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# ON THE LIE DERIVATIVE OF FORMS OF BIDEGREE

BUI CAO VAN

Abstract. The aim of this work is to study some properties of the Lie derivative of forms of bidegree on a $n-$dimensional complex manifold $M$. More precisely, we give sufficient conditions such that the Lie derivative $L_X\omega$ of bidegree $(p,p)$ form $\omega$ along the vector field $X$ is also the bidegree $(p,p)$ form.

## 1. Introduction

The Lie derivative on differential forms is important operation. This is a generalization of the notion of directional derivative of a function. The Lie differentiation theory plays an important role in studying automorphisms of differential geometric structures. Moreover, the Lie derivative also is an essential tool in the Riemannian geometry. The Lie derivative of forms and its application was investigated by many authors (see [5], [6], [7], [8], [11], [12], [13] and the references given therein). Recently, the authors of [1] constructed the Lie derivative of the real currents on Riemann manifolds and given some applications on Lie groups. The main goal of the present work is to investigate some properties of the Lie derivative of differential forms of bidegree. More precisely, we shall give a sufficient conditions such that the Lie derivative $L_X\omega$ of bidegree $(p,p)$ form $\omega$ along vector field $X$ is also the bidegree $(p,p)$ form (Theorem 3.5). We would like to emphasize that our interest for studying the Lie derivative of forms of bidegree stems from the ideas for construction of Lie derivative of the currents of bidegree. It will be useful in studying of the pluripotential theory.

## 2. Preliminaries

Let $M$ be a $n$-dimensional complex manifold and let $(U,\{z_1,z_2,...,z_n\}), z_j = x_j+iy_j, j=\overline{1,n}$ be local complex coordinates on an open $U\subset M$. If we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ then the linear forms $dx_j$, $dy_j$ can be written in a unique way as linear combinations with complex coefficients of $dz_j = dx_j + idy_j$ and $d\overline{z_j} = dx_j - idy_j$. For each $j = 1,...,n$, we use operators

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\Big);\ \ \frac{\partial}{\partial \overline{z}_j} := \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\Big),$$

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the second operator is called Cauchy-Riemann operator. Let $I = \{j_1, ..., j_r\} \subset \{1, ..., n\}$ be multi-index. We denote $dz_I = dz_{j_1} \wedge ... \wedge dz_{j_r}$, $d\bar{z}_I = d\bar{z}_{j_1} \wedge ... \wedge d\bar{z}_{j_r}$. Then any $k$-form $\omega$ on $M$ can be written locally in a unique way as

$$\omega = \sum_{|J|+|K|=k} \varphi_{JK} dz_J \wedge d\bar{z}_K,$$

where $\varphi_{JK} \in C^\infty(M, \mathbb{C})$ are functions with complex values. We say that $\omega$ is *a form of bidegree* $(p, q)$ if $\varphi_{JK} = 0$ when either $|J| \neq p$ and $|K| \neq q$.

Note that, every differential form of degree $k$ can be written uniquely as sum of differential forms of types $(p, q)$, where $p + q = k$. We denote by $\Omega^{(p,q)}(M, \mathbb{C})$ the space of differential forms of bidegree (p, q) on $M$ and $\Omega^k(M, \mathbb{C}) = \bigoplus\limits_{p+q=k} \Omega^{(p,q)}(M, \mathbb{C})$. If $\omega \in \Omega^{(p,q)}(M, \mathbb{C})$ then $\omega$ can be written as

$$\omega = \sum_{|J|=p,\, |K|=q} \omega_{JK} dz_J \wedge d\bar{z}_K.$$

Next, we need recalling the differential operators. For each differential $(p, q)-$form $\omega$, we set

$$\partial\omega = \sum_{|J|=p,\, |K|=q} \partial\varphi_{JK} \wedge dz_J \wedge d\bar{z}_K = \sum_{|J|=p,\, |K|=q} \sum_{j=1}^{n} \frac{\partial\varphi_{IJ}}{\partial z_j} dz_j \wedge dz_J \wedge d\bar{z}_K;$$

$$\overline{\partial}\omega = \sum_{|J|=p,\, |K|=q} \overline{\partial}\varphi_{JK} \wedge dz_J \wedge d\bar{z}_K = \sum_{|J|=p,\, |K|=q} \sum_{j=1}^{n} \frac{\partial\varphi_{IJ}}{\partial \bar{z}_j} d\bar{z}_j \wedge dz_J \wedge d\bar{z}_K,$$

and

$$d\omega = \sum_{|J|=p, |K|=q} d\varphi_{JK} \wedge dz_J \wedge d\bar{z}_K.$$

It is easy to see that if $\omega$ is a form of bidegree of $(p, q)$ then $\partial\omega$ and $\overline{\partial}\omega$ are forms of bidegree of $(p+1, q)$ and $(p, q+1)$ respectively. As in the case of real differential forms, if the degree of $\omega$ is positive, i.e if $\omega$ is not a scalar valued function, then the operator $d$ of exterior differentiation has to be distinguished from the differential $d$. In particular, the former satisfies $d^2 = dd = 0$. Furthermore, $d = \partial + \overline{\partial}$ and hence

$$0 = d^2\omega = \partial^2\omega + \left(\partial\overline{\partial} + \overline{\partial}\partial\right)\omega + \overline{\partial}^2\omega$$

Since all three terms are of different types, we conclude that $\partial^2 = 0; \overline{\partial}^2 = 0; \partial\overline{\partial} = -\overline{\partial}\partial$

Another important, differential operator that will be used the paper is the operator $d^c$ defined by $d^c = i\left(\overline{\partial} - \partial\right)$. Note that $dd^c = 2i\partial\overline{\partial}$ and that, if $u \in \mathcal{C}^2(U)$, then

$$dd^c u = 2i \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k.$$

The set of vector fields on M is denoted by $\mathfrak{B}_{\mathbb{C}}(M)$ and the set of holomorphic functions on M is denoted by $\mathcal{O}(M)$. A vector field $X = (X_1, ..., X_n)$ is called a holomorphic vector field if the functions $X_j : M \to \mathbb{C}$ are holomorphic, $X_j \in \mathcal{O}(M)$. The set of holomorphic vector fields on $M$ is denoted by $\mathfrak{B}_{hol}^{(1,0)}(M)$. We denote set of holomorphic p-forms on $M$ by let $\Omega_{hol}^p(M)$, whose coordinate functions are holomorphic; that is, if $U \subseteq M$ is a coordinate neighborhood, with holomorphic local coordinates $(z_1, z_2, ..., z_n)$ then $\omega \in \Omega^p_{hol}(M)$ implies

$$\omega = \sum_{1 \leq i_1 \leq ... \leq i_p \leq n} \omega_{i_1 ... i_p} dz_{i_1} \wedge dz_{i_2} \wedge ... \wedge dz_{i_p}, \omega_{i_1 ... i_p} \in \mathcal{O}(M).$$

$f \in \Omega^0_{hol}(M) = \mathcal{O}(M)$ if and only if $\overline{\partial} f = 0$. More generally,

$$\omega \in \Omega^p_{hol}(M) \Leftrightarrow \omega \in \Omega^{(p,0)}(M) \cap \ker \overline{\partial}.$$

Next, we recall the concept and basis properties of Lie derivative of forms along vector field $X$.

**Definition 2.1.** Suppose that $X \in \mathfrak{B}_{\mathbb{C}}(M), \omega \in \Omega^k(M, \mathbb{C})$ and let $\{\varphi_t\}$ be an one-parameter group of transformations on M generated by $X$. The map

$$\begin{aligned} L_X\omega : M &\to \bigwedge^k T_pM \\ p &\mapsto (L_X\omega)_p \end{aligned}$$

is called the Lie derivative of $\omega$ with respect to $X$ and is denoted by $L_X\omega$, where $(L_X\omega)_p$ is defined by:

$$(L_X\omega)_p = \lim_{t \to 0} \frac{(\varphi_t)^*\omega_p - \omega_p}{t} = \frac{d}{dt}\Big((\varphi_t)^*\omega_p\Big)\Big|_{t=0}, \forall p \in M,$$

where $(\varphi_t)^*\omega$ is the pull-back of $\omega$ along $\varphi_t$.

For $f \in C^\infty(M, \mathbb{C}), \varphi_t^* f = f o \varphi_t$ also belongs to $C^\infty(M, \mathbb{C})$. Then the Lie derivative of $f$ with respect to $X$, is denoted by $L_X f$ and is defined by:

$$\begin{aligned} (L_X f)_x &= \lim_{t \to 0} \frac{f(\varphi_t(x)) - f(x)}{t} = \lim_{t \to 0} \frac{f(\varphi(x,t)) - f(x)}{t} \\ &= \lim_{t \to 0} \frac{f(\varphi_x(t)) - f(\varphi_x(0))}{t} = \frac{d}{dt}(\varphi_x f)|_{t=0} = X_x[f] = (X[f])(x) \end{aligned}$$

Obviously, $L_X : \Omega^k(M, \mathbb{C}) \to \Omega^k(M, \mathbb{C})$ is a linear map.

We now state a number of properties of Lie derivatives without proofs. Most of these proofs are fairly straightforward computations, often tedious, and can be found in most texts, including Warner [14], Morita [15] and Gallot, Hullin and Lafontaine [16].

**Proposition 2.2.** *For every vector field $X \in \mathfrak{B}_{\mathbb{C}}(M)$, the following properties hold:*

i) *For all $\omega \in \Omega^k(M, \mathbb{C})$ and all $\mu \in \Omega^r(M, \mathbb{C})$,*

$$L_X(\omega \wedge \mu) = (L_X\omega) \wedge \mu + \omega \wedge (L_X\mu),$$

*that is, $L_X$ is a derivation.*

ii) *For all $\omega \in \Omega^k(M, \mathbb{C})$, for all $X, X_1, X_2, ..., X_k \in \mathfrak{B}_{\mathbb{C}}(M)$,*

$$(L_X\omega)(X_1, ..., X_k) = L_X(\omega(X_1, ..., X_k)) - \sum_{i=1}^k \omega(X_1, ..., L_X X_i, ..., X_k).$$

iii) *The Lie derivative commutes with d: $L_X \circ d = d \circ L_X$.*

**Definition 2.3.** For every vector field $X \in \mathfrak{B}_{\mathbb{C}}(M)$, for all $k \geq 1$, there is a linear map, $i_X : \Omega^{(k)}(M, \mathbb{C}) \to \Omega^{k-1}(M, \mathbb{C})$, defined so that, for all $\omega \in \Omega^k(M, \mathbb{C})$, for all $p \in M$, for all $u_1, u_2, ..., u_{k-1} \in T_pM$,

$$(i_X\omega)_p(u_1, u_2, ..., u_{k-1}) = \omega_p(X_p, u_1, u_2, ..., u_{k-1}).$$

Next, we give some more interested properties of $L_X$. For this, we have to define the interior multiplication by a vector field $i_X$.

Obviously, $i_X$ is $C^\infty(M, \mathbb{C})$-linear in $M$ and it is easy to check that $i_X\omega$ is indeed a smooth $(k-1)$-form. When $k = 0$, we set $i_X\omega = 0$. Observe that $i_X\omega$ is also given by $(i_X\omega)_p = i_{X_p}\omega_p, \forall p \in M$, where $i_{X_p}$ is the interior product (or insertion operator). As a consequence, the operator $i_X$ is an anti-derivation of degree -1, that is, we have $i_X^2 = 0; i_Xi_Y + i_Yi_X = 0; i_X(\omega\wedge\mu) = i_X\omega\wedge\mu + (-1)^k\omega\wedge i_X\mu; i_X(\omega+\mu) = i_X\omega + i_X\mu; i_{X+Y}\omega = i_X\omega + i_Y\omega; i_Xd\varphi = L_X\varphi; i_{\varphi X}\omega = \varphi i_X\omega; i_X(\varphi\omega) = \varphi i_X\omega$, for all $X, Y \in \mathfrak{B}_{\mathbb{C}}(M)$, for all $\omega, \mu \in \Omega^k(M, \mathbb{C})$, for all $\varphi \in C^\infty(M, \mathbb{C})$.

The following proposition is the well known result.

**Theorem 2.4.** *(Cartan's formula) For every vector field $X \in \mathfrak{B}_{\mathbb{C}}(M)$ and for every $\omega \in \Omega^k(M, \mathbb{C})$, we have*

$$L_X\omega = di_X\omega + i_Xd\omega,$$

*that is, $L_X = doi_X + i_Xod$.*

## 3. The Lie derivative of differential form of bidegree

Let $\omega \in \Omega^{(p,p)}(M, \mathbb{C})$. In general case, $L_X\omega \notin \Omega^{(p,p)}(M, \mathbb{C})$, the illustrated example will be mentioned late. Our main goal is to find conditions on $X$ such that $L_X\omega$ belongs to $\Omega^{(p,p)}(M, \mathbb{C})$.

The following theorem is a key for proving the main result of this section.

**Theorem 3.1.** *Let $\omega = \sum_{j,k=1}^{n} \varphi_{jk}dz_j \wedge d\overline{z}_k \in \Omega^{(1,1)}(M, \mathbb{C}), X = (X_1, X_2, ..., X_n) \in \mathfrak{B}_{\mathbb{C}}(M)$. Then we have*

$$L_X\omega = \sum_{j,k=1}^{n} \left[X\left[\varphi_{jk}\right]dz_j \wedge d\overline{z}_k + \varphi_{jk}\left(dX_j \wedge d\overline{z}_k - d\overline{X}_k \wedge dz_j\right)\right]. \tag{3.1}$$

*Proof.* For every $Y \in \mathfrak{B}_{\mathbb{C}}(M)$, we have

$$\begin{aligned}
(i_X\omega)(Y) = \omega(X, Y) &= \sum_{j,k=1}^{n} \varphi_{jk}dz_j \wedge d\overline{z}_k\,(X, Y) = \sum_{j,k=1}^{n} \varphi_{jk} \begin{vmatrix} dz_j(X) & dz_j(Y) \\ d\overline{z}_k(X) & d\overline{z}_k(Y) \end{vmatrix} \\
&= \sum_{j,k=1}^{n} \varphi_{jk} \begin{vmatrix} X_j & Y_j \\ \overline{X}_k & \overline{Y}_k \end{vmatrix} = \sum_{j,k=1}^{n} \varphi_{jk}\left(X_j\overline{Y}_k - \overline{X}_kY_j\right) \\
&= \sum_{j,k=1}^{n} \varphi_{jk}\left(X_jd\overline{z}_k(Y) - \overline{X}_kdz_j(Y)\right) \\
&= \left(\sum_{j,k=1}^{n} \varphi_{jk}\left(X_jd\overline{z}_k - \overline{X}_kdz_j\right)\right)(Y)\,, \forall Y \in \mathfrak{B}(M).
\end{aligned}$$

Thus $i_X\omega = \sum\limits_{j,k=1}^{n} \varphi_{jk}\left(X_j d\bar{z}_k - \overline{X}_k dz_j\right)$. Hence

$$
\begin{aligned}
d(i_X\omega) &= \sum_{j,k=1}^{n} \left(d(\varphi_{jk}X_j)\wedge d\bar{z}_k - d(\varphi_{jk}\overline{X}_k)\wedge dz_j\right)\\
&= \sum_{j,k=1}^{n}\sum_{l=1}^{n}\left[\frac{\partial(\varphi_{jk}X_j)}{\partial z_l}dz_l + \frac{\partial(\varphi_{jk}X_j)}{\partial \bar{z}_l}d\bar{z}_l\right]\wedge d\bar{z}_k\\
&\qquad - \sum_{j,k=1}^{n}\sum_{l=1}^{n}\left[\frac{\partial\left(\varphi_{jk}\overline{X}_k\right)}{\partial z_l}dz_l + \frac{\partial\left(\varphi_{jk}\overline{X}_k\right)}{\partial \bar{z}_l}d\bar{z}_l\right]\wedge dz_j\\
&= \sum_{j,k=1}^{n}\left[\sum_{l=1}^{n}\frac{X_j\partial\varphi_{jk}}{\partial z_l}dz_l\wedge d\bar{z}_k + \sum_{l=1}^{n}\frac{\varphi_{jk}\partial X_j}{\partial z_l}dz_l\wedge d\bar{z}_k + \sum_{l=1}^{n}\frac{X_j\partial\varphi_{jk}}{\partial \bar{z}_l}d\bar{z}_l\wedge d\bar{z}_k + \right.\\
&\qquad + \sum_{l=1}^{n}\frac{\varphi_{jk}\partial X_j}{\partial \bar{z}_l}d\bar{z}_l\wedge d\bar{z}_k - \sum_{l=1}^{n}\frac{\overline{X}_k\partial\varphi_{jk}}{\partial z_l}dz_l\wedge dz_j - \sum_{l=1}^{n}\frac{\varphi_{jk}\partial\overline{X}_k}{\partial z_l}dz_l\wedge dz_j -\\
&\qquad \left. - \sum_{l=1}^{n}\frac{\overline{X}_k\partial\varphi_{jk}}{\partial \overline{z_l}}d\overline{z_l}\wedge dz_j - \sum_{l=1}^{n}\frac{\varphi_{jk}\partial\overline{X}_k}{\partial z_l}d\overline{z_l}\wedge dz_j\right].
\end{aligned}
\tag{3.2}
$$

On other hand, for every vector fields $Y, Z \in \mathfrak{B}_{\mathbb{C}}(M)$, we have

$$
\begin{aligned}
&(i_X d\omega)(Y,Z) = d\omega(X,Y,Z) = \sum_{j,k=1}^{n} d\varphi_{jk}\wedge dz_j\wedge d\bar{z}_k\,(X,Y,Z)\\
&= \sum_{j,k=1}^{n}\begin{vmatrix} d\varphi_{jk}(X) & d\varphi_{jk}(Y) & d\varphi_{jk}(Z)\\ dz_j(X) & dz_j(Y) & dz_j(Z)\\ d\bar{z}_k(X) & d\bar{z}_k(Y) & d\bar{z}_k(Z)\end{vmatrix} = \sum_{j,k=1}^{n}\begin{vmatrix} X[\varphi_{jk}] & Y[\varphi_{jk}] & Z[\varphi_{jk}]\\ X_j & Y_j & Z_j\\ \overline{X}_k & \overline{Y}_k & \overline{Z}_k\end{vmatrix}\\
&= \sum_{j,k=1}^{n}\left[X[\varphi_{jk}]\begin{vmatrix} Y_j & Z_j\\ \overline{Y}_k & \overline{Z}_k\end{vmatrix} - X_j\begin{vmatrix} Y[\varphi_{jk}] & Z[\varphi_{jk}]\\ \overline{Y}_k & \overline{Z}_k\end{vmatrix} + \overline{X}_k\begin{vmatrix} Y[\varphi_{jk}] & Z[\varphi_{jk}]\\ Y_j & Z_j\end{vmatrix}\right]\\
&= \sum_{j,k=1}^{n}\left[X[\varphi_{jk}]\begin{vmatrix} dz_j & dz_j\\ d\bar{z}_k & d\bar{z}_k\end{vmatrix} - X_j\begin{vmatrix} d\varphi_{jk} & d\varphi_{jk}\\ d\bar{z}_k & d\bar{z}_k\end{vmatrix} + \overline{X}_k\begin{vmatrix} d\varphi_{jk} & d\varphi_{jk}\\ dz_j & dz_j\end{vmatrix}\right](Y,Z)\\
&= \left[\sum_{j,k=1}^{n}\left(X[\varphi_{jk}]\,dz_j\wedge d\bar{z}_k - X_j d\varphi_{jk}\wedge d\bar{z}_k + \overline{X}_k d\varphi_{jk}\wedge dz_j\right)\right](Y,Z),\forall Y,Z\in\mathfrak{B}_{\mathbb{C}}(M).
\end{aligned}
$$

Thus

$$\begin{aligned}
i_X d\omega = & \sum_{j,k=1}^{n} \left( X\left[\varphi_{jk}\right] dz_j \wedge d\bar{z}_k - X_j d\varphi_{jk} \wedge d\bar{z}_k + \overline{X}_k d\varphi_{jk} \wedge dz_j \right) \\
= & \sum_{j,k=1}^{n} \left[ \sum_{l=1}^{n} \left( \frac{\partial \varphi_{jk}}{\partial z_l} X_l + \frac{\partial \varphi_{jk}}{\partial \bar{z}_l} \overline{X_l} \right) dz_l \wedge d\bar{z}_k - X_j \sum_{l=1}^{n} \left( \frac{\partial \varphi_{jk}}{\partial z_l} dz_l + \frac{\partial \varphi_{jk}}{\partial \bar{z}_l} d\bar{z}_l \right) \wedge d\bar{z}_k + \right. \\
& \left. + \overline{X_k} \sum_{l=1}^{n} \left( \frac{\partial \varphi_{jk}}{\partial z_l} dz_l + \frac{\partial \varphi_{jk}}{\partial \bar{z}_l} d\bar{z}_l \right) \wedge dz_j \right] \\
= & \sum_{j,k=1}^{n} \left[ \sum_{l=1}^{n} \frac{X_l \partial \varphi_{jk}}{\partial z_l} dz_l \wedge d\bar{z}_k + \overline{X}_l \sum_{l=1}^{n} \frac{\partial \varphi_{jk}}{\partial \bar{z}_l} dz_j \wedge d\bar{z}_k - \sum_{l=1}^{n} \frac{X_j \partial \varphi_{jk}}{\partial z_l} dz_l \wedge d\bar{z}_k - \right. \\
& \left. - \sum_{l=1}^{n} \frac{X_j \partial \varphi_{jk}}{\partial \bar{z}_l} d\bar{z}_l \wedge d\bar{z}_k + \sum_{l=1}^{n} \frac{\overline{X}_k \partial \varphi_{jk}}{\partial z_l} dz_l \wedge dz_j + \sum_{l=1}^{n} \frac{\overline{X}_k \partial \varphi_{jk}}{\partial \bar{z}_l} d\bar{z}_l \wedge dz_j \right].
\end{aligned} \tag{3.3}$$

From (3.2), (3.3) and applying Cartan's formula, we obtain

$$\begin{aligned}
L_X\omega = & d(i_X\omega) + i_X(d\omega) \\
= & \sum_{j,k=1}^{n} \left[ \sum_{l=1}^{n} \frac{X_j \partial \varphi_{jk}}{\partial z_l} dz_l \wedge d\bar{z}_k + \sum_{l=1}^{n} \frac{\varphi_{jk} \partial X_j}{\partial z_l} dz_l \wedge d\bar{z}_k + \sum_{l=1}^{n} \frac{X_j \partial \varphi_{jk}}{\partial \bar{z}_l} d\bar{z}_l \wedge d\bar{z}_k + \right. \\
& + \sum_{l=1}^{n} \frac{\varphi_{jk} \partial X_j}{\partial \bar{z}_l} d\bar{z}_l \wedge d\bar{z}_k - \sum_{l=1}^{n} \frac{\overline{X}_k \partial \varphi_{jk}}{\partial z_l} dz_l \wedge dz_j - \sum_{l=1}^{n} \frac{\varphi_{jk} \partial \overline{X}_k}{\partial z_l} dz_l \wedge dz_j - \\
& - \sum_{l=1}^{n} \frac{\bar{X}_k \partial \varphi_{jk}}{\partial \overline{z_l}} d\overline{z_l} \wedge dz_j - \sum_{l=1}^{n} \frac{\varphi_{jk} \partial \bar{X}_k}{\partial z_l} d\overline{z_l} \wedge dz_j + \sum_{l=1}^{n} \frac{X_l \partial \varphi_{jk}}{\partial z_l} dz_l \wedge d\bar{z}_k + \\
& + \bar{X}_l \sum_{l=1}^{n} \frac{\partial \varphi_{jk}}{\partial \bar{z}_l} dz_j \wedge d\bar{z}_k - \sum_{l=1}^{n} \frac{X_j \partial \varphi_{jk}}{\partial z_l} dz_l \wedge d\bar{z}_k - \\
& \left. - \sum_{l=1}^{n} \frac{X_j \partial \varphi_{jk}}{\partial \bar{z}_l} d\bar{z}_l \wedge d\bar{z}_k + \sum_{l=1}^{n} \frac{\overline{X}_k \partial \varphi_{jk}}{\partial z_l} dz_l \wedge dz_j + \sum_{l=1}^{n} \frac{\overline{X}_k \partial \varphi_{jk}}{\partial \bar{z}_l} d\bar{z}_l \wedge dz_j \right] \\
= & \sum_{j,k=1}^{n} \left[ \sum_{l=1}^{n} \left( X_l \frac{\partial \varphi_{jk}}{\partial z_l} + \overline{X_l} \frac{\partial \varphi_{jk}}{\partial \bar{z}_l} \right) dz_j \wedge d\bar{z} + \sum_{l=1}^{n} \varphi_{jk} \left( \frac{\partial X_j}{\partial z_l} dz_l + \frac{\partial X_j}{\partial \bar{z}_l} d\overline{z_l} \right) \wedge d\bar{z}_k \right. \\
& \left. - \varphi_{jk} \sum_{l=1}^{n} \left( \frac{\partial \overline{X_k}}{\partial z_l} dz_l + \frac{\partial \overline{X_k}}{\partial \bar{z}_l} d\bar{z}_l \right) \wedge dz_j \right] \\
= & \sum_{j,k=1}^{n} \left[ X\left[\varphi_{jk}\right] dz_j \wedge d\bar{z}_k + \varphi_{jk} \left( dX_j \wedge d\bar{z}_k - d\overline{X}_k \wedge dz_j \right) \right].
\end{aligned}$$

□

We easily get the following corollary.

**Corollary 3.2.** *Let* $X = (X_1, X_2, ..., X_n) \in \mathfrak{B}_{hol}^{(1,0)}(M)$ *be a holomorphic vector field on* $M$ *and* $\omega = \sum_{j,k=1}^{n} \varphi_{jk} dz_j \wedge d\bar{z}_k \in \Omega^{(1,1)}(M, \mathbb{C})$. *Then* $L_X\omega \in \Omega^{(1,1)}(M, \mathbb{C})$

*and $L_X\omega$ is defined by formula:*

$$L_X\omega = \sum_{j,k=1}^{n}\left[L_X\varphi_{jk}dz_j\wedge d\bar{z}_k + \sum_{l=1}^{n}\varphi_{jk}\frac{\partial X_j}{\partial z_l}dz_l\wedge d\bar{z}_k - \sum_{l=1}^{n}\varphi_{jk}\frac{\partial \bar{X}_k}{\partial \bar{z}_l}d\bar{z}_l\wedge dz_j\right]. \tag{3.4}$$

*Proof.* Since $X = (X_1, X_2, ..., X_n) \in \mathfrak{B}^{(1,0)}_{hol}(M)$ be a holomorphic vector field on $M$, thus $\frac{\partial X_j}{\partial \bar{z}_l} = 0; \frac{\partial \bar{X}_k}{\partial z_l} = 0, \forall j,k,l = \overline{1,n}$. Hence, applying Theorem 3.1, we have

$$\begin{aligned}
L_X\omega &= \sum_{j,k=1}^{n}\left[X\left[\varphi_{jk}\right]dz_j\wedge d\overline{z}_k + \varphi_{jk}\left(dX_j\wedge d\overline{z}_k - d\overline{X}_k\wedge dz_j\right)\right] \\
&= \sum_{j,k=1}^{n}\Bigg[X\left[\varphi_{jk}\right]dz_j\wedge d\overline{z}_k + \varphi_{jk}\Bigg(\sum_{l=1}^{n}\frac{\partial X_j}{\partial z_l}dz_l\wedge d\overline{z} + \sum_{l=1}^{n}\frac{\partial X_j}{\partial \overline{z}_l}d\overline{z}_l\wedge d\overline{z}_k - \\
&\qquad - \sum_{l=1}^{n}\frac{\partial \bar{X}_k}{\partial z_l}dz_l\wedge dz_j - \sum_{l=1}^{n}\frac{\partial \bar{X}_k}{\partial \bar{z}_l}d\bar{z}_l\wedge dz_j\Bigg)\Bigg] \\
&= \sum_{j,k=1}^{n}\left[L_X\varphi_{jk}dz_j\wedge d\overline{z}_k + \sum_{l=1}^{n}\varphi_{jk}\frac{\partial X_j}{\partial z_l}dz_l\wedge d\bar{z}_k - \sum_{l=1}^{n}\varphi_{jk}\frac{\partial \bar{X}_k}{\partial \bar{z}_l}d\bar{z}_l\wedge dz_j\right].
\end{aligned}$$

So that $L_X\omega \in \Omega^{(1,1)}(M,\mathbb{C})$. □

**Example 3.3.** Let $M = \mathbb{C}^2, \omega = z_1^2 dz_1\wedge d\overline{z}_1, X = (z_1, z_2)$. Calculate the Lie derivative of differential form $\omega$?

Applying Theorem 3.1, we obtain

$$\begin{aligned}
L_X\omega &= \sum_{j,k=1}^{2}\left[X\left[\varphi_{jk}\right]dz_j\wedge d\overline{z}_k + \varphi_{jk}\left(dX_j\wedge d\overline{z}_k - d\overline{X}_k\wedge dz_j\right)\right] \\
&= X\left[z_1^2\right]dz_1\wedge d\overline{z}_1 + z_1^2\left(dX_1\wedge d\overline{z}_1 - d\bar{X}_1\wedge dz_1\right) \\
&= 2z_1^2 dz_1\wedge d\overline{z}_1 + z_1^2\left(dz_1\wedge d\overline{z}_1 - d\overline{z}_1\wedge dz_1\right) = 4z_1^2 dz_1\wedge d\overline{z}_1
\end{aligned}$$

**Example 3.4.** Let $M = \mathbb{C}^2, \omega = z_1 z_2 dz_2\wedge d\overline{z}_2, X = (\overline{z}_2, \overline{z}_1)$. Calculate the Lie derivative of differential form $\omega$?

Note that $X$ is not a holomorphic vector field. Applying Theorem 3.1, we obtain

$$\begin{aligned}
L_X\omega &= \sum_{j,k=1}^{2}\left[X\left[\varphi_{jk}\right]dz_j\wedge d\overline{z}_k + \varphi_{jk}\left(dX_j\wedge d\overline{z}_k - d\overline{X}_k\wedge dz_j\right)\right] \\
&= X\left[z_1 z_2\right]dz_2\wedge d\overline{z}_2 + z_1 z_2\left(dX_2\wedge d\overline{z}_2 - d\bar{X}_2\wedge dz_2\right) \\
&= \sum_{l=1}^{2}\left[\frac{\partial\left(z_1 z_2\right)}{\partial z_l}X_l + \frac{\partial\left(z_1 z_2\right)}{\partial \overline{z_l}}\overline{X_l}\right]dz_2\wedge d\bar{z}_2 + z_1 z_2\left(d\overline{z_1}\wedge d\bar{z}_2 - dz_1\wedge dz_2\right) \\
&= \left(z_2 X_1 + 0\right)dz_2\wedge d\bar{z}_2 + \left(z_1 X_2 + 0\right)dz_2\wedge d\bar{z}_2 + z_1 z_2 d\overline{z_1}\wedge d\bar{z}_2 - z_1 z_2 dz_1\wedge dz_2 \\
&= \left(\left|z_1\right|^2 + \left|z_2\right|^2\right)dz_2\wedge d\bar{z}_2 + z_1 z_2 d\bar{z}_1\wedge d\bar{z}_2 - z_1 z_2 dz_1\wedge dz_2.
\end{aligned}$$

This shows that

$$L_X\omega \notin \Omega^{(1,1)}(\mathbb{C}^2,\mathbb{C}).$$

Now, we state the main result of this section.

**Theorem 3.5.** *If $X = (X_1, X_2, ..., X_n) \in \mathfrak{B}^{(1,0)}_{hol}(M)$ is a holomorphic vector field on $M$ and $\omega = \sum\limits_{|J|=|K|=p} \varphi_{JK} dz_J \wedge d\bar{z}_K \in \Omega^{(p,p)}(M, \mathbb{C})$ be the differential forms of bidegree $(p, p)$ then $L_X\omega \in \Omega^{(p,p)}(M, \mathbb{C})$.*

*Proof.* We will prove Theorem 3.5 by induction. The case n = 1 is easily followed by Corollary 3.2. Assume that there is a $k \geq 1$ such that $L_X\omega \in \Omega^{(k,k)}(M, \mathbb{C}), \forall \omega \in \Omega^{(k,k)}(M, \mathbb{C})$. We shall prove that the formula

$$L_X\omega \in \Omega^{(k+1,k+1)}(M, \mathbb{C}), \forall \omega \in \Omega^{(k+1,k+1)}(M, \mathbb{C}).$$

Indeed, let

$$\begin{aligned}\omega &= \sum_{|J|=k+1} \varphi_{j_1.j_2...j_{k+1}} dz_{j_1} \wedge dz_{j_2} \wedge ... \wedge dz_{j_k} \wedge dz_{j_{k+1}} \wedge d\bar{z}_{j_1} \wedge d\bar{z}_{j_2} \wedge ... \wedge d\bar{z}_{j_k} \wedge d\bar{z}_{j_{k+1}} \\ &= (-1)^k \sum_{|J|=k+1} \varphi_{j_1.j_2...j_{k+1}} dz_{j_1} \wedge dz_{j_2} \wedge ... \wedge dz_{j_k} \wedge d\bar{z}_{j_1} \wedge d\bar{z}_{j_2} \wedge ... \wedge d\bar{z}_{j_k} \wedge \left(dz_{j_{k+1}} \wedge d\bar{z}_{j_{k+1}}\right),\end{aligned}$$

where $J = (1 \leq j_1 \leq ... \leq j_{k+1} \leq n)$. We have

$$\begin{aligned}L_X\omega &= (-1)^k \sum_{|J|=k+1} \left[L_X \left(\left(\varphi_{j_1.j_2...j_{k+1}} dz_{j_1} \wedge ... \wedge dz_{j_k} \wedge d\bar{z}_{j_1} \wedge ... \wedge d\bar{z}_{j_k}\right) \wedge \left(dz_{j_{k+1}} \wedge d\bar{z}_{j_{k+1}}\right)\right)\right] \\ &= (-1)^k \sum_{|J|=k+1} \left[\left(L_X \left(\varphi_{j_1.j_2...j_{k+1}} dz_{j_1} \wedge ... \wedge dz_{j_k} \wedge d\bar{z}_{j_1} \wedge ... \wedge d\bar{z}_{j_k}\right)\right) \wedge \left(dz_{j_{k+1}} \wedge d\bar{z}_{j_{k+1}}\right) + \right. \\ &\qquad \left. + \left(\varphi_{j_1.j_2...j_{k+1}} dz_{j_1} \wedge ... \wedge dz_{j_k} \wedge d\bar{z}_{j_1} \wedge ... \wedge d\bar{z}_{j_k}\right) \wedge L_X \left(dz_{j_{k+1}} \wedge d\bar{z}_{j_{k+1}}\right)\right].\end{aligned}$$

It follows that

$$L_X \left(\varphi_{j_1.j_2...j_{k+1}} dz_{j_1} \wedge ... \wedge dz_{j_k} \wedge d\bar{z}_{j_1} \wedge ... \wedge d\bar{z}_{j_k}\right) \in \Omega^{(k,k)}(M, \mathbb{C})$$

,

$$dz_{j_{k+1}} \wedge d\bar{z}_{j_{k+1}} \in \Omega^{(1,1)}(M, \mathbb{C}),$$

$$\varphi_{j_1.j_2...j_{k+1}} dz_{j_1} \wedge ... \wedge dz_{j_k} \wedge d\bar{z}_{j_1} \wedge ... \wedge d\bar{z}_{j_k} \in \Omega^{(k,k)}(M, \mathbb{C})$$

and

$$L_X \left(dz_{j_{k+1}} \wedge d\bar{z}_{j_{k+1}}\right) \in \Omega^{(1,1)}(M, \mathbb{C}).$$

Note that, if $\omega \in \Omega^{(k,k)}(M, \mathbb{C})$ and $\mu \in \Omega^{(l,l)}(M, \mathbb{C})$ then $\omega \wedge \mu \in \Omega^{(k+l,k+l)}(M, \mathbb{C})$. Therefore

$$L_X\omega \in \Omega^{(k+1,k+1)}(M, \mathbb{C}), \forall \omega \in \Omega^{(k+1,k+1)}(M, \mathbb{C}).$$

□

The following result give an slight computation in the special case.

**Proposition 3.6.** *Let $X = (X_1, X_2, ..., X_n) \in \mathfrak{B}^{(1,0)}_{hol}(M)$ is a holomorphic vector field and $X$ is bounded on $M$; $\omega = \sum\limits_{j,k=1}^{n} \varphi_{jk} dz_j \wedge d\bar{z}_k \in \Omega^{(1,1)}(M, \mathbb{C})$ be the differential forms of bidegree (1,1). Then we have*

$$L_X\omega = \sum_{j,k=1}^{n} (L_X\varphi_{jk}) dz_j \wedge d\bar{z}_k. \tag{3.5}$$

*Proof.* By using Corollary 3.2, we obtain

$$L_X\omega = \sum_{j,k=1}^{n}\left[L_X\varphi_{jk}dz_j\wedge d\bar{z}_k + \sum_{l=1}^{n}\varphi_{jk}\frac{\partial X_j}{\partial z_l}dz_l\wedge d\bar{z}_k - \sum_{l=1}^{n}\varphi_{jk}\frac{\partial \bar{X}_k}{\partial \bar{z}_l}d\bar{z}_l\wedge dz_j\right].$$

Since $X$ is a holomorphic vector field and $X$ is bounded on $M$, therefore applying Liouvile, then $X$ is constant vector field. Hence $L_X\omega = \sum\limits_{j,k=1}^{n}(L_X\varphi_{jk})\,dz_j\wedge d\bar{z}_k$. □

*Remark* 3.7. Suppose that $X = (X_1, X_2, ..., X_n) \in \mathfrak{V}_{hol}^{(1,0)}(M)$ is a holomorphic vector field on $M$.

i) If $f \in \mathcal{O}(M)$ then $L_X f \in \mathcal{O}(M)$.

ii) If $\omega \in \Omega_{hol}^{p}(M)$ then $L_X\omega \in \Omega_{hol}^{p}(M)$.

*Proof.* i) Since $X = (X_1, X_2, ..., X_n) \in \mathfrak{V}_{hol}^{(1,0)}(M)$ is a holomorphic vector field and $f \in \mathcal{O}(M)$ is a holomorphic function on $M$, we have $\frac{\partial f}{\partial \bar{z}_j} = 0, \forall j = \overline{1,n}$ and $\dfrac{\partial f}{\partial z_j}, \forall j = \overline{1,n}$ are holomorphic functions on $M$. Therefore

$$L_X f = \sum_{j=1}^{n}\left(\frac{\partial f}{\partial z_j}X_j + \frac{\partial f}{\partial \bar{z}_j}\overline{X}_j\right) = \sum_{j=1}^{n}\frac{\partial f}{\partial z_j}X_j$$

Hence $\dfrac{\partial\,(L_X f)}{\partial \bar{z}_k} = \sum\limits_{j=1}^{n}\left(\dfrac{\partial^2 f}{\partial z_j\partial \overline{z_k}}X_j + \dfrac{\partial f}{\partial z_j}\dfrac{\partial X_j}{\partial \overline{z_k}}\right) = 0, \forall k = \overline{1,n}$. Hence $L_X f \in \mathcal{O}(M)$.

ii) The result easily follows from i). □

*Remark* 3.8. If $u \in C^{\infty}(M, \mathbb{C})$ then

$$L_X dd^c u = 2i\sum_{j,k=1}^{n}\frac{\partial^2\,(L_X u)}{\partial z_j\partial \bar{z}_k}dz_j\wedge d\bar{z}_k. \tag{3.6}$$

*Proof.* Since $\partial o L_X = L_X o\partial$ and $\overline{\partial} o L_X = L_X o\overline{\partial}$, we obtain $d^c o L_X = L_X o d^c$. Hence, applying Proposition 2.2, we obtain $dd^c o L_X = L_X o dd^c$. This implies that

$$L_X dd^c u = dd^c\,(L_X u) = 2i\sum_{j,k=1}^{n}\frac{\partial^2\,(L_X u)}{\partial z_j\partial \bar{z}_k}dz_j\wedge d\bar{z}_k.$$

□

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Bui Cao Van,
Department of Mathematics, Vinh University,182 Le Duan, Vinh City, Vietnam
*E-mail address*: buicaovan@gmail.com