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SOME RESULTS ON A GENERALIZED HYPERGEOMETRIC k-FUNCTIONS

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Abstract. In this paper, we define further generalized hypergeometric k-functions, using a special case of Wright hypergeometric function. Some of the differential properties, integral representation, contiguous relations and differential formulas of the generalized hypergeometric k-functions ${}_2R_{1,k}(a,b;c;\tau;z)$ (where $k>0$) are established.

1. Introduction

The hypergeometric function ${}_2F_1(a,b;c;z)$ plays an important role in mathematical analysis and its applications. Most of the special functions encountered in physics, engineering and probability theory are special cases of hypergeometric functions see ([8],[9],[12],[13], [16],[17], [27]). Wright [29] has extended the generalization of the hypergeometric function in the following form

$$ {}_p\Psi_q(z)=\sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+\beta n)\cdots\Gamma(\alpha_p+\beta_p n)}{\Gamma(\rho_1+\mu_1 n)\cdots\Gamma(\rho_q+\mu_q n)}, \tag{1.1}$$

where β_r and μ_s are real positive numbers such that

$$1+\sum_{s=1}^{q}\mu_s-\sum_{r=1}^{p}\beta_r>0.$$

When β_r and μ_s are equal to 1, equation (1.1) is differ from generalized hypergeometric function ${}_pF_q(z)$ by a constant multiplier only. This generalized form of hypergeometric function has been established by Malovichko [14]. But Dotsenko [4] considered one of the interesting cases which has the following form

$$ {}_2R_1^{\omega,\mu}(z)={}_2R_1(a,b;c;\omega;\mu;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+\frac{\omega}{\mu}n)}{\Gamma(c+\frac{\omega}{\mu}n)}\frac{z^n}{n!} \tag{1.2}$$

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and its integral representation is expressed in the form

$$ {}_2R_1^{\omega,\mu}(z) = \frac{\mu\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{\mu b-1}(1-t^{\mu})^{c-b-1}(1-zt^{\omega})^{-a}dt \tag{1.3} $$

where $Re(c) > Re(b) > 0$. In 2001, Virchenko *et al.* [28] have investigated by direct observation, the function ${}_2R_1^{\omega,\mu}(z)$ is not symmetric with respect to the parameters a and b. In the same paper, they defined the said Wright type hypergeometric function ${}_2R_1^{\tau}(z)$ in the following form

$$ {}_2R_1^{\tau}(z) = {}_2R_1(a,b;c;\tau;z) = \frac{\Gamma(c)}{\Gamma(c-b)} \sum_{n=0}^{\infty} \frac{(a)_n\Gamma(b+\tau n)}{\Gamma(c+\tau n)} \frac{z^n}{n!}; \quad \tau > 0, \quad |z| < 1 \tag{1.4} $$

and its integral representation is defined as

$$ {}_2R_1^{\tau}(z) = {}_2R_1(a,b;c;\tau;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt^{\tau})^{-a}dt \tag{1.5} $$

or

$$ {}_2R_1^{\tau}(z) = {}_2R_1(a,b;c;\tau;z) = \frac{\Gamma(c)}{\tau\Gamma(b)\Gamma(c-b)} \int_0^1 t^{\frac{b}{\tau}-1}(1-t^{\frac{1}{\tau}})^{c-b-1}(1-zt)^{-a}dt. \tag{1.6} $$

The same authors have also defined the following contiguous function relations for ${}_2R_1^{\tau}(z)$

$$ (b-a\tau)R = bR(b+1) - a\tau R(a+1) \tag{1.7} $$

$$ (c-a\tau-1)R = (c-1)R(c-1) - a\tau R(a+1) \tag{1.8} $$

$$ (c-b-1)R = (c-1)R(c-1) - bR(b+1) \tag{1.9} $$

$$ cR = (c-b)R(c+1) - bR(b+1) \tag{1.10} $$

where for simplicity $R = {}_2R_1^{\tau}(z) = R(a,b;c;\tau;z)$ and $R(a+1) = R(a+1,b;c;\tau;z)$ etc., have been used. For more details about the theory of Wright type hypergeometric series and for its properties, see ([25]-[27],[30]).

In 2007, Diaz and Pariguan [6] have introduced and proved some identities of gamma k-function, beta k-function and Pochhammer k-symbol. They have deduced an integral representation of gamma k-function and beta k-function respectively given by

$$ \Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma(\frac{z}{k}) = \int_0^{\infty} t^{z-1}e^{-\frac{z^k}{k}}dt, \quad Re(z) > 0, k > 0 \tag{1.11} $$

$$ B_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}dt, \quad Re(x) > 0, Re(y) > 0. \tag{1.12} $$

They have also provided the following some useful and applicable relations

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)} \tag{1.13}$$

$$(z)_{n,k} = \frac{\Gamma_k(z+nk)}{\Gamma_k(z)} \tag{1.14}$$

where $(z)_{n,k} = (z)(z+k)(z+2k)\cdots(z+(n-1)k); \quad (z)_{0,k} = 1$ and $k > 0$ and

$$\sum_{n=0}^{\infty}(\alpha)_{n,k}\frac{z^n}{n!} = (1-kz)^{\frac{-\alpha}{k}}. \tag{1.15}$$

The Researchers ([1]-[3], [5],[7],[11],[15] have proved a number of properties and Kokologiannaki [10] has also taken up zeta k-function as

$$\zeta(z,s) = \sum_{n=0}^{\infty}\frac{1}{(z+nk)^s}, \quad k, z > 0, s > 1 \tag{1.16}$$

$$m^{mj}(\frac{z}{m})_{j,k}(\frac{z+k}{m})_{j,k}\cdots(\frac{z+(m-1)k}{m})_{j,k} = (z)_{mj,k} \tag{1.17}$$

$$(z)_{mj,k} = \frac{\Gamma_k(z+mjk)}{\Gamma_k(z)}$$

$$\sum_{j=0}^{\infty}\frac{z^j}{j!} = e^z. \tag{1.18}$$

$$\tag{1.19}$$

For more details about the theory of special k-functions like gamma k-function, beta k-function, hypergeometric k-function, solutions of hypergeometric k-differential equations, contiguous k-function relations, inequalities with applications and integral representations involving gamma and beta k-functions, contiguous function relations and integral representation for Appell k-series and so forth (See [18]-[23]). In 2012, Mubeen and Habibullah [24] have defined an integral representation of some hypergeometric k-functions as

$$F_k[(a,k),(b,k);(c,k);z] = \frac{\Gamma_k(c)}{k\Gamma_k(b)\Gamma_k(c-b)}\int_0^1 t^{\frac{b}{k}-1}(1-t)^{\frac{c-b}{k}-1}(1-ktz)^{\frac{-a}{k}}dt.$$

2. Wright type hypergeometric k-functions

In this section, we define the said hypergeometric k-functions and their integral representation in terms of a new parameter k where $k > 0$.

2.1. **Extended hypergeometric k-series.** The extended hypergeometric k-series is defined in the following form as

$$ {}_p\Psi_{q,k}(z) = \sum_{n=0}^{\infty} \frac{\Gamma_k(\alpha_1+\beta_1 nk)\cdots\Gamma_k(\alpha+\beta_p nk)}{\Gamma_k(\rho_1+\mu_1 nk)\cdots\Gamma_k(\rho_1+\mu_q nk)} \frac{z^n}{n!} \tag{2.1} $$

where β_r, μ_s and k are real positive numbers such that

$$ 1+\sum_{s=1}^{q}\mu_s-\sum_{r=1}^{p}\beta_r>0. $$

Equation (2.1) differs from the generalized hypergeometric k-function ${}_pF_{q,k}(z)$ only by a constant multiplier.

2.2. **Wright type hypergeometric k-function.** The Wright type hypergeometric k-function is defined in the following form

$$ {}_2R_{1,k}^{\omega,\mu}(z) = {}_2R_{1,k}(a,b;c;\omega;\mu;z) = \frac{\Gamma_k(c)}{\Gamma_k(a)\Gamma_k(b)} \sum_{n=0}^{\infty} \frac{\Gamma_k(a+nk)\Gamma_k(b+\frac{\omega}{\mu}nk)}{\Gamma_k(c+\frac{\omega}{\mu}nk)} \frac{z^n}{n!}, \quad k>0. \tag{2.2} $$

Theorem 2.1. *If $Re(c) > Re(b) > 0$, then the function ${}_2R_{1,k}^{\omega,\mu}(z)$ can be expressed in the following form*

$$ {}_2R_{1,k}^{\omega,\mu}(z) = \frac{\mu\Gamma_k(c)}{k\Gamma_k(b)\Gamma_k(c-b)} \int_0^1 t^{\mu\frac{b}{k}-1}(1-t^{\mu})^{\frac{c-b}{k}-1}(1-zt^{\omega})^{\frac{-a}{k}}\,dt, \quad k>0. \tag{2.3} $$

Proof. Let us consider

$$ \begin{aligned} {}_2R_{1,k}^{\omega,\mu}(z) &= \frac{\Gamma_k(c)}{\Gamma_k(a)\Gamma_k(b)} \sum_{n=0}^{\infty} \frac{\Gamma_k(a+nk)\Gamma_k(b+\frac{\omega}{\mu}nk)}{\Gamma_k(c+\frac{\omega}{\mu}nk)} \frac{z^n}{n!} \\ &= \frac{\Gamma_k(c)}{\Gamma_k(a)\Gamma_k(b)\Gamma_k(c-b)} \sum_{n=0}^{\infty} \frac{\Gamma_k(a+nk)\Gamma_k(b+\frac{\omega}{\mu}nk)\Gamma_k(c-b)}{\Gamma_k(c+\frac{\omega}{\mu}nk)} \frac{z^n}{n!} \\ &= \frac{\Gamma_k(c)}{\Gamma_k(a)\Gamma_k(b)\Gamma_k(c-b)} \sum_{n=0}^{\infty} \Gamma_k(a+nk)B_k(b+\frac{\omega}{\mu}nk, c-b)\frac{z^n}{n!} \\ &= \frac{\Gamma_k(c)}{\Gamma_k(a)\Gamma_k(b)\Gamma_k(c-b)} \sum_{n=0}^{1} \Gamma_k(a+nk)[\frac{1}{k}\int_0^1 t^{\frac{b}{k}+\frac{\omega}{\mu}n-1}(1-t)^{\frac{c-b}{k}-1}dt]\frac{z^n}{n!} \end{aligned} $$

$$ = \frac{\Gamma_k(c)}{k\Gamma_k(a)\Gamma_k(b)\Gamma_k(c-b)} [\sum_{n=0}^{1}\Gamma_k(a+nk)\frac{z^n t^{\frac{\omega}{\mu}n}}{n!}] \int_0^1 t^{\frac{b}{k}-1}(1-t)^{\frac{c-b}{k}-1}dt. \tag{2.4} $$

Now since

$$ (1-kzt)^{-\frac{a}{k}} = \frac{1}{\Gamma_k(a)} \sum_{n=0}^{\infty} \Gamma_k(a+nk)\frac{z^n}{n!} \tag{2.5} $$

and taking into account

$$ (1-kzt^{\frac{\omega}{\mu}})^{-\frac{a}{k}} = \frac{1}{\Gamma_k(a)} \sum_{n=0}^{\infty} \Gamma_k(a+nk)\frac{z^n t^{\frac{\omega}{\mu}n}}{n!}. \tag{2.6} $$

Hence by substituting (2.6) in (2.4), we obtain

$$ {}_2R^{\omega,\mu}_{1,k}(z) = \frac{\Gamma_k(c)}{k\Gamma_k(b)\Gamma_k(c-b)} \int_0^1 t^{\frac{b}{k}-1}(1-t)^{\frac{c-b}{k}-1}(1-kzt^{\frac{\omega}{\mu}})^{-\frac{a}{k}}dt. \quad (2.7) $$

Thus after a simplification, we get the required result as:

$$ {}_2R^{\omega,\mu}_{1,k}(z) = \frac{\mu\Gamma_k(c)}{k\Gamma_k(b)\Gamma_k(c-b)} \int_0^1 t^{\mu\frac{b}{k}-1}(1-t^{\mu})^{\frac{c-b}{k}-1}(1-kzt^{\omega})^{-\frac{a}{k}}dt. $$

□

3. The function ${}_2R^{\tau}_{1,k}(z)$

The function ${}_2R^{\omega,\mu}_{1,k}(z)$ is not symmetric with respect to the parameters a and b. So by substituting $\frac{\omega}{\mu} = \tau > 0$ in (2.2), then we have the following form

$$ {}_2R^{\tau}_{1,k}(z) ={}_2 R_{1,k}(a,b;c;\tau;z) = \frac{\Gamma_k(c)}{\Gamma_k(a)\Gamma_k(b)} \sum_{n=0}^{\infty} \frac{\Gamma_k(a+nk)\Gamma_k(b+\tau nk)}{\Gamma_k(c+\tau nk)} \frac{z^n}{n!}, \quad k>0, \quad \tau>0. (3.1) $$

Its integral representation is expressed in the following form:

$$ {}_2R^{\tau}_{1,k}(z) = \frac{\Gamma_k(c)}{k\Gamma_k(b)\Gamma_k(c-b)} \int_0^1 t^{\frac{b}{k}-1}(1-t)^{\frac{c-b}{k}-1}(1-kzt^{\tau})^{-\frac{a}{k}}dt \quad (3.2) $$

and by change of variable, we obtain

$$ {}_2R^{\tau}_{1,k}(z) = \frac{\Gamma_k(c)}{\tau k\Gamma_k(b)\Gamma_k(c-b)} \int_0^1 t^{\frac{b}{\tau k}-1}(1-t^{\frac{1}{\tau}})^{\frac{c-b}{k}-1}(1-kzt)^{-\frac{a}{k}}dt. \quad (3.3) $$

3.1. Definition.
We define the contiguous function to ${}_2R^{\tau}_{1,k}(z)$ as a function which is obtained by increasing or decreasing one of the parameters by $\pm k$ where $k > 0$. For simplicity, we use the following notations

$$ {}_2R_{1,k}(a,b;c;\tau;z) = R_k, \quad {}_2R_{1,k}(a+k,b;c;\tau;z) = R_k(a+k), \quad {}_2R_{1,k}(a,b+k;c;\tau;z) = R_k(b+k). $$

Lemma 3.1. *For ${}_2R^{\tau}_{1,k}(z)$ and its contiguous functions, the following relations satisfy*

$$ (b-a\tau)R_k = bR_k(b+k) - a\tau R_k(a+k) \quad (3.4) $$

$$ (c-a\tau-k)R_k = (c-k)R_k(c-k) - a\tau R_k(a+k) \quad (3.5) $$

$$ (c-b-k)R_k = cR_k(c-k) - bR_k(a+k) \quad (3.6) $$

$$ cR_k = (c-b)R_k(c+k) - bR_k(b+k;c+k) \quad (3.7) $$

$$ \Gamma_k(b)\Gamma_k(c+\tau k)R_k = \Gamma_k(b)\Gamma_k(c+\tau k)R_k(a+k) - kz\Gamma_k(c)\Gamma_k(c+\tau k)R_k(a+k;b+k;c+k). \quad (3.8) $$

Proof. To prove the first relation (3.4), we have

$$\begin{aligned} bR_k(b+k) &= \frac{b\Gamma_k(c)}{\Gamma_k(a)\Gamma_k(b+k)}\sum_{n=0}^{\infty}\frac{\Gamma_k(a+nk)\Gamma_k(b+k+\tau nk)}{\Gamma_k(c+\tau nk)}\frac{z^n}{n!} \\ &= \frac{\Gamma_k(c)}{\Gamma_k(a)\Gamma_k(b)}\sum_{n=0}^{\infty}\frac{\Gamma_k(a+nk)\Gamma_k(b+\tau nk)}{\Gamma_k(b+\tau nk)}\frac{z^n}{n!}(b+\tau nk) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} a\tau R_k(a+k) &= \frac{a\Gamma_k(c)}{\Gamma_k(a)\Gamma_k(a+k)}\sum_{n=0}^{\infty}\frac{\Gamma_k(a+k+nk)\Gamma_k(b+\tau nk)}{\Gamma_k(b+\tau nk)}\frac{z^n}{n!} \\ &= \frac{\Gamma_k(c)}{\Gamma_k(a)\Gamma_k(a)}\sum_{n=0}^{\infty}\frac{\Gamma_k(a+nk)\Gamma_k(b+\tau nk)}{\Gamma_k(a+nk)}\frac{\tau z^n}{n!}(a+nk). \end{aligned} \tag{3.10}$$

Subtracting (3.10) from (3.9), we get the required relation (3.4). Now to prove relation (3.5), we have

$$\begin{aligned} (c-k)R_k(c-k) &= \frac{(c-k)\Gamma_k(c-k)}{\Gamma_k(a)\Gamma_k(b)}\sum_{n=0}^{\infty}\frac{\Gamma_k(a+nk)\Gamma_k(b+k+\tau nk)}{\Gamma_k(c-k+\tau nk)}\frac{z^n}{n!} \\ &= \frac{\Gamma_k(c)}{\Gamma_k(a)\Gamma_k(b)}\sum_{n=0}^{\infty}\frac{\Gamma_k(a+nk)\Gamma_k(b+k+\tau nk)}{\Gamma_k(c+\tau nk)}\frac{z^n}{n!}(c+\tau nk) \end{aligned} \tag{3.11}$$

and

$$a\tau R_k(a+k) = \frac{\Gamma_k(c)}{\Gamma_k(a)\Gamma_k(a)}\sum_{n=0}^{\infty}\frac{\Gamma_k(a+nk)\Gamma_k(b+\tau nk)}{\Gamma_k(a+nk)}\frac{\tau z^n}{n!}(a+nk). \tag{3.12}$$

Thus subtracting (3.12) from (3.11), we get the desired relation. In the same manner, we can prove (3.6)-(3.8). ☐

Lemma 3.2. *If $\tau \in \mathbb{N}$ $(\tau = n)$, then the following relation holds*

$$\begin{aligned} &{}_2R_{1,k}(a,b;c;n;z) \\ &= A_{n+1}F_{n,k}[(a,k),(\frac{b}{n},k),\cdots,(\frac{b+(n-1)k}{n},k);(\frac{c}{n},k),(\frac{c+k}{n},k),\cdots,(\frac{c+(n-1)k}{n},k);z], \end{aligned} \tag{3.13}$$

where

$$A = n^{-\frac{\delta}{k}}\frac{\Gamma_k(c)\Gamma_k(\frac{b}{n})\Gamma_k(\frac{b+k}{n})\cdots\Gamma_k(\frac{b+(n-1)k}{n}))}{\Gamma_k(b)\Gamma_k(\frac{c}{n})\Gamma_k(\frac{c+k}{n})\cdots\Gamma_k(\frac{b+(n-1)k}{n})}, \quad \delta = c-b.$$

Proof. Let us consider

$$\begin{aligned} &{}_{n+1}F_{n,k}[(a,k),(\frac{b}{n},k),\cdots,(\frac{b+(n-1)k}{n},k);(\frac{c}{n},k),(\frac{c+k}{n},k),\cdots,(\frac{c+(n-1)k}{n},k);z] \\ &= \frac{\Gamma_k(\frac{c}{n})\cdots\Gamma_k(\frac{c+(n-1)k}{n})}{\Gamma_k(a)\Gamma_k(\frac{b}{n})\cdots\Gamma_k(\frac{b+(n-1)k}{n})}\sum_{n=0}^{\infty}\frac{\Gamma_k(a+nk)\Gamma_k(\frac{b}{n}+nk)\Gamma_k(\frac{b+k}{n}+nk)\cdots\Gamma_k(\frac{b+(n-1)k}{n}+nk))}{\Gamma_k(\frac{c}{n}+nk)\Gamma_k(\frac{c+k}{n}+nk)\cdots\Gamma_k(\frac{c+(n-1)k}{n}+nk)}\frac{z^n}{n!} \\ &= \frac{n^{\frac{c}{k}}\Gamma_k(\frac{c}{n})\cdots\Gamma_k(\frac{c+(n-1)k}{n})}{n^{\frac{b}{k}}\Gamma_k(a)\Gamma_k(\frac{b}{n})\cdots\Gamma_k(\frac{b+(n-1)k}{n})}\sum_{n=0}^{\infty}\frac{\Gamma_k(a+nk)\Gamma_k(b+n^2k)}{\Gamma_k(c+n^2k)}\frac{z^n}{n!}. \end{aligned} \tag{3.14}$$

By substituting (3.14) in right hand side of (3.13), we get

$$\frac{\Gamma_k(c)}{\Gamma_k(a)\Gamma_k(b)}\sum_{n=0}^{\infty}\frac{\Gamma_k(a+nk)\Gamma_k(b+n^2k)}{\Gamma_k(c+n^2k)}\frac{z^n}{n!} ={}_2 R_{1,k}(a,b;c;n;z).$$

□

4. Differentiation formulas

In this section, we derive some basic differentiation formulas by the help of following lemmas.

Lemma 4.1. *If $k>0$, then*

$$\frac{d}{dz}[{}_2R_{1,k}(a,b;c;\tau;z)] = a\frac{\Gamma_k(c)\Gamma_k(b+\tau k)}{\Gamma_k(b)\Gamma_k(c+\tau k)}\quad {}_2R_{1,k}(a+k,b+\tau k;c+\tau k;\tau;z). \tag{4.1}$$

Proof. Consider

$$\frac{d}{dz}[{}_2R_{1,k}(a,b;c;\tau;z)] = \frac{\Gamma_k(c)}{\Gamma_k(b)\Gamma_k(a)}\frac{d}{dz}\sum_{n=0}^{\infty}\frac{\Gamma_k(a+nk)\Gamma_k(b+\tau nk)}{\Gamma_k(c+\tau nk)}\frac{z^n}{n!}.$$

Thus, we can write

$$\frac{d}{dz}[{}_2R_{1,k}(a,b;c;\tau;z)] = \frac{\Gamma_k(c)}{\Gamma_k(b)\Gamma_k(a)}\sum_{n=1}^{\infty}\frac{\Gamma_k(a+nk)\Gamma_k(b+\tau nk)}{\Gamma_k(c+\tau nk)}\frac{z^{n-1}}{(n-1)!}. \tag{4.2}$$

Now replace $n-1$ by n in (4.2), we obtain

$$\begin{aligned}\frac{d}{dz}[{}_2R_{1,k}(a,b;c;\tau;z)] &= \frac{\Gamma_k(c)}{\Gamma_k(b)\Gamma_k(a)}\sum_{n=1}^{\infty}\frac{\Gamma_k(a+k+nk)\Gamma_k(b+\tau k+\tau nk)}{\Gamma_k(c+\tau k+\tau nk)}\frac{z^n}{n!}\\ &= a\frac{\Gamma_k(c)\Gamma_k(c+\tau k)\Gamma_k(b+\tau k)}{\Gamma_k(b)\Gamma_k(b+\tau k)\Gamma_k(a+k)\Gamma_k(c+\tau k)}\\ &\times \sum_{n=1}^{\infty}\frac{\Gamma_k(a+k+nk)\Gamma_k(b+\tau k+\tau nk)}{\Gamma_k(c+\tau k+\tau nk)}\frac{z^n}{n!}\\ &= a\frac{\Gamma_k(c)\Gamma_k(b+\tau k)}{\Gamma_k(b)\Gamma_k(c+\tau k)}\quad {}_2R_{1,k}(a+k,b+\tau k;c+\tau k;\tau;z).\end{aligned}$$

□

Lemma 4.2. *If $k>0$, then*

$$\frac{d}{dz}[z^{\frac{a}{k}}\quad {}_2R_{1,k}(a,b;c;\tau;z) = \frac{1}{k}[az^{\frac{a}{k}-1}\quad {}_2R_{1,k}(a+k,b;c;\tau;z)]. \tag{4.3}$$

Proof. Let us consider

$$\begin{aligned}\frac{d}{dz}[z^{\frac{a}{k}}\quad {}_2R_{1,k}(a,b;c;\tau;z)] &= \frac{\Gamma_k(c)}{\Gamma_k(a)\Gamma_k(b)}\frac{d}{dz}\sum_{n=0}^{\infty}\frac{\Gamma_k(a+nk)\Gamma_k(b+\tau nk)}{\Gamma_k(c+\tau nk)}\frac{z^{n+\frac{a}{k}}}{n!}\\ &= \frac{\Gamma_k(c)}{\Gamma_k(a)\Gamma_k(b)}\sum_{n=0}^{\infty}\frac{\Gamma_k(a+nk)\Gamma_k(b+\tau nk)}{\Gamma_k(c+\tau nk)}\frac{z^{n+\frac{a}{k}-1}}{n!}(\frac{a}{k}+n)\\ &= \frac{1}{k}[az^{\frac{a}{k}-1}\quad {}_2R_{1,k}(a+k,b;c;\tau;z)].\end{aligned}$$

□

Similarly the following differentiation formulas holds for $k > 0$

$$\frac{d^n}{dz^n}[{}_2R_{1,k}(a,b;c;\tau;z)] = \frac{\Gamma_k(a+nk)\Gamma_k(b+\tau nk)\Gamma_k(c)}{\Gamma_k(a)\Gamma_k(b)\Gamma_k(c+\tau nk)} \quad {}_2R_{1,k}(a+nk, b+\tau nk; c+\tau nk; \tau; z) \tag{4.4}$$

$$\frac{d^n}{dz^n}[z^{\frac{a}{k}+n-1} \quad {}_2R_{1,k}(a,b;c;\tau;z)] = \frac{\Gamma_k(a+nk)}{k\Gamma_k(a)} z^{\frac{a}{k}-1} \quad {}_2R_{1,k}(a+nk,b;c;\tau;z) \tag{4.5}$$

$$a {}_2R_{1,k}(a+k,b;c;\tau;z) = (kz\frac{d}{dz}+a) \quad {}_2R_{1,k}(a,b;c;\tau;z). \tag{4.6}$$

To prove the result (4.6), we have

$$\begin{aligned}
a[{}_2R_{1,k}(a+k,b;c;\tau;z) -{}_2 R_{1,k}(a,b;c;\tau;z)] \\
&= \frac{\Gamma_k(c)}{\Gamma_k(b)} \sum_{n=0}^{\infty} [\frac{a\Gamma_k(a+k+nk)\Gamma_k(b+\tau nk)}{\Gamma_k(a+k)\Gamma_k(c+\tau nk)} \\
&- \frac{a\Gamma_k(a+nk)\Gamma_k(b+\tau nk)}{\Gamma_k(a)\Gamma_k(c+\tau nk)}] \frac{z^n}{n!} \\
&= \frac{\Gamma_k(c)}{\Gamma_k(a)\Gamma_k(b)} \sum_{n=0}^{\infty} \frac{\Gamma_k(a+nk)\Gamma_k(b+\tau nk)}{\Gamma_k(a)\Gamma_k(c+\tau nk)} [a+nk-a] \frac{z^n}{n!} \\
&= \frac{\Gamma_k(c)}{\Gamma_k(a)\Gamma_k(b)} \sum_{n=1}^{\infty} \frac{\Gamma_k(b+\tau nk)}{\Gamma_k(a)\Gamma_k(c+\tau nk)} k \frac{z^n}{(n-1)!} \\
&= kz\frac{d}{dz} \quad {}_2R_{1,k}(a,b;c;\tau;z).
\end{aligned}$$

This implies that

$$a {}_2R_{1,k}(a+k,b;c;\tau;z) = (kz\frac{d}{dz}+a) {}_2R_{1,k}(a,b;c;\tau;z).$$

5. Integral formulas of ${}_2R^{\tau}_{1,k}(z)$

In this section, we derive some integral formulas in term of k, where $k > 0$.

Theorem 5.1. *If $Re(c-b) > 1 - \frac{1}{\tau k}$, $Re(c-b) > 0$, then ${}_2R^{\tau}_{1,k}(z)$ can be expressed in the following integral forms:*

$$\begin{aligned}
{}_2R_{1,k}(a,b;c,\frac{1}{\tau};z) = \frac{2\Gamma_k(c)}{\tau k \Gamma_k(b)\Gamma_k(c-b)} \int_0^{\infty} \frac{(\sinh\phi)^{2\frac{b}{\tau k}-1}(\cosh\phi+1)^{\frac{1}{\tau}+\frac{a}{k}-\frac{(b+c)}{\tau k}}}{[1+kz+(1-kz)\cosh\phi]^{\frac{a}{k}}} \\
\times [(\cosh\phi+1)^{\frac{1}{\tau}} - (\cosh\phi-1)^{\frac{1}{\tau}}]^{\frac{c-b}{k}-1} d\phi
\end{aligned} \tag{5.1}$$

$$\begin{aligned}
{}_2R_{1,k}(a,b;c,\frac{1}{\tau};z) = \frac{4\Gamma_k(c)}{\tau k \Gamma_k(b)\Gamma_k(c-b)} \int_0^{\infty} \frac{(\cosh\phi)^{\frac{2}{\tau}-\frac{2c}{\tau k}+\frac{2a}{k}-1}(\cosh\phi-1)^{\frac{(b+c)}{\tau k}-\frac{a}{k}-\frac{1}{\tau}}}{[1+kz+(1-kz)\cosh\phi]^{\frac{a}{k}}} \\
\times [(\cosh\phi+1)^{\frac{1}{\tau}} - (\cosh\phi-1)^{\frac{1}{\tau}}]^{\frac{c-b}{k}-1} d\phi.
\end{aligned} \tag{5.2}$$

Proof. To prove (5.1), using the substitution $t^\tau = \tanh^2\frac{\phi}{2}$ in (3.2) then

$$\begin{aligned}{}_2R_{1,k}(a,b;c,\frac{1}{\tau};z) &= \frac{2\Gamma_k(c)}{\tau k\Gamma_k(b)\Gamma_k(c-b)}\int_0^\infty \frac{(\tanh^2\frac{\phi}{2})^{\frac{1}{\tau}(\frac{b}{k}-1)}(1-\tanh^2\frac{\phi}{2})^{\frac{1}{\tau}(\frac{c-b}{k}-1)}}{(1-kz(\tanh^2\frac{\phi}{2}))^{-\frac{a}{k}}}\\ &\times (\tanh^2\frac{\phi}{2})^{\frac{1}{\tau}-1}\tanh\frac{\phi}{2}\frac{1}{\cosh^2\frac{\phi}{2}}d\phi.\end{aligned}$$

Now taking into account that

$$\cosh\phi - 1 = \frac{\sinh^2\phi}{\cosh\phi+1},$$

and after simplification, we get

$$\begin{aligned}{}_2R_{1,k}(a,b;c,\frac{1}{\tau};z) &= \frac{2\Gamma_k(c)}{\tau k\Gamma_k(b)\Gamma_k(c-b)}\int_0^\infty \frac{(\sinh\phi)^{2\frac{b}{\tau k}-1}(\cosh\phi+1)^{\frac{1}{\tau}+\frac{a}{k}-\frac{(b+c)}{\tau k}}}{[1+kz+(1-kz)\cosh\phi]^{\frac{a}{k}}}\\ &\times [(\cosh\phi+1)^{\frac{1}{\tau}}-(\cosh\phi-1)^{\frac{1}{\tau}}]^{\frac{c-b}{k}-1}d\phi.\end{aligned}$$

Similarly, using the substitution $t^\tau = \tanh^2\frac{\phi}{2}$ in (3.2) and then taking the following into account, we will get the required integral (5.2)

$$\cosh\phi + 1 = \frac{\sinh^2\phi}{\cosh\phi-1}.$$

□

Theorem 5.2. *If $Re(c) > Re(b) > 0$, then the following relation holds:*

$${}_2R_{1,k}(a,b;c;\tau;z) = \frac{\Gamma_k(c)}{k\Gamma_k(b)\Gamma_k(c-b)}\int_0^\infty s^{\frac{b}{k}}(1+s)^{-\frac{c}{k}}[1-kz(\frac{s}{s+1})^\tau]^{-\frac{a}{k}}ds. \quad (5.3)$$

Proof. Let us consider (3.2)

$${}_2R^\tau_{1,k}(z) = \frac{\Gamma_k(c)}{k\Gamma_k(b)\Gamma_k(c-b)}\int_0^1 t^{\frac{b}{k}-1}(1-t)^{\frac{c-b}{k}-1}(1-kzt^\tau)^{-\frac{a}{k}}dt.$$

Now replacing t by $\frac{s}{s+1}$, then $dt = \frac{1}{(s+1)^2}ds$. Thus, we can write

$$\begin{aligned}{}_2R^\tau_{1,k}(z) &= \frac{\Gamma_k(c)}{k\Gamma_k(b)\Gamma_k(c-b)}\int_0^\infty (\frac{s}{s+1})^{\frac{b}{k}-1}(1-(\frac{s}{s+1}))^{\frac{c-b}{k}-1}[1-kz(\frac{s}{s+1})^\tau]^{-\frac{a}{k}}\frac{1}{(s+1)^2}ds\\ &= \frac{\Gamma_k(c)}{k\Gamma_k(b)\Gamma_k(c-b)}\int_0^\infty s^{\frac{b}{k}-1}(1+s)^{-\frac{c}{k}}[1-kz(\frac{s}{s+1})^\tau]^{-\frac{a}{k}}ds.\end{aligned}$$

□

Corollary 5.3. *The substitution* $s = \sinh^2\phi$ *in (5.3) leads to the following integral representation as*

$$ {}_2R^{\tau}_{1,k}(z) = \frac{2\Gamma_k(c)}{k\Gamma_k(b)\Gamma_k(c-b)}\int_0^\infty (\sinh\phi)^{\frac{2b}{k}-1}(\cosh\phi)^{-\frac{2c}{k}+1}[1-kz(\tanh\phi)^{2\tau}]^{-\frac{a}{k}}d\phi. \tag{5.4}$$

The following integral representation can be easily derived from theorem 5.2

$$ {}_2R^{\tau}_{1,k}(z) = \frac{2\Gamma_k(c)}{\tau k\Gamma_k(b)\Gamma_k(c-b)}\int_0^{\frac{\pi}{2}} \frac{(\sin\lambda)^{\frac{2b}{\tau k}-1}(1-\sin^{\frac{2}{\tau}}\lambda)^{\frac{c-b}{k}-1}}{(1-kz\sin^2\lambda)^{\frac{a}{k}}\cos\lambda}d\lambda \tag{5.5}$$

$$ {}_2R^{\tau}_{1,k}(z) = \frac{2\Gamma_k(c)}{\tau k\Gamma_k(b)\Gamma_k(c-b)}\int_0^{\pi} \frac{(\sinh\frac{\lambda}{2})^{\frac{2b}{\tau k}-1}(1-\sin^{\frac{2}{\tau}}\frac{\lambda}{2})^{\frac{c-b}{k}-1}}{(1-k\frac{z}{2}+k\frac{z}{2}\cos\lambda)^{\frac{a}{k}}\cos\frac{\lambda}{2}}d\lambda \tag{5.6}$$

$$ {}_2R^{\tau}_{1,k}(z) = \frac{2\Gamma_k(c)}{\tau k\Gamma_k(b)\Gamma_k(c-b)}\int_0^{\infty} \frac{(\tanh\lambda)^{\frac{2b}{\tau k}-1}(1-\tanh^{\frac{2}{\tau}}\lambda)^{\frac{c-b}{k}-1}}{(1-kz\tanh^2\lambda)^{\frac{a}{k}}\cosh^2\lambda}d\lambda. \tag{5.7}$$

To prove (5.5), we may write theorem 5.2 as

$$ {}_2R_{1,k}(a,b;c;\tau;z) = \frac{\Gamma_k(c)}{\tau k\Gamma_k(b)\Gamma_k(c-b)}\int_0^\infty s^{\frac{b}{\tau k}}(1+s)^{-\frac{c}{\tau k}}[1-kz(\frac{s}{s+1})]^{-\frac{a}{k}}ds.$$

Now by replacing $s = \tan^2\lambda$, then after simplification we get the required integral representation. Similarly we can prove (5.6) and (5.7).

Conclusion. In this paper, the authors introduced the τ-Gauss hypergeometric functions in term of a new parameter $k > 0$. The substitution $k = 1$ will leads to the results of Virchenko *et al.* [28].

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Conflict of Interests

The author(s) declare(s) that there is no conflict of interests regarding the publication of this article.

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