.

(b) $\max |c_2 - \lambda c_1^2| = 2\ \max\{1, |2\lambda - 1|\}$, *for any complex number* λ.

Lemma 1.2. ([8]) *The power series* (1.5) *converges in* $\mathbb{D}$ *to a function in* $\mathcal{P}$, *if and only if the Toeplitz determinants*

$$T_n(p) = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \quad n \in \mathbb{N}$$

and $c_{-n} = \overline{c}_n$, *are all nonnegative. The only exception is when* $p(z)$ *has the form*

$$p(z) = \sum_{\nu=1}^{m} \rho_\nu \frac{1+\epsilon_\nu z}{1-\epsilon_\nu z}, \quad m \geq 1,$$

where $\rho_\nu > 0$, $|\epsilon_\nu| = 1$, *and* $\epsilon_k \neq \epsilon_l$ *if* $k \neq l$; $k, l = 1, 2, \cdots, m$; *we have then* $T_n(p) > 0$ *for* $n < m-1$ *and* $T_n(p) = 0$ *for* $n \geq m$.

This necessary and sufficient condition is due to Carathéodory and Toeplitz and can be found in [8]. Note that for $n = 2$

$$T_2(p) = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} = 8 + 2\,\Re\{c_1^2 \overline{c}_2\} - 2|c_2|^2 - 4|c_1|^2 \geq 0,$$

is equivalent to

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{1.6}$$

for some x with $|x| \leq 1$. Further, if

$$T_3(p) = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix},$$

then $T_3(p) \geq 0$ is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{1.7}$$

Solving (1.7) with the help of (1.6), we get

$$4c_3 = c_1^3 + 2c_1x(4 - c_1^2) - c_1x^2(4 - c_1^2) + 2(4 - c_1^2)(1 - |x|^2)z, \tag{1.8}$$

for some x and z with $|x| \leq 1$ and $|z| \leq 1$.

Lemma 1.3. ([17, Theorem 1]) *If* $f \in \mathcal{I}$ *and its inverse* f^{-1} *having the form* (1.2). *Then*

$$|\gamma_2| \leq 1, \quad |\gamma_3| \leq \frac{4}{3}, \quad |\gamma_4| \leq \frac{13}{6}, \quad |\gamma_5| \leq \frac{59}{15} \quad \text{and} \quad |\gamma_6| \leq \frac{344}{45}. \tag{1.9}$$

The bounds in (1.9) *are best possible.*

2. Main Results

Theorem 2.1. *Let* $f \in \mathcal{I}$ *be of the form* (1.1) *and its inverse* f^{-1} *be given by* (1.2). *Then*

$$|\gamma_2\gamma_3 - \gamma_4| \leq \frac{13\sqrt{78}}{108} \quad \text{and} \quad |\gamma_2\gamma_4 - \gamma_3^2| \leq \frac{137}{288}. \tag{2.1}$$

Proof. Let $f \in \mathcal{I}$ be of the form (1.1) and its inverse f^{-1} is given by (1.2). Then it is well known [17, Eq. 3.4] that

$$\gamma_2 = -\frac{1}{2}c_1, \quad \gamma_3 = \frac{1}{6}(3c_1^2 - 2c_2) \quad \text{and} \quad \gamma_4 = \frac{1}{24}(20c_1c_2 - 15c_1^3 - 6c_3), \tag{2.2}$$

which gives

$$\begin{cases} |\gamma_2\gamma_3 - \gamma_4| = \dfrac{1}{24}|9c_1^3 - 16c_1c_2 + 6c_3| \\ \text{and} \\ |\gamma_2\gamma_4 - \gamma_3^2| = \dfrac{1}{144}|9c_1^4 - 12c_1^2c_2 - 16c_2^2 + 18c_1c_3|. \end{cases} \tag{2.3}$$

By using (1.6) and (1.8) from Lemma 1.2 in (2.3), we obtain

$$\begin{cases} |\gamma_2\gamma_3 - \gamma_4| = \dfrac{1}{48}|5c_1^3 + (4-c_1^2)\{-10c_1x - 3c_1x^2 + 6(1-|x|^2)z\}| \\ \text{and} \\ |\gamma_2\gamma_4 - \gamma_3^2| = \dfrac{1}{288}|7c_1^4 + (4-c_1^2)\{-10c_1^2x - 8x^2(4-c_1^2) \\ \qquad\qquad\qquad -9c_1^2x^2 + 18c_1(1-|x|^2)z\}|. \end{cases} \tag{2.4}$$

If $p(z) \in \mathcal{P}$, then $p(e^{i\alpha}z) \in \mathcal{P}$. We can always select a real α in $p(e^{i\alpha}z)$ so that $c_ne^{i\alpha n} \geq 0$. Hence we may suppose that $c_n \geq 0$ $(n \in \mathbb{N})$. Further, the power series (1.5) converges in $\mathbb{D}$ to a function in $\mathcal{P}$, if and only if the Toeplitz determinants $T_n(p)$ and $c_{-n} = \bar{c}_n$, are all nonnegative, i.e. c_1 is real, $c_1 \geq 0$, and by Lemma 1.1, it is clear that $|c_1| \leq 2$. Therefore, letting $c_1 = c$, we may assume without restriction that $c \in [0, 2]$. Hence, applying triangle inequality with $\mu = |x|$, we obtain

$$\begin{cases} |\gamma_2\gamma_3 - \gamma_4| \leq \dfrac{1}{48}[5c^3 + (4-c^2)\{10c\mu + 3c\mu^2 + 6(1-\mu^2)\}] \\ \qquad\qquad := A(c,\mu) \\ \text{and} \\ |\gamma_2\gamma_4 - \gamma_3^2| \leq \dfrac{1}{288}[7c^4 + (4-c^2)\{10c^2\mu + 32\mu^2 + c^2\mu^2 + 18c(1-\mu^2)\}] \\ \qquad\qquad := B(c,\mu). \end{cases}$$

Now to prove our results, we need to maximize the values of A and B over the region $\Omega = \{(c,\mu) : 0 \leq c \leq 2 \text{ and } 0 \leq \mu \leq 1\}$. For this, first differentiating A with respect to μ and c, we obtain

$$\frac{\partial A}{\partial \mu} = \frac{1}{48}\left[(4-c^2)\{10c + 6\mu(c-2)\}\right], \tag{2.5}$$

and

$$\frac{\partial A}{\partial c} = \frac{1}{48}\left[40\mu + 12\mu^2 + 12(\mu^2-1)c + 3(5 - 10\mu - 3\mu^2)c^2\right]. \tag{2.6}$$

A critical point of $A(c,\mu)$ must satisfy $\frac{\partial A}{\partial \mu} = 0$ and $\frac{\partial A}{\partial c} = 0$. The condition $\frac{\partial A}{\partial \mu} = 0$ gives $c = \pm 2$ or $\mu = \frac{5c}{3(2-c)}$. Points (c,μ) satisfying such conditions are not interior point of Ω. So the maximum cannot attain in the interior of Ω. Now to see on the boundary, taking the boundary line $L_1 = \{(0,\mu) : 0 \leq \mu \leq 1\}$, we have $A(0,\mu) = (1-\mu^2)/2$, and its maximum on this line is equal to $1/2$, which is attained at the point $(0,0)$. On the boundary line $L_2 = \{(2,\mu) : 0 \leq \mu \leq 1\}$, we have $A(2,\mu) = 5/6$, which is a constant. On the boundary line $L_3 = \{(c,0) : 0 \leq c \leq 2\}$, we have $A(c,0) = (5c^3 - 6c^2 + 24)/48$, and the maximum on this line is $5/6$, which is attained at the point $(2,0)$. On the line $L_4 = \{(c,1) : 0 \leq c \leq 2\}$, we have

$A(c,1) = (52c - 8c^3)/48$, and the maximum on this line is $13\sqrt{78}/108$ which is attained at the point $(\sqrt{13/6}, 1)$. Comparing these results, we observe that

$$A(0,0) < A(2,\mu) < A(\sqrt{13/6}, 1).$$

Thus we get

$$\max_{\Omega} A(c,\mu) = A(\sqrt{13/6}, 1) = 13\sqrt{78}/108.$$

Using the same procedure to find the maxima for $B(c,\mu)$, we get

$$\max_{\Omega} B(c,\mu) = B(\sqrt{3/2}, 1) = 137/288.$$

This completes the proof of Theorem 2.1. □

Theorem 2.2. *Let $f \in \mathcal{I}$ be of the form* (1.1) *and its inverse f^{-1} be given by* (1.2)*. Then for any complex number μ,*

$$|\gamma_3 - \mu\gamma_2^2| \leq \frac{2}{3}\, max\left\{1, \frac{|4-3\mu|}{2}\right\}. \tag{2.7}$$

Proof. Using (2.2), we get

$$|\gamma_3 - \mu\gamma_2^2| = \frac{1}{3}\left|c_2 - \frac{6-3\mu}{4}c_1^2\right|. \tag{2.8}$$

The result now follows from Lemma 1.1. □

If we take $\mu = 1$ in Theorem 2.2, we get

Corollary 2.3. *Let $f \in \mathcal{I}$ be of the form* (1.1) *and its inverse f^{-1} be given by* (1.2)*. Then*

$$|\gamma_3 - \gamma_2^2| \leq \frac{2}{3}. \tag{2.9}$$

Theorem 2.4. *Let $f \in \mathcal{I}$ be of the form* (1.1) *and its inverse f^{-1} be given by* (1.2)*. Then*

$$|H_{3,1}(f^{-1})| \leq \frac{10551 + 845\sqrt{78}}{3240}.$$

Proof. By definition,

$$H_{3,1}(f^{-1}) = \begin{vmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \gamma_2 & \gamma_3 & \gamma_4 \\ \gamma_3 & \gamma_4 & \gamma_5 \end{vmatrix} = \gamma_3(\gamma_2\gamma_4 - \gamma_3^2) - \gamma_4(\gamma_4 - \gamma_2\gamma_3) + \gamma_5(\gamma_3 - \gamma_2^2).$$

Now by using, Lemma 1.3, Theorem 2.1, Corollary 2.3 and the triangle inequality, we get

$$\begin{aligned} |H_{3,1}(f^{-1})| &\leq |\gamma_3||\gamma_2\gamma_4 - \gamma_3^2| + |\gamma_4||\gamma_2\gamma_3 - \gamma_4| + |\gamma_5||\gamma_3 - \gamma_2^2| \\ &\leq \frac{4}{3}\,\frac{137}{288} + \frac{13}{6}\,\frac{13\sqrt{78}}{108} + \frac{59}{15}\,\frac{2}{3} \\ &= \frac{10551 + 845\sqrt{78}}{3240}. \end{aligned}$$

□

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