Bulletin of Mathematical Analysis and Applications
ISSN: 1821-1291, URL: http://www.bmathaa.org
Volume 8 Issue 4(2016), Pages 72-77.

IMPLICIT CONTRACTIVE MAPPINGS IN SPHERICALLY COMPLETE ULTRAMETRIC SPACES

MALIHE HAJIMOJTAHED, ALIREZA KAMEL MIRMOSTAFAEE*

Abstract. In this paper, we apply implicit functions to establish a general fixed point theorem in spherically complete ultrametric spaces which enable us to extend some known results. In particular, we will show that in a spherically complete space X a self-mapping T satisfies

$$d(Tx,Ty) < \max\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx), d(x,y)\}$$

for each $x, y \in X$ with $x \neq y$, then T has a unique fixed point. This improves Gajic's fixed point theorem in spherically complete ultrametric spaces.

1. Introduction

A metric space (X, d) is said to be an ultrametric space, if it satisfies the strong triangle inequality

$$d(x,y) \leq \max\{d(x,z), d(z,y)\} \quad (x, y \in X).$$

Sometimes the associated metric is also called a non-Archimedean metric or super-metric.

An ultrametric space (X, d) is said to be spherically complete if every shrinking collection of balls in X has a nonempty intersection. Clearly, every spherically complete ultrametric space is complete with respect to the topology induced by its metric. But the converse is not true in general. For example, the completion $\mathbb{C}_p$ of the algebraic closure of the field of rational p-adic numbers is complete. However, it is not spherically complete cf. [18], pp. 134-145.

Let X be a nonempty set and $T : X \to X$ be a function. A point $x \in X$ is said to be a *fixed point* of T provided that $Tx = x$. The function T is called *contraction* if there exists a constant $r < 1$ such that

$$d(Tx,Ty) \leq r d(x,y) \quad (x, y \in X). \tag{1.1}$$

T is said to be *non-expansive* if (1.1) holds for $r = 1$. T is called *contractive* if we replace the inequality (1.1) with strict inequality and $r = 1$. Clearly, every contraction mapping is contractive and every contractive mapping is non-expansive. However, the converse is not true [4].

2000 *Mathematics Subject Classification.* 47H10, 54E50, 32P05.
Key words and phrases. Fixed points, contractions, implicit functions, ultrametric spaces.

*Corresponding author.
Submitted June 3, 2016. Published November 7, 2016.
Communicated by Lahcene Mezrag.

In 1922, S. Banach [5] established an important fixed point theorem known as "Banach Contraction Principle" states that every contraction mapping of a complete metric space into itself has a unique fixed point.

The Banach contraction principle has been extended by some mathematicians (see e.g. [1, 7, 9, 10, 11, 15, 17]). In particular, Ćirić proved the following.

Theorem 1.1. [7] *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping such that for some $0 \leq r < 1$,*

$$d(Tx, Ty) \leq r \max\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(x, y)\} \quad (x, y \in X).$$

Then T has a unique fixed point.

In 1973, Hardy and Rogers gave another remarkable extension of Banach's fixed point theorem as follows.

Theorem 1.2. [9] *Let X be a complete metric space with metric d, and let $T : X \to X$ be a function with the following property:*

$$d\big(Tx, Ty\big) \leq a\ d\big(x, Tx\big) + b\ d\big(y, Ty\big) + c\ d\big(x, Ty\big) + e\ d\big(y, Tx\big) + f\ d(x, y),$$

where $0 \leq a, b, c, e, f < 1$ and $a + b + c + e + f < 1$. Then T has a unique fixed point.

It is known that a contractive mapping $T : \mathbb{R} \to \mathbb{R}$ need not have a fixed point. For example, let $T : \mathbb{R} \to \mathbb{R}$ be defined by

$$Tx = Ln(1 + e^x) \quad (x \in \mathbb{R}).$$

Then $T'x = \frac{e^x}{1+e^x} < 1$ for all $x \in \mathbb{R}$. Therefore T is a contractive mapping. It is easy to verify that the equation $Tx = x$ is equivalent to $1 + e^x = e^x$, which is absurd. Therefore T has no fixed point [16]. However, in spherically complete non-Archimedean spaces, we have the following result.

Theorem 1.3. [12] *Let X be a non-Archimedean spherically complete normed space. If $T : X \to X$ is a contractive mapping, then T has a unique fixed point.*

In 2001, Gajic obtained the following extension of Theorem 1.3.

Theorem 1.4. [8] *Let (X, d) be a spherically complete non-Archimedean metric space. If $T : X \to X$ is such that for any $x, y \in X, x \neq y$,*

$$d(Tx, Ty) < max\{d(x, y), d(x, Tx), d(y, Ty)\},$$

then T has a unique fixed point.

In 1997, V. Popa [13] initiated a study of implicit contractive type conditions to give a simple proofs for some classical fixed point theorems. Since then, some authors studied various types of implicit contractions see e. g. [2, 3, 6, 14]. In this paper, we will define a new class of implicit functions to prove a general fixed point theorem in spherically complete ultrametric spaces. This result enable us to generalize Hardy-Rogers and Gajic fixed point theorems in spherically complete ultrametric spaces.

2. Main results

In this section, we define a class of implicit functions to establish a general fixed point theorem on spherically complete ultrametric spaces.

Let $\mathcal{G}$ denote the set of all functions $g : [0,\infty)^5 \to [0,\infty)$ with the following properties.

(1) $g(1,1,1,1,1) = h \leq 1$,

(2) $x_i, y_i \in [0,\infty)$ and $x_i \leq y_i$ for $1 \leq i \leq 5$, implies that

$$g(x_1, x_2, x_3, x_4, x_5) \leq g(y_1, y_2, y_3, y_4, y_5),$$

(3) $g(\alpha x_1, \alpha x_2, \alpha x_3, \alpha x_4, \alpha x_5) \leq \alpha g(x_1, x_2, x_3, x_4, x_5)$ whenever $\alpha \geq 0$ and $x_i \in [0,\infty)$ for $1 \leq i \leq 5$.

Here we provide some examples of this class of implicit functions.

Example 2.1. *Let $g : [0,\infty)^5 \to [0,\infty)$ be defined by*

$$g(x_1, x_2, x_3, x_4, x_5) = max\{x_1, x_2, x_3, x_4, x_5\}$$

for all $(x_1, x_2, x_3, x_4, x_5) \in [0,\infty)^5$. Then g satisfies the conditions (1), (2) and (3). Therefore $g \in \mathcal{G}$.

Example 2.2. *Define $g : [0,\infty)^5 \to [0,\infty)$ by*

$$g(x_1, x_2, x_3, x_4, x_5) = ax_1 + bx_2 + cx_3 + ex_4 + fx_5, \quad (x_1, x_2, x_3, x_4, x_5 \in [0,\infty)^5),$$

where $a, b, c, e, f \in [0,1]$ and $a + b + c + e + f \leq 1$. One can easily check that g satisfies the conditions (1), (2) and (3). Thus $g \in \mathcal{G}$.

In order to state the main result of this section, we need to the following auxiliary result.

Lemma 2.3. *If $g \in \mathcal{G}$ and $u, v \in [0,\infty)$ are such that*

$$u < \max\{g(v, u, \max(u,v), 0, v),\ g(v, u, v, v, \max(u,v))\}$$

then $u < v$.

Proof. Suppose that $u \geq v$, then

$$\begin{aligned} u &< g(u,u,u,u,u) \leq ug(1,1,1,1,1) \\ &= uh \leq u, \end{aligned}$$

which is a contradiction. Thus $u < v$. □

Now, we are ready to state the main result of this section.

Theorem 2.4. *Let (X, d) be a spherically complete ultrametric space. Let $g \in \mathcal{G}$ and $T : X \to X$ satisfy*

$$d(Tx, Ty) < g\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(x, y)\} \tag{2.1}$$

for each $x, y \in X$ with $x \neq y$. Then T has a unique fixed point.

Proof. For each $x \in X$, let B_x be the closed ball centered at x with the radius $d(x, Tx)$. Let $\mathcal{A} = \{B_x : x \in X\}$. Define a partial order $\preceq$ on $\mathcal{A}$ as follows.

$$B_x \preceq B_y \text{ if and only if } B_x \subseteq B_y.$$

Let $\mathcal{A}_1$ be a totally ordered subfamily of $\mathcal{A}$. Since (X,d) is spherically complete, $B=\bigcap_{B_a\in\mathcal{A}_1} B_a \neq \emptyset$. Let b be an arbitrary element of B and $B_a\in\mathcal{A}_1$. Then $b\in B_a$. Therefore $d(b,a)\leq d(a,Ta)$. By the strong triangle inequality, we have

$$\begin{aligned} d(b,Tb) &\leq \max\{d(b,a),d(a,Ta),d(Ta,Tb)\} \\ &\leq \max\{d(a,Ta),d(Ta,Tb)\}. \end{aligned}$$

Therefore

$$d(b,Tb)\leq d(a,Ta) \text{ or } d(b,Tb)\leq d(Ta,Tb).$$

We claim that $d(b,Tb)\leq d(a,Ta)$. Suppose that $d(b,Tb)>d(a,Ta)$. By the above inequality, $d(b,Tb)\leq d(Ta,Tb)$. Therefore, we have

$$\begin{aligned} d(b,Tb) &\leq d(Ta,Tb) < g\big(d(a,Ta),d(b,Tb),d(a,Tb),d(b,Ta),d(a,b)\big) \\ &\leq g\big(d(b,Tb),d(b,Tb),\max\{d(a,b),d(b,Tb)\},\max\{d(a,b),d(a,Ta)\},d(a,b)\big) \end{aligned}$$

Since $d(a,b)\leq d(a,Ta)<d(b,Tb)$, by the above inequality, we have

$$\begin{aligned} d(b,Tb) &< g\big(d(b,Tb),d(b,Tb),d(b,Tb),d(b,Tb),d(b,Tb)\big) \\ &\leq d(b,Tb)g(1,1,1,1,1)\leq d(b,Tb). \end{aligned}$$

This contradiction shows that $d(b,Tb)\leq d(a,Ta)$. We claim that $B_b\subseteq B_a$. In fact, for every $x\in B_b$, we have

$$d(x,b)\leq d(b,Tb)\leq d(a,Ta).$$

It follows that

$$d(x,a)\leq \max\{d(x,b),d(b,a)\}\leq d(a,Ta).$$

Therefore $x\in B_a$. This proves our claim. Since B_a was an arbitrary element of $\mathcal{A}_1$, B_b is a lower bound for $\mathcal{A}_1$ in $\mathcal{A}$. By Zorn's lemma, $\mathcal{A}$ has a minimal element. Let B_z be a minimal element of $\mathcal{A}$ and $w=Tz$. We will show that w is a unique fixed point of T. If $Tw\neq w$, we have

$$\begin{aligned} d(w,Tw)=d(Tz,Tw) &< g\big(d(z,Tz),d(w,Tw),d(z,Tw),d(w,Tz),d(z,w)\big) \\ &\leq g\big(d(z,Tz),d(w,Tw),\max\{d(z,Tz),d(Tz,Tw)\},0,d(z,Tz)\big). \end{aligned}$$

By Lemma 2.3, $d(w,Tw)<d(z,Tz)$. Hence $Tz\notin B_w$. This means that $d(w,Tw)< d(w,Tz)$. Since $Tz=w$, we have $d(w,Tw)<0$. This contradiction shows that $Tw=w$ is a fixed point of T.

Suppose that w_1 and w_2 are distinct fixed points of T. Then

$$\begin{aligned} d(w_1,w_2) &= d(Tw_1,Tw_2) < g\big(d(w_1,Tw_1),d(w_2,Tw_2),d(w_1,Tw_2),d(w_2,Tw_1),d(w_1,w_2)\big) \\ &\leq g\big(0,0,\max\{d(w_1,w_2),d(w_2,Tw_2)\},\max\{d(w_1,w_2),d(w_1,Tw_1)\},d(w_1,w_2)\big) \\ &\leq d(w_1,w_2)g(1,1,1,1,1)\leq d(w_1,w_2). \end{aligned}$$

It follows that $d(w_1,w_2)<d(w_1,w_2)$, which is a contradiction. This proves the uniqueness of the fixed point of T. □

Theorem 2.4 enable us to obtain the following generalization of Theorem 1.2 provided that the metric space is spherically complete non-Archimedean.

Corollary 2.5. *Let (X, d) be a spherically complete ultrametric space and T be a self-mapping on X. Let for each $x, y \in X$ with $x \neq y$,*

$$d(Tx, Ty) < a\ d(x, Tx) + b\ d(y, Ty) + c\ d(x, Ty) + e\ d(y, Tx) + f\ d(x, y).$$

Then T has a unique fixed point provided that $0 \leq a, b, c, e, f \leq 1$ and $a + b + c + e + f \leq 1$.

Proof. Apply Theorem 2.4 for the function g in Example 2.2. □

The following result, which is an extension of Theorem 1.4, improves Ćirić 's fixed point theorem in spherically complete ultrametric spaces.

Corollary 2.6. *Let (X, d) be a spherically complete ultrametric space and $T : X \to X$ be a mapping. Suppose that for all $x, y \in X$ with $x \neq y$,*

$$d(Tx, Ty) < \max\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx), d(x, y)\}.$$

Then T has a unique fixed point.

Proof. The result follows from Example 2.1 and Theorem 2.4. □

The following result follows immediately from Corollary 2.6 and the ultrametric inequality.

Corollary 2.7. *Let (X, d) be a spherically complete ultrametric space. Then a self-mapping $T : X \to X$ has a unique fixed point provided that one of the following conditions satisfies.*

(a) $d(Tx, Ty) \neq \max\{d(Tx, x), d(x, y), d(Ty, y)\}$ *for each $x, y \in X$ with $x \neq y$.*
(b) $d(Tx, Ty) \neq \max\{d(Tx, y), d(x, y), d(Ty, x)\}$ *for each $x, y \in X$ with $x \neq y$.*
(c) $d(Tx, Ty) \neq \max\{d(Tx, y), d(Ty, y)\}$ *for each $x, y \in X$ with $x \neq y$.*
(d) $d(Tx, Ty) \neq \max\{d(Tx, x), d(Ty, x)\}$ *for each $x, y \in X$ with $x \neq y$.*

References

[1] A. Aghajani and R. Allahyari, *Fixed-point theorems for multivalued generalized nonlinear contractive maps in partial metric spaces,* Ukranian Math. J. **66**(1), (2014), 1–15.
[2] S. M. A. Aleomraninejad, S. Rezapour and N. Shahzad, *On fixed point generalizations of suzuki's method,* Appl. Math. Lett. **24**, 1037-1040 (2011).
[3] I. Altun, *Fixed point and homotopy results for multivalued maps satisfying an implicit relation,* J. Fixed Point Theory Appl. **9** (2011), 125–134.
[4] S. Almezel, H. Ansari and M. A. Khamsi, *Topics in fixed point theory,* Springer-Verag, 2014.
[5] S. Banach, *Surles operations dans les ensembles abstraits et leur application aux equations integrals*, Fund, Math., **3** (1922), 133-181.
[6] V. Berinde, *Approximating fixed points of implicit almost contractions,* Hacet. J. Math. Stat. **40** (2012), 93-102
[7] L. B. Ćirić, *Generalized contractions and fixed-point theorems,* Publ. Inst. Math. (Beograd) (N.S.) **12**(26) (1971),1 9–26
[8] Lj. Gajic, *On ultrametric spaces*, Novi Sad J. Math. **31**, 2 (2001), 69–71.
[9] G. Hardy and T. Rogers, *A generalization of fixed point theorem of Reich,* Canad. Math. Bull. **16**(2) (1973), 201–206.
[10] M. Javahernia1, A. Razani1, and Farshid Khojasteh, *Common fixed point of the generalized Mizoguchi-Takahashis type contractions,* l. Fixed Point Theory Appl.(2014), 2014:195, 12 pages.
[11] A. Latif, M. Eshaghi Gordji, E. Karapinar and W. Sintunavarat, *Fixed point results for generalized (α, ψ)-Meir-Keeler contractive mappings and applications*, J. Ineq. Appl.(2014), 2014:68, 11 pages.
[12] C. Petalas and F. Vidalis, *A fixed point theorem in non-Archimedean vector spaces*, Proc. Amer. Math. Soc. **118** (1993), 819-821.

[13] V. Popa, *Fixed point theorems for implicit contractive mappings,* Stud. Cerc. St. Ser. Mat. Univ. Bacău **7** (1997), 127-133.

[14] V. Popa, *Some fixed point theorems for compatible mappings satisfying an implicit relation,* Demonstratio Math. **32** (1999), 157-163.

[15] O. Popescu, *A new type of contractive multivalued operators,* Bull. Sci. Math. **137** (2013), 30–44.

[16] E. Rakotch, *A note on contractive mappings,* Proc. Amer. Math. Soc. **13** (1962), 459– 465.

[17] B. E. Rhoades, *A comparison of various definitions of contractive mappings,* Amer Math. Soc. Vol. **224** (1977), 257–290.

[18] A. M. Robert, *A course in p-adic analysis,* vol. 198 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000.

Malihe Hajimojtahed
Department of Pure Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Mashhad, Iran
E-mail address: `malihehajimojtahed42@yahoo.com`

Alireza Kamel Mirmostafaee
Center of Excellence in Analysis on Algebraic Structures, Department of Pure Mathematics, Ferdowsi University of Mashhad, P. O. Box 1159, Mashhad 91775, Mashhad, Iran
E-mail address: `mirmostafaei@um.ac.ir`