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COMMON FIXED POINT THEOREMS FOR JS-CONTRACTIONS

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Abstract. The aim of this article is to study JS-contractions and to establish some new common fixed point theorems for these contractions in the setup of complete metric spaces. Presented theorems are generalizations of recent fixed point theorems due to Hussain et al. [Fixed Point Theory and Applications (2015) 2015:185]. An example is also given to support our generalized result.

1. Introduction and Preliminaries

Over the past two decades, fixed point theory become a distinguished mathematical theory which is a pretty mixture of analysis, topology, and geometry. It is an interdisciplinary theory in which the existence of linear and nonlinear problems is frequently transformed into fixed point problems, for example, the existence of solutions to partial differential equations, the existence of solutions to integral equations, and the existence of periodic orbits in dynamical systems. This makes fixed point theory a contemporary area and a subject of active scientific research, constantly evolving and growing in a perpetual progress.

The Banach Contraction Principle is one of the cornerstones in the development of Nonlinear Analysis, in general, and metric fixed point theory, in particular. The method of successive approximation introduced by Liouville in 1837 and systematically developed by Picard in 1890 culminated in formulation of Banach Contraction Principle by Polish Mathematician Stefan Banach in 1922. This theorem provides an illustration of the unifying power of functional analytic methods and usefulness of fixed point theory in analysis. Extensions of the Banach contraction principle have been obtained either by generalizing the domain of the mapping or by extending the contractive condition on the mappings see [1, 2, 4, 5, 6, 7, 9, 10, 12, 13, 14, 15, 16, 18, 19, 20, 21].

Very recently, Jleli and Samet [11] introduced a new type of contraction and established some new fixed point theorems for such contraction in the context of generalized metric spaces.

Definition 1.1. *Let $\psi : [0, \infty) \to [1, \infty)$ be a function satisfying:*

(ψ_1) *ψ is nondecreasing;*

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(ψ_2) *for each sequence* $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \psi(\alpha_n) = 1$ *if and only if* $\lim_{n\to\infty}(\alpha_n) = 0$;

(ψ_3) *there exists* $0 < k < 1$ *and* $l \in (0, \infty]$ *such that* $\lim_{a\to 0^+} \frac{\psi(\alpha)-1}{\alpha^k} = l$.

Let Θ denote the family of functions ψ satisfying the above assertions.

Definition 1.2. [11] *A mapping $T : X \to X$ is said to be JS-contraction if there exists the above function ψ and a constant $\alpha \in [0, 1)$ such that*

$$d(Tx, Ty) \neq 0 \Longrightarrow \psi(d(Tx, Ty)) \leq [\psi(d(x, y))]^\alpha \tag{1.1}$$

for all $x, y \in X$.

Using the above JS-Contraction, they gave the following result as a generalization of Banach Contraction Principle.

Theorem 1.3. [11] *Let (X, d) be a complete metric space and let $T : X \to X$ be a JS-contraction. Then T has a unique fixed point.*

Hussain et al. [21] modified and extended the above family Θ of functions $\psi : [0, \infty) \to [1, \infty)$ and proved the following fixed point theorem for ψ-contractive condition in the setting of complete metric spaces.

(ψ_1') ψ is nondecreasing and $\psi(t) = 1$ if and only if $t = 0$;

(ψ_4) $\psi(a + b) \leq \psi(a) + \psi(b)$ for all $a, b > 0$.

To be consistent with Hussain et al. [21], we denote by Ψ the set of all functions $\psi : [0, \infty) \to [1, \infty)$ satisfying the conditions ($\psi_1' - \psi_4$).

Theorem 1.4. [8] *Let (X, d) be a complete metric space and $T : X \to X$ be a self-mapping. If there exists a function $\psi \in \Psi$ and positive real numbers k_1, k_2, k_3 and k_4 with $0 \leq k_1 + k_2 + k_3 + 2k_4 < 1$ such that*

$$\psi(d(Tx, Ty)) \leq [\psi(d(x, y))]^{k_1} \cdot [\psi(d(x, Tx))]^{k_2} \cdot [\psi(d(y, Ty))]^{k_3} \cdot [\psi((d(x, Ty) + d(y, Tx))]^{k_4} \tag{1.2}$$

for all $x, y \in X$, then T has a unique fixed point.

This result is not only a generalization of the Banach Contraction Principle but also Kannan fixed point theorem and Chatterjea Fixed Point Theorem. In this paper, we prove common fixed point results for a pair of self mappings satisfying a generalized ψ-contractive condition in the framework of complete metric spaces.

2. Main Results

Very recently, Ahmad et al. [3] defined the family $M(S, T)$ of all functions $a : X \times X \to [0, 1)$ with following assertions

$$a(TSx, y) \leq a(x, y) \text{ and } a(x, STy) \leq a(x, y)$$

and the family $N(S, T)$ of all functions $\beta : X \to [0, 1)$ such that for all $x, y \in X$ with

$$\beta(TSx) \leq \beta(x)$$

on a metric space (X, d) and for two self mappings $S, T : X \to X$. In this section, we prove a common fixed point theorem for self mappings regarding $\psi-$contractions. The following proposition plays an important role in the proofs of our main theorems.

Proposition 2.1. [3]*Let (X,d) be a metric space and $S,T : X \to X$ be self-mappings. Let $x_0 \in X$, we define the sequence $\{x_n\}$ by $x_{2n+1} = Sx_{2n}$, $x_{2n+2} = Tx_{2n+1}$ for all integers $n \geq 0$.*

If $a \in M(S,T)$, then $a(x_{2n}, y) \leq a(x_0, y)$ and $a(x, x_{2n+1}) \leq a(x, x_1)$ for all $x, y \in X$ and integers $n \geq 0$.

Now we state our main theorem.

Theorem 2.2. *Let (X,d) be a complete metric space and let $S,T : X \to X$ be self-mappings. If there exists mappings $a_1, a_2, a_3, a_4 \in M(S,T)$ such that for all $x, y \in X$:*

(a) $a_1(x,y) + a_2(x,y) + a_3(x,y) + 2a_4(x,y) < 1$
(b)

$$\begin{aligned} & \psi(d(Sx,Ty)) \\ \leq\ & [\psi(d(x,y))]^{a_1(x,y)} \cdot [\psi(d(x,Sx))]^{a_2(x,y)} \cdot [\psi(d(y,Ty))]^{a_3(x,y)} \cdot [\psi(d(x,Ty)) + d(y,Sx))]^{a_4(x,y)} \end{aligned}$$

where $\psi \in \Psi$, then S and T have a unique common fixed point.

Proof. Let $x_0 \in X$, we define the sequence $\{x_n\}$ by

$$x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1}$$

for all integers $n \geq 0$. From Proposition 2.1, for all integers $n \geq 0$, we have

$$\begin{aligned} 1 &< \psi(d(x_{2n}, x_{2n+1})) = \psi(d(Tx_{2n-1}, Sx_{2n})) = \psi(d(Sx_{2n}, Tx_{2n-1})) \\ &\leq [\psi(d(x_{2n}, x_{2n-1}))]^{a_1(x_{2n},x_{2n-1})} \cdot [\psi(d(x_{2n}, Sx_{2n}))]^{a_2(x_{2n},x_{2n-1})} \\ &\quad [\psi(d(x_{2n-1}, Tx_{2n-1}))]^{a_3(x_{2n},x_{2n-1})} \cdot \\ &\quad \cdot [\psi(d(x_{2n}, Tx_{2n-1}) + d(x_{2n-1}, Sx_{2n})]^{a_4(x_{2n},x_{2n-1})} \\ &= [\psi(d(x_{2n}, x_{2n-1}))]^{a_1(x_{2n},x_{2n-1})} \cdot [\psi(d(x_{2n}, x_{2n+1}))]^{a_2(x_{2n},x_{2n-1})} \\ &\quad \cdot [\psi(d(x_{2n-1}, x_{2n}))]^{a_3(x_{2n},x_{2n-1})} \cdot [\psi(d(x_{2n-1}, x_{2n+1}))]^{a_4(x_{2n},x_{2n-1})} \\ &\leq [\psi(d(x_{2n}, x_{2n-1}))]^{a_1(x_{2n},x_{2n-1})} \cdot [\psi(d(x_{2n}, x_{2n+1}))]^{a_2(x_{2n},x_{2n-1})} \\ &\quad \cdot [\psi(d(x_{2n-1}, x_{2n}))]^{a_3(x_{2n},x_{2n-1})} \cdot [\psi(d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1}))]^{a_4(x_{2n},x_{2n-1})} \\ &\leq [\psi(d(x_{2n}, x_{2n-1}))]^{a_1(x_0,x_{2n-1})} \cdot [\psi((d(x_{2n}, x_{2n+1}))]^{a_2(x_0,x_{2n-1})} \\ &\quad [\psi(d(x_{2n-1}, x_{2n}))]^{a_3(x_0,x_{2n-1})} \cdot [\psi(d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1}))]^{a_4(x_0,x_{2n-1})} \\ &\leq [\psi(d(x_{2n}, x_{2n-1}))]^{a_1(x_0,x_1)} \cdot [\psi(d(x_{2n}, x_{2n+1}))]^{a_2(x_0,x_1)} \\ &\quad \cdot [\psi(d(x_{2n-1}, x_{2n}))]^{a_3(x_0,x_1)} \cdot [\psi(d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1}))^{a_4(x_0,x_1)} \\ &\leq [\psi(d(x_{2n}, x_{2n-1}))]^{a_1(x_0,x_1)} \cdot [\psi(d(x_{2n}, x_{2n+1}))]^{a_2(x_0,x_1)} \\ &\quad \cdot [\psi(d(x_{2n-1}, x_{2n}))]^{a_3(x_0,x_1)} \cdot [\psi(d(x_{2n-1}, x_{2n})]^{a_4(x_0,x_1)} \cdot [\psi(d(x_{2n}, x_{2n+1}))]^{a_4(x_0,x_1)} \\ &= [\psi(d(x_{2n}, x_{2n-1}))]^{a_1(x_0,x_1)+a_3(x_0,x_1)+a_4(x_0,x_1)} \cdot [\psi(d(x_{2n}, x_{2n+1}))]^{a_2(x_0,x_1)+a_4(x_0,x_1)}. \end{aligned}$$

Thus

$$\psi(d(x_{2n}, x_{2n+1})) \leq [\psi(d(x_{2n}, x_{2n-1}))]^{\frac{a_1(x_0,x_1)+a_3(x_0,x_1)+a_4(x_0,x_1)}{1-a_2(x_0,x_1)-a_4(x_0,x_1)}}. \tag{2.1}$$

Similarly, we have

$$\begin{aligned}
1 &< \psi(d(x_{2n+1},x_{2n+2})) = \psi(d(Sx_{2n},Tx_{2n+1}))\\
&\leq [\psi(d(x_{2n},x_{2n+1}))]^{a_1(x_{2n},x_{2n+1})}\cdot[\psi(d(x_{2n},Sx_{2n}))]^{a_2(x_{2n},x_{2n+1})}\\
&\quad [\psi(d(x_{2n+1},Tx_{2n+1}))]^{a_3(x_{2n},x_{2n+1})}\cdot[\psi(d(x_{2n},Tx_{2n+1})+d(x_{2n+1},Sx_{2n}))]^{a_4(x_{2n},x_{2n+1})}\\
&= [\psi(d(x_{2n},x_{2n+1}))]^{a_1(x_{2n},x_{2n+1})}\cdot[\psi(d(x_{2n},x_{2n+1}))]^{a_2(x_{2n},x_{2n+1})}\\
&\quad \cdot[\psi(d(x_{2n+1},x_{2n+2}))]^{a_3(x_{2n},x_{2n+1})}\cdot[\psi(d(x_{2n},x_{2n+2}))]^{a_4(x_{2n},x_{2n+1})}\\
&\leq [\psi(d(x_{2n},x_{2n+1}))]^{a_1(x_{2n},x_{2n+1})}\cdot[\psi(d(x_{2n},x_{2n+1}))]^{a_2(x_{2n},x_{2n+1})}\\
&\quad \cdot[\psi(d(x_{2n+1},x_{2n+2}))]^{a_3(x_{2n},x_{2n+1})}\cdot[\psi(d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2}))]^{a_4(x_{2n},x_{2n+1})}\\
&\leq [\psi(d(x_{2n},x_{2n+1}))]^{a_1(x_0,x_{2n+1})}\cdot[\psi(d(x_{2n},x_{2n+1}))]^{a_2(x_0,x_{2n+1})}\\
&\quad [\psi(d(x_{2n+1},x_{2n+2}))]^{a_3(x_0,x_{2n+1})}\cdot\\
&\quad [\psi((d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2}))]^{a_4(x_0,x_{2n+1})}\\
&\leq [\psi(d(x_{2n},x_{2n+1}))]^{a_1(x_0,x_1)}.[\psi(d(x_{2n},x_{2n+1}))]^{a_2(x_0,x_1)}\\
&\quad [\psi(d(x_{2n+1},x_{2n+2}))]^{a_3(x_0,x_1)}\cdot[\psi((d(x_{2n},x_{2n+1})+d(x_{2n+1},x_{2n+2}))]^{a_4(x_0,x_1)}\\
&\leq [\psi(d(x_{2n},x_{2n+1}))]^{a_1(x_0,x_1)}.[\psi(d(x_{2n},x_{2n+1}))]^{a_2(x_0,x_1)}\\
&\quad [\psi(d(x_{2n+1},x_{2n+2}))]^{a_3(x_0,x_1)}\cdot[\psi((d(x_{2n},x_{2n+1})]^{a_4(x_0,x_1)}\cdot[\psi(d(x_{2n+1},x_{2n+2}))]^{a_4(x_0,x_1)}\\
&= [\psi(d(x_{2n},x_{2n+1}))]^{a_1(x_0,x_1)+a_3(x_0,x_1)+a_4(x_0,x_1)}\cdot[\psi(d(x_{2n},x_{2n+1}))]^{a_2(x_0,x_1)+a_4(x_0,x_1)}..
\end{aligned}$$

Thus

$$\psi(d(x_{2n+1},x_{2n+2})) \leq [\psi(d(x_{2n},x_{2n+1}))]^{\frac{a_1(x_0,x_1)+a_3(x_0,x_1)+a_4(x_0,x_1)}{1-a_2(x_0,x_1)-a_4(x_0,x_1)}}. \tag{2.2}$$

Let $\lambda = \frac{a_1(x_0,x_1)+a_3(x_0,x_1)+a_4(x_0,x_1)}{1-a_2(x_0,x_1)-a_4(x_0,x_1)} < 1$. Then from (2.1) and (2.2), we get

$$1 < \psi(d(x_n,x_{n+1})) \leq [\psi(d(x_{n-1},x_n))]^{\lambda} \leq [\psi(d(x_{n-2},x_{n-1}))]^{\lambda^2} \leq \cdots \leq [\psi(d(x_0,x_1))]^{\lambda^n}.$$

It gives

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0.$$

From the condition (ψ_3), there exist $0<k<1$ and $l \in (0,\infty]$ such that

$$\lim_{n\to\infty}\frac{\psi(d(x_n,x_{n+1}))-1}{d(x_n,x_{n+1})^k} = l.$$

Suppose that $l<\infty$. In this case, let $B=\frac{l}{2}>0$. From the definition of the limit, there exists $n_1\in\mathbb{N}$ such that

$$\left|\frac{\psi(d(x_n,x_{n+1}))-1}{d(x_n,x_{n+1})^k} - l\right| \leq B$$

for all $n>n_1$. This implies that

$$\frac{\psi(d(x_n,x_{n+1}))-1}{d(x_n,x_{n+1})^k} \geq l-B = \frac{l}{2} = B$$

for all $n>n_1$. Then

$$n(d(x_n,x_{n+1}))^k \leq An[\psi(d(x_n,x_{n+1}))-1.$$

Then there exists $n_1\in\mathbb{N}$ such that

$$d(x_n,x_{n+1}) \leq \frac{1}{n^{1/k}}$$

for all $n > n_1$. Now we prove that $\{x_n\}$ is a Cauchy sequence. For $m > n > n_1$ we have,

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/k}}.$$

As $0 < k < 1$, then $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ converges. Therefore, $d(x_n, x_m) \to 0$ as $m, n \to \infty$. Thus we proved that $\{x_n\}$ is a Cauchy sequence in X. The completeness of X ensures that there exist $z \in X$ such that, $x_n \to z$ as $n \to \infty$. First we show that z is fixed point of S. We suppose on the contary that $z \neq Sz$, then by Proposition (2.1), we have

$$\begin{aligned}
1 &< \psi(d(Sz, x_{2n+2})) = \psi(d(Sz, Tx_{2n+1})) \\
&\leq [\psi(d(z, x_{2n+1}))]^{a_1(z,x_{2n+1})} \cdot [\psi(d(z, Sz))]^{a_2(z,x_{2n+1})} \\
&\quad \cdot [\psi(d(x_{2n+1}, Tx_{2n+1}))]^{a_3(z,x_{2n+1})} \cdot [\psi(d(z, Tx_{2n+1}) + d(x_{2n+1}, Sz))]^{a_4(z,x_{2n+1})} \\
&= [\psi(d(z, x_{2n+1}))]^{a_1(z,x_{2n+1})} \cdot [\psi(d(z, Sz))]^{a_2(z,x_{2n+1})} \\
&\quad [\psi(d(x_{2n+1}, x_{2n+2}))]^{a_3(z,x_{2n+1})} \cdot [\psi(d(z, x_{2n+2}) + d(x_{2n+1}, Sz))]^{a_4(z,x_{2n+1})} \\
&\leq [\psi(d(z, x_{2n+1}))]^{a_1(z,x_1)} \cdot [\psi(d(z, Sz))]^{a_2(z,x_1)} \cdot [\psi(d(x_{2n+1}, x_{2n+2}))]^{a_3(z,x_1)} \\
&\quad [\psi(d(z, x_{2n+2}) + d(x_{2n+1}, Sz))]^{a_4(z,x_1)}.
\end{aligned}$$

Letting $n \to +\infty$, in the previous inequality, we get

$$1 < \psi(d(Sz, z)) \leq [\psi(d(z, Sz))]^{a_2(z,x_1)+a_4(z,x_1)} < \psi(d(z, Sz))$$

which is a contradiction. Thus we have $z = Sz$. We also show that z is a fixed point of T, so suppose on the contary that $z \neq Tz$, then by by Proposition (2.1), we have

$$\begin{aligned}
1 &< \psi(d(x_{2n+1}, Tz)) = \psi(d(Sx_{2n}, Tz)) \\
&\leq [\psi(d(x_{2n}, z))]^{a_1(x_{2n},z)} \cdot [\psi(d(x_{2n}, Sx_{2n}))]^{a_2(x_{2n},z)} \\
&\quad \cdot [\psi(d(z, Tz))]^{a_3(x_{2n},z)} \cdot [\psi(d(x_{2n}, Tz) + d(z, Sx_{2n}))]^{a_4(x_{2n},z)} \\
&= [\psi(d(x_{2n}, z))]^{a_1(x_{2n},z)} \cdot [\psi(d(x_{2n}, x_{2n+1}))]^{a_2(x_{2n},z)} \cdot [\psi(d(z, Tz))]^{a_3(x_{2n},z)} \\
&\quad [\psi(d(x_{2n}, Tz) + d(z, x_{2n+1}))]^{a_4(x_{2n},z)} \\
&\leq [\psi(d(x_{2n}, z))]^{a_1(x_0,z)} \cdot [\psi(d(x_{2n}, x_{2n+1}))]^{a_2(x_0,z)} \cdot [\psi(d(z, Tz))]^{a_3(x_0,z)} \\
&\quad [\psi(d(x_{2n}, Tz))]^{a_4(x_0,z)} \cdot [\psi(d(z, x_{2n+1}))]^{a_4(x_0,z)}
\end{aligned}$$

Letting $n \to +\infty$, in the previous inequality, we get

$$1 < \psi(d(z, Tz)) \leq [\psi(d(z, Tz))]^{a_3(x_0,z)+a_4(x_0,z)} < [\psi(d(z, Tz))]$$

which is a contradiction. Hence $z = Tz$. Therefore, z is a common fixed point of S and T.

Now we show the uniqueness. Suppose that there exist another common fixed point u of S and T that is $u = Su = Tu$. Assume that $Su \neq Tz$, then from (b) we have

$$\begin{aligned}
1 &< \psi(d(u, z)) = \psi(d(Su, Tz)) \\
&\leq [\psi(d(u, z))]^{a_1(u,z)} \cdot [\psi(d(u, Su))]^{a_2(u,z)} \cdot [\psi(d(z, Tz))]^{a_3(u,z)} \cdot [\psi(d(u, Tz) + d(z, Su))]^{a_4(u,z)} \\
&\leq [\psi(d(u, z))]^{a_1(u,z)} \cdot [\psi(d(u, z))]^{a_4(u,z)} \cdot [\psi(d(z, u))]^{a_4(u,z)} \\
&= [\psi(d(u, z))]^{a_1(u,z)+2a_4(u,z)} < [\psi(d(u, z))]
\end{aligned}$$

which is a contradiction to the fact that $Su \neq Tz$. Thus $Su = Tz$. Thus S and T have a unique common fixed point, which ends the proof. □

Consequently, we have the following result.

Corollary 2.3. *Let (X,d) be a complete metric space and $S : X \to X$ be self-mapping. If there exist mappings $a_1, a_2, a_3, a_4 \in M(S,T)$ such that for all $x, y \in X$:*

(a) $a_1(x,y) + a_2(x,y) + a_3(x,y) + 2a_4(x,y) < 1$
(b)

$$\begin{aligned} & \psi(d(Sx,Sy)) \\ \leq\ & [\psi(d(x,y))]^{a_1(x,y)} \cdot [\psi(d(x,Sx))]^{a_2(x,y)} \cdot [\psi(d(y,Sy))]^{a_3(x,y)} \cdot [\psi(d(x,Sy)) + d(y,Sx))]^{a_4(x,y)} \end{aligned}$$

where $\psi \in \Psi$, then S has a unique fixed point.

Proof. Taking $S = T$ in Theorem 2.2. □

Theorem 2.4. *Let (X,d) be a complete metric space and $S,T : X \to X$ be self-mappings. If there exist mappings $a_1, a_2, a_3, a_4 \in M(S,T)$ such that for all $x, y \in X$:*

(a) $a_1(x,y) + a_2(x,y) + a_3(x,y) + 2a_4(x,y) < 1$
(b)

$$\begin{aligned} & \sqrt{d(Sx,Ty)} \\ \leq\ & a_1(x,y)\sqrt{d(x,y)} + a_2(x,y)\sqrt{d(x,Sx))} + a_3(x,y)\sqrt{d(y,Ty)} + a_4(x,y)\sqrt{d(x,Ty) + d(y,Sx)} \end{aligned}$$

then S and T have a unique common fixed point.

Proof. Taking $\psi(t) = e^{\sqrt{t}}$ in Theorem 2.2. □

Corollary 2.5. *Let (X,d) be a complete metric space and $S : X \to X$ be self-mapping. If there exist mappings $a_1, a_2, a_3, a_4 \in M(S,T)$ such that for all $x, y \in X$:*

(a) $a_1(x,y) + a_2(x,y) + a_3(x,y) + 2a_4(x,y) < 1$
(b)

$$\begin{aligned} & \sqrt{d(Sx,Sy)} \\ \leq\ & a_1(x,y)\sqrt{d(x,y))} + a_2(x,y)\sqrt{d(x,Sx))} + a_3(x,y)\sqrt{d(y,Sy)} + a_4(x,y)\sqrt{d(x,Sy) + d(y,Sx)} \end{aligned}$$

then S has a unique fixed point.

Proof. Taking $\psi(t) = e^{\sqrt{t}}$ in above Corollary. □

Remark 2.6. *Notice that condition (2.1) is equivalent to*

$$\begin{aligned} & d(Sx,Ty) \leq {a_1}^2(x,y)d(x,y) \\ & +{a_2}^2(x,y)d(x,Sx) + {a_3}^2(x,y)d(y,Ty) + {a_4}^2(x,y)[d(x,Ty) + d(y,Sx)] \\ & 2a_1(x,y)a_2(x,y)\sqrt{d(x,y)d(x,Sx)} + 2a_1(x,y)a_3(x,y)\sqrt{d(x,y)d(y,Ty)} \\ & +2a_1(x,y)a_4(x,y)\sqrt{d(x,y)[d(x,Ty) + d(y,Sx)]} + 2a_2(x,y)a_3(x,y)\sqrt{d(x,Sx)d(y,Ty)} \\ & +2a_2(x,y)a_4(x,y)\sqrt{d(x,Sx)[d(x,Ty) + d(y,Sx)]} \\ & +2a_3(x,y)a_4(x,y)\sqrt{d(y,Ty)[d(x,Ty) + d(y,Sx)]}. \end{aligned}$$

Theorem 2.7. *Let (X,d) be a complete metric space and $S,T{:}X \to X$ be self-mappings. If there exist mappings $\beta_1, \beta_2, \beta_3, \beta_4 \in N(S,T)$ such that for all $x, y \in X$:*

(a) $\beta_1(x)+\beta_2(x)+\beta_3(x)+2\beta_4(x)=1$;
(b)

$$\psi(d(Sx,Ty)) \leq [\psi(d(x,y))]^{\beta_1(x)} \cdot [\psi(d(x,Sx))]^{\beta_2(x)} \cdot [\psi(d(y,Ty))]^{\beta_3(x)} \cdot [\psi(d(x,Ty)+d(y,Sx))]^{\beta_4(x)}$$

where $\psi \in \Psi$, then S and T have a unique common fixed point.

Proof. Define $a_1, a_2, a_3, a_4 : X \times X \to [0,1)$ by $a_1(x,y)=\beta_1(x)$, $a_2(x,y)=\beta_2(x)$, $a_3(x,y)=\beta_3(x)$ and $a_4(x,y)=\beta_4(x)$ for all $x,y \in X$. Then for all $x,y \in X$;
(a)

$$a_1(TSx,y)=\beta_1(TSx) \leq \beta_1(x)=a_1(x,y) \text{ and } a_1(x,STy)=\beta_1(x)=a_1(x,y)$$

$$a_2(TSx,y)=\beta_2(TSx) \leq \beta_2(x)=a_2(x,y) \text{ and } a_2(x,STy)=\beta_2(x)=a_2(x,y),$$

$$a_3(TSx,y)=\beta_3(TSx) \leq \beta_3(x)=a_3(x,y) \text{ and } a_3(x,STy)=\beta_3(x)=a_3(x,y);$$

$$a_4(TSx,y)=\beta_4(TSx) \leq \beta_4(x)=a_3(x,y) \text{ and } a_4(x,STy)=\beta_4(x)=a_4(x,y);$$

(b)

$$a_1(x,y)+a_2(x,y)+a_3(x,y)+2a_4(x,y)=\beta_1(x)+\beta_2(x)+\beta_3(x)+2\beta_4(x)<1,$$

c)

$$\begin{aligned}
\psi(d(Sx,Ty)) &\leq [\psi(d(x,y))]^{\beta_1(x)} \cdot [\psi(d(x,Sx))]^{\beta_2(x)} \cdot [\psi(d(y,Ty))]^{\beta_3(x)} \\
&\quad [\psi(d(x,Ty)+d(y,Sx))]^{\beta_4(x)} \\
&= [\psi(d(x,y))]^{a_1(x,y)} \cdot [\psi(d(x,Sx))]^{a_2(x,y)} \cdot [\psi(d(y,Ty))]^{a_3(x,y)} \\
&\quad [\psi(d(x,Ty)+d(y,Sx))]^{a_4(x,y)}.
\end{aligned}$$

By Theorem 2.2, S and T have unique common fixed point. □

By letting $\beta_1(\cdot)=\beta_1$, $\beta_2(\cdot)=\beta_2$, $\beta_3(\cdot)=\beta_3$ and $\beta_4(\cdot)=\beta_4$ in Corollary 2.4, we get the following result.

Corollary 2.8. *Let (X,d) be a complete metric space and $S,T{:}X \to X$ be self-mappings. If there exist non negative reals $\beta_1,\beta_2,\beta_3,\beta_4 \in [0,1)$ with $\beta_1+\beta_2+\beta_3+2\beta_4<1$ such that for all $x,y \in X$:*

$$\psi(d(Sx,Ty)) \leq [\psi(d(x,y))]^{\beta_1} \cdot [\psi(d(x,Sx))]^{\beta_2} \cdot [\psi(d(y,Ty))]^{\beta_3} \cdot [\psi(d(x,Ty)+d(y,Sx))]^{\beta_4}$$

where $\psi \in F$, then S and T have a unique common fixed point.

By setting $S=T$ in the above Corollary, we get Theorem 3.1 of [5].

Corollary 2.9. [5] *Let (X,d) be a complete metric space and $T:X \to X$ be a self-mapping. If there exists non negative reals $\beta_1,\beta_2,\beta_3,\beta_4 \in [0,1)$ with $\beta_1+\beta_2+\beta_3+2\beta_4<1$ such that for all $x,y \in X$:*

$$\psi(d(Tx,Ty)) \leq [\psi(d(x,y))]^{\beta_1} \cdot [\psi(d(x,Tx))]^{\beta_2} \cdot [\psi(d(y,Ty))]^{\beta_3} \cdot [\psi(d(x,Ty)+d(y,Tx))]^{\beta_4}$$

where $\psi \in \Psi$, then T has a unique fixed point.

Taking $\beta_2=\beta_3=\beta_4$ and $\beta_1=\beta$ in the above Corollary, we get Corollary 2.1 of [11].

Corollary 2.10. *Let (X,d) be a complete metric space and $T : X \to X$ be a self-mapping. If there exists non negative real $\beta \in [0,1)$ such that for all $x, y \in X$:*

$$\psi(d(Tx,Ty)) \leq [\psi(d(x,y))]^{\beta}$$

where $\psi \in \Psi$, then T has a unique fixed point.

Theorem 2.11. *Let (X,d) be a complete metric space and $S,T : X \to X$ be self-mappings. If there exists the mappings $\beta_1, \beta_2, \beta_3, \beta_4 \in N(S,T)$ such that*

(a) $\beta_1(x) + \beta_2(x) + \beta_3(x) + 2\beta_4(x) < 1$
(b)

$$\begin{aligned} &\sqrt{d(Sx,Ty)} \\ \leq \quad &\beta_1(x)\sqrt{d(x,y)} + \beta_2(x)\sqrt{d(x,Sx)}) + \beta_3(x)\sqrt{d(y,Ty)} + \beta_4(x)\sqrt{d(x,Ty)+d(y,Sx)} \end{aligned}$$

for all $x, y \in X$, then S and T have a unique common fixed point.

Remark 2.12. *Notice that condition (2.4), is equivalent to*

$$\begin{aligned} &d(Sx,Ty) \leq {\beta_1}^2(x)d(x,y) \\ &+\beta_2^2(x)d(x,Sx) + \beta_3^2(x)d(y,Ty) + \beta_4^2(x)[d(x,Ty)+d(y,Sx)] \\ &2\beta_1(x)\beta_2(x)\sqrt{d(x,y)d(x,Sx)} + 2\beta_1(x)\beta_3(x)\sqrt{d(x,y)d(y,Ty)} \\ &+2\beta_1(x)\beta_4(x)\sqrt{d(x,y)[d(x,Ty)+d(y,Sx)]} + 2\beta_2(x)\beta_3(x)\sqrt{d(x,Sx)d(y,Ty)} \\ &+2\beta_2(x)\beta_4(x)\sqrt{d(x,Sx)[d(x,Ty)+d(y,Sx)]} \\ &+2\beta_3(x)\beta_4(x)\sqrt{d(y,Ty)[d(x,Ty)+d(y,Sx)]} \end{aligned}$$

Corollary 2.13. *Let (X,d) be a complete metric space and $S : X \to X$ be self-mapping. If there exists the mappings $\beta_1, \beta_2, \beta_3, \beta_4 \in N(S,T)$ such that*

(a) $\beta_1(x) + \beta_2(x) + \beta_3(x) + 2\beta_4(x) < 1$
(b)

$$\begin{aligned} &\sqrt{d(Sx,Sy)} \\ \leq \quad &\beta_1(x)\sqrt{d(x,y)}) + \beta_2(x)\sqrt{d(x,Sx)}) + \beta_3(x)\sqrt{d(y,Sy)} + \beta_4(x)\sqrt{d(x,Sy)+d(y,Sx)} \end{aligned}$$

for all $x, y \in X$, then S has a unique fixed point.

Corollary 2.14. *Let (X,d) be a complete metric space and $S : X \to X$ be self-mapping. If there exist the non negative real numbers $\beta_1, \beta_2, \beta_3, \beta_4$ such that*

(a) $\beta_1 + \beta_2 + \beta_3 + 2\beta_4 < 1$
(b)

$$\begin{aligned} &\sqrt{d(Sx,Sy)} \\ \leq \quad &\beta_1\sqrt{d(x,y)}) + \beta_2\sqrt{d(x,Sx)}) + \beta_3\sqrt{d(y,Sy)} + \beta_4\sqrt{d(x,Sy)+d(y,Sx)} \end{aligned}$$

for all $x, y \in X$, then S has a unique fixed point.

Corollary 2.15. *Let (X,d) be a complete metric space and $S : X \to X$ be self-mapping. If there exists a constant $\beta \in [0,1)$ and such that*

$$d(Sx,Sy) \leq \beta d(x,y)$$

for all $x, y \in X$, then S has a unique fixed point.

Corollary 2.16. *Let (X,d) be a complete metric space and $S : X \to X$ be self-mapping. If there exist mappings $\beta_1, \beta_2, \beta_3, \beta_4 \in N(S,T)$ such that for all $x, y \in X$:*

(a) $\beta_1(x) + \beta_2(x) + \beta_3(x) + 2\beta_4(x) < 1$
(b)
$$\begin{aligned} & \sqrt[n]{d(Sx,Sy)} \\ \leq\ & \beta_1(x)\sqrt[n]{d(x,y)} + \beta_2(x)\sqrt[n]{d(x,Sx)} + \beta_3(x)\sqrt[n]{d(y,Sy)} + \beta_4(x)\sqrt[n]{d(x,Sy)+d(y,Sx)} \end{aligned}$$

then S has a unique fixed point.

Corollary 2.17. *Let (X,d) be a complete metric space and $S : X \to X$ be self-mapping. If there exist the non negative real numbers $\beta_1, \beta_2, \beta_3, \beta_4$ such that for all $x, y \in X$:*

(a) $\beta_1 + \beta_2 + \beta_3 + 2\beta_4 < 1$
(b)
$$\begin{aligned} & \sqrt[n]{d(Sx,Sy)} \\ \leq\ & \beta_1\sqrt[n]{d(x,y)} + \beta_2\sqrt[n]{d(x,Sx)} + \beta_3\sqrt[n]{d(y,Sy)} + \beta_4\sqrt[n]{d(x,Sy)+d(y,Sx)} \end{aligned}$$

then S has a unique fixed point.

Now, let us introduce the following example which shows and support our deduced results.

Example 2.18. *Consider the sequence*
$S_1 = 1 \times 2$
$S_2 = 1 \times 2 + 2 \times 3$
$S_3 = 1 \times 2 + 2 \times 3 + 3 \times 4$
$S_n = 1 \times 2 + 2 \times 3 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$.

Let $X = \{S_n : n \in \mathbb{N}\}$ and $d(x,y) = |x-y|$. Then (X,d) is a complete metric space. Define the mapping $T : X \to X$ by,

$$T(S_1) = S_1, \quad T(S_n) = S_{n-1}, \quad \text{for all } n \geq 2.$$

Clearly, the Banach contraction is not satisfied. In fact, we can check easily that

$$\lim_{n\to\infty}\frac{d(T(S_n),T(S_1))}{d(S_n,S_1)} = \lim_{n\to\infty}\frac{d(S_{n-1},S_1)}{d(S_n,S_1)} = \lim_{n\to\infty}\frac{(n-1)n(n+1)-6}{n(n+1)(n+2)-6} = 1.$$

Let us consider the mapping $\psi : (0,\infty) \to (1,\infty)$ defined by $\psi(t) = e^{\sqrt{te^t}}$. Then it is easy to show that $\psi \in \Psi$. We shall prove that T satisfies the condition of the result 2.10, that is,

$$d(T(S_n),T(S_m)) \neq 0 \Longrightarrow e^{\sqrt{d(T(S_n),T(S_m))e^{d(T(S_n),T(S_m))}}} \leq e^{\beta\sqrt{d(S_n,S_m)e^{d(S_n,S_m)}}}$$

for some $\beta \in (0,1)$. The above condition is equivalent to

$$d(T(S_n),T(S_m)) \neq 0 \Longrightarrow d(T(S_n),T(S_m))e^{d(T(S_n),T(S_m))} \leq \beta^2 d(S_n,S_m)e^{d(S_n,S_m)}.$$

So, we have to check that

$$d(T(S_n),T(S_m)) \neq 0 \Longrightarrow \frac{d(T(S_n),T(S_m))e^{d(T(S_n),T(S_m))-d(S_n,S_m)}}{d(S_n,S_m)} \leq \beta^2.$$

We consider two cases.

Case 01: *For* $1 = n < m$, *we have*

$$|T(S_m) - T(S_1)| = |S_{m-1} - S_1| = 2 \times 3 + \ldots + (m-1)(m)$$

and

$$d(S_m, S_1) = |S_m - S_1| = 2 \times 3 + \ldots + (m)(m+1).$$

Thus

$$\frac{d(T(S_m), T(S_n))e^{d(T(S_m),T(S_n))-d(S_m,S_n)}}{d(S_m, S_n)} = \frac{e^{-m(m+1)}}{m(m+1)} \leq e^{-1}.$$

Case 02*: For* $m > n > 1$, *we have*

$$|T(S_m) - T(S_n)| = (2n-1)(2n) + (2n+1)(2n+2) + ... + (2m-3)(2m-2)$$

and

$$|S_m - S_n| = (2n+1)(2n+2) + (2n+3)(2n+4) + ... + (2m-1)(2m).$$

Since $m > n > 1$, *we have*

$$\frac{d(T(S_m), T(S_n))e^{d(T(S_m),T(S_n))-d(S_m,S_n)}}{d(S_m, S_n)} = \frac{(2n-1)(2n)e^{(2n-1)(2n)-(2m-1)(2m)}}{(2m-1)(2m)} \leq e^{-1}$$

Hence all the conditions of result (2.10) are satisfied and S_1 *is a unique fixed point of* T.

Competing interests

The authors declare that they have no competing interests.

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