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ON THE STABILITY AND INSTABILITY OF FUNCTIONAL VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS OF FIRST ORDER

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Abstract. This paper is concerned with non-linear Volterra integro-differential equation (VIDE) with constant time-lag, τ :

$$x'(t) = P(t)f(x(t)) - \int_{t-\tau}^{t} K(t,s)f(x(s))ds.$$

Via Lyapunov functionals and basic inequalities, sufficient conditions are given for the exponential stability (ES) and instability (I) of the trivial solution of the former (VIDE). We introduce two new results for the above topics for the trivial solution of that (VIDE). Our conditions involve the nonlinear generalization and extensions of those found in the literature. The results to be obtained are new and complements that in the literature.

1. Introduction

Mathematical models are powerful tools used to describe real world problems in mathematical language and concepts. In the relative literature, one of famous mathematical models is known as the (VIDE), which appeared after its establishment by Vito Volterra, in 1926. Today, that kind of model has many important and interesting applications in physics, biology and engineering, etc.. (see Wazwaz [22] and the references therein).

In recent years, qualitative problems related to (VIDEs) have been extensively studied. For the researches of such (VIDEs), we refer the reader to Adıvar and Raffoul [1], Becker [2], Burton [3, 4, 5], Burton and Haddock [6], Burton and Mahfoud [7], Graef et al. [9], Gripenberg et al. [10], Furumochi and Matsuoka [8], Hara et al. [11], Miller [12], Raffoul [13, 14], Raffoul and Unal [15], Staffans [16], Tunç [17, 18, 19, 20, 21, 22], Vanualailai and Nakagiri [23], Wang [24], Wazwaz [25], Zhang [26] and many papers and books in their references.

By this information we mean that it is worth and deserve investigating qualitative behaviors of solutions, stability, instability, boundedness, convergence, globally existence of solutions, etc., of (VIDEs).

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In 2009, Becker [2] considered the scalar linear (VIDE)

$$x'(t) = -a(t)x(t) + \int_0^t b(t,s)x(s)ds. \tag{1.1}$$

Becker [2] investigated some qualitative behaviors of solutions of (VIDE) (1.1) by Lyapunov's functional approach. He also gave examples to illustrate the obtained results.

Later, in 2012, Adivar and Raffoul [1] considered the following linear (VIDE) with constant time-lag, $\tau > 0$,

$$x'(t) = p(t)x(t) - \int_{t-\tau}^t q(t,s)x(s)ds. \tag{1.2}$$

The authors discussed (ES), (I) of the trivial solution and the existence of some inequalities with respect to the solutions of (VIDE) (1.2) by means of the Lyapunov functionals.

In this paper, motivated by the ideas in [1] and [2], we consider the non-linear (VIDE)

$$x'(t) = P(t)f(x(t)) - \int_{t-\tau}^t K(t,s)f(x(s))ds, \tag{1.3}$$

where $t \geq 0$, $\tau > 0$ is a constant time-lag such that $t-\tau \geq 0$, $K : [0,\infty)\times[-\tau,\infty) \to \Re$, $P : [0,\infty) \to \Re$ and $f : \Re \to \Re$ with $f(0) = 0$ are continuous functions with $\Omega := \{(t,s) : 0 \leq \tau \leq s \leq t < \infty\}$.

Let

$$f_1(x) = \begin{cases} \frac{f(x)}{x}, & x \neq 0 \\ f'(0), & x = 0. \end{cases}$$

Then, (VIDE) (1.3) is equivalent to

$$x'(t) = P(t)f_1(x(t))x(t) - \int_{t-\tau}^t K(t,s)f_1(x(s))x(s)ds.$$

It is clear that Adivar and Raffoul [1] and Becker [2] discussed linear (VIDEs) without and with time-lag, respectively. However, (VIDE) (1.3) is non-linear and time-lag generalization of (VIDEs) (1.1) and (1.3). In reality, it may be followed that (VIDE) (1.3) includes and improves (VIDEs) (1.1) and (1.2) studied by [1] and [2]. In fact, let take zero instead of $t-\tau$, $P(t) = -a(t)$, $f(x) = x$ and $K(t,s) = -b(t,s)$ and $P(t) = p(t)$, $f(x) = x$ and $K(t,s) = q(t,s)$, respectively. Hence, (VIDE) (1.3) becomes reduced to (VIDE) (1.1) and (VIDE) (1.2), respectively.

In the present work, we search (ES) and (I) of the trivial solution of (VIDE) (1.3). The results to be obtained here are different from that given in the literature (see ([1]-[23]) and the references thereof). Namely, (VIDE) (1.3) and the assumptions described are distinct in from that in ([1]-[25]). This case is an improvement of the problems discussed in [1] and [2], and it shows the novelty and originality of the paper.

2. Preliminaries

Let $x(t) = x(t,t_0,\phi)$ be a solution of (VIDE) (1.3) on $[t_0-\tau,\infty)$ such that $x(t) = \phi(t)$ on $\phi \in [t_0-\tau, t_0]$, $t_0 \geq 0$. For $\phi \in C[0,t_0]$, let $|\phi|_{t_0} := \sup\{|\phi(t)| : 0 \leq t \leq t_0\}$.

Let

$$A(t,s) = \int_{t-s}^{\tau} K(u+s,s)du,$$

where $t \in [0,\infty)$ and $s \in [-\tau,\infty)$.

It is clear that (VIDE) (1.3) can be written as

$$x'(t) = P(t)f(x) - A(t,t)f(x) + \frac{d}{dt}\int_{t-\tau}^{t} A(t,s)f(x(s))ds. \tag{2.1}$$

Let

$$Q(t) = P(t) - A(t,t).$$

Hence, it may be seen from (VIDE) (2.1) that

$$x'(t) = Q(t)f(x) + \frac{d}{dt}\int_{t-\tau}^{t} A(t,s)f(x(s))ds. \tag{2.2}$$

For the sake of brevity, if one follows the way in [1], it can be easily shown that the following estimates are true:

(i) $A(t,t-\tau) \equiv 0, \quad t \geq 0,$

(ii) $A(t,s)K(t,s) \geq 0, \quad t \in [0,\infty), s \in [-\tau,\infty),$

(iii) $A^2\left(t - \frac{(\alpha-1)\tau}{\alpha}, \xi\right) \geq A^2(t,\xi) \geq 0, \quad 1 < \alpha \leq 2, t \in [0,\infty), s \in [t-\tau,t],$

$$\begin{aligned}\int_{-\tau}^{0}\int_{t+s}^{t} & A(t,\xi)\frac{\partial A(t,\xi)}{\partial t} f^2(x(\xi))d\xi ds \\ &= -\int_{t-\tau}^{t}\int_{-\tau}^{\xi-t} A(t,\xi)K(t,\xi)f^2(x(\xi))dsd\xi \\ &= -\int_{t-\tau}^{t}\int_{-\tau}^{\xi-t} A(t,\xi)K(t,\xi)f^2(x(\xi))dsd\xi \leq 0\end{aligned}$$

and

$$\frac{\partial A(t,s)}{\partial t} = -K(t,s).$$

3. Exponential stability

First, we describe a new auxiliary functional $V = V(t)$ by

$$V = \left[x(t) - \int_{t-\tau}^{t} A(t,s)f(x(s))ds\right]^2 + \int_{-\tau}^{0}\int_{t+s}^{t} A^2(t,\xi)f^2(\xi)d\xi ds. \tag{3.1}$$

We benefit from the former auxiliary functional to prove the stability result of this paper.

Before stating the first main result, we give some lemmas as auxiliary results.

Lemma 3.1. *Let assumptions (ii), (iii) and*

$$-\frac{1}{2\tau} \leq Q(t)f_1(x)$$

hold. If $f_1(x) \geq 1$, then

$$\frac{d}{dt}V(t) \leq Q(t)V(t),$$

where $V(t)$ is given by (3.1).

Proof. Take $x(t) = x(t, t_0, \phi)$ as a solution of (VIDE) (1.3) such that $\phi \in C[-\tau, 0)$. If we differentiate $V(t)$ along the solutions of (VIDE) (1.3), then it follows that

$$\begin{aligned}
\frac{d}{dt}V &= \frac{d}{dt}\left[x(t) - \int_{t-\tau}^{t} A(t,s) f(x(s))ds\right]^2 \\
&\quad + \frac{d}{dt}\int_{-\tau}^{0}\int_{t+s}^{t} A^2(t,\xi) f^2(\xi) d\xi ds \\
&= 2\left[x(t) - \int_{t-\tau}^{t} A(t,s) f(x(s))ds\right] \times \left[x'(t) - \frac{d}{dt}\int_{t-\tau}^{t} A(t,s) f(x(s))ds\right] \\
&\quad + \tau A^2(t,t) f^2(x) - \int_{-\tau}^{0} A^2(t,t+s) f^2(x(t+s))ds \\
&\quad + \int_{-\tau}^{0}\int_{t+s}^{t} 2A(t,\xi)\frac{\partial A(t,\xi)}{\partial t} f^2(x(\xi)) d\xi ds \\
&= 2\left[x(t) - \int_{t-\tau}^{t} A(t,s) f(x(s))ds\right] \\
&\quad \times \left[Q(t)f(x) + \frac{d}{dt}\int_{t-\tau}^{t} A(t,s) f(x(s))ds - \frac{d}{dt}\int_{t-\tau}^{t} A(t,s) f(x(s))ds\right] \\
&\quad + \tau A^2(t,t) f^2(x) - \int_{-\tau}^{0} A^2(t,t+s) f^2(x(t+s))ds \\
&\quad + \int_{-\tau}^{0}\int_{t+s}^{t} 2A(t,\xi)\frac{\partial A(t,\xi)}{\partial t} f^2(x(\xi)) d\xi ds \\
&= Q(t)\left[f_1(x)x^2 - 2f_1(x)x\int_{t-\tau}^{t} A(t,s) f_1(x(s))x(s)ds\right] \\
&\quad + \tau A^2(t,t) f^2(x) - \int_{-\tau}^{0} A^2(t,t+s) f^2(x(t+s))ds \\
&\quad + \int_{-\tau}^{0}\int_{t+s}^{t} 2A(t,\xi)\frac{\partial A(t,\xi)}{\partial t} f^2(x(\xi)) d\xi ds \ + Q(t)f_1(x)x^2 \\
&= Q(t)f_1(x)\left[x - \int_{t-\tau}^{t} A(t,s) f(x(s))ds\right]^2 \\
&\quad - Q(t)f_1(x)[\left[\int_{t-\tau}^{t} A(t,s) f(x(s))ds\right]^2 \\
&\quad + \tau A^2(t,t) f^2(x) - \int_{-\tau}^{0} A^2(t,t+s) f^2(x(t+s))ds \\
&\quad + \int_{-\tau}^{0}\int_{t+s}^{t} 2A(t,\xi)\frac{\partial A(t,\xi)}{\partial t} f^2(x(\xi)) d\xi ds \ + Q(t)f_1(x)x^2.
\end{aligned}$$

Then, in view of the last estimate and the assumptions of Lemma 3.1, we have

$$\begin{aligned}\frac{d}{dt}V \quad &\leq \quad Q(t)f_1(x)V(t) - Q(t)f_1(x)\left[\int_{t-\tau}^{t} A(t,s)f(x(s))ds\right]^2 \\ &\quad + \left[\tau A^2(t,t) + Q(t)f_1(x)\right]x^2(t) - \int_{-\tau}^{0} A^2(t,t+s)f^2(x(t+s))ds \\ &\quad - Q(t)f_1(x)\int_{-\tau}^{0}\int_{t+s}^{t} A^2(t,\xi)f^2(\xi)d\xi ds. \qquad (3.2)\end{aligned}$$

Via the Schwartz inequality, it may be followed that

$$\int_{-\tau}^{0}\int_{t+s}^{t} A^2(t,\xi)f^2(x(\xi))d\xi ds \leq \tau\int_{t-\tau}^{t} A^2(t,s)f^2(x(s))ds,$$

$$\left[\int_{t-\tau}^{t} A(t,s)f(x(s))ds\right]^2 \leq \tau\int_{t-\tau}^{t} A^2(t,s)f^2(x(s))ds$$

and

$$\int_{-\tau}^{0} A^2(t,t+s)f^2(x(t+s))ds = \int_{t-\tau}^{t} A^2(t,s)f^2(x(s))ds.$$

Then, from (3.2), we reach that

$$\begin{aligned}\frac{d}{dt}V \quad &\leq \quad Q(t)f_1(x)V(t) - Q(t)f_1(x)\tau\int_{t-\tau}^{t} A^2(t,s)x^2(s)ds \\ &\quad + \left[\tau A^2(t,t) + Q(t)f_1(x)\right]x^2 - \int_{t-\tau}^{t} A^2(t,s)f^2(x(s))ds \\ &\quad - Q(t)f_1(x)\tau\int_{t-\tau}^{t} A^2(t,s)f^2(x(s))ds \\ &= \quad Q(t)f_1(x)V(t) + [\tau A^2(t,t) + Q(t)f_1(x)]x^2 \\ &\quad + [-2\tau Q(t)f_1(x) - 1]\int_{t-\tau}^{t} A^2(t,s)f^2(x(s))ds. \qquad (3.3)\end{aligned}$$

Hence, we can conclude from (3.3) that

$$\frac{d}{dt}V \leq Q(t)V(t).$$

So, the conclusion of Lemma 3.1 follows. □

Theorem 3.2. *Let $1 < \alpha \leq 2$. If assumptions of Lemma 3.1 hold, then, the inequality*

$$|x(t)|^2 \leq 2\left(\frac{2\alpha-1}{\alpha-1}\right)V(t_0)\exp\left(\int_{t_0}^{t-(\alpha-1)\tau/\alpha}[P(s) - A(s,s)]ds\right)$$

is true for $t \geq t_0 + (\alpha-1)\alpha^{-1}\tau$, in which $x(t) = x(t, t_0, \psi)$ is a solution (VIDE) (1.3). If

$$A(t,t) - P(t) \geq \rho, (\rho \in \Re, \rho > 0), \text{ for all } t \geq t_0,$$

then the trivial solution of (VIDE) (1.3) is (ES).

Proof. Under the assumptions of Lemma 3.1, and following the way done by Adivar and Raffoul [1], it can be easily arrived at the result of Theorem 3.2. Hence, we would not like to give the details of the proof. □

Remark. If $P(t) - A(t,t)$ is in $L^1[0,\infty)$, then the solution $x(t)$ of (VIDE) (1.3) is (ES) provided that $t \geq t_0 + (\alpha-1)\alpha^{-1}\tau$, where $L^1[0,\infty)$ is the space of Lebesgue integrable functions.

Remark. Since $A(t,t) - P(t) \geq \rho > 0$, then it clear that $x(t)$ is (ES). In fact, it follows that

$$\begin{aligned} |x(t)|^2 &\leq 2\left(\frac{2\alpha-1}{\alpha-1}\right)V(t_0)\exp\left(-\int_{t_0}^{t-(\alpha-1)\tau/\alpha}[A(s,s)-P(s)]ds\right) \\ &\leq 2\left(\frac{2\alpha-1}{\alpha-1}\right)V(t_0)\exp\left(-\rho(t-(\alpha-1)\alpha^{-1}\tau)\right). \end{aligned}$$

This shows the desired result.

4. Instability

In this section, we give an instability result for the trivial solution of (VIDE) (1.3). Before, we state our result, we define a new Lyapunov functional $V_1 = V_1(t)$ by

$$V_1 = \left[x(t) - \int_{t-\tau}^{t} A(t,s)f(x(s))ds\right]^2 - \lambda_1\int_{t-\tau}^{t} A^2(t,s)f^2(x(s))ds, \tag{4.1}$$

where $\lambda_1 > 0$ is a constant. We choose that constant later, and V_1 is defined for $x \in C[-r,\infty)$.

Lemma 4.1. *Let assumptions (ii), $\lambda_1 > \tau > 0$ and $\lambda_1 A^2(t,t) \leq Q(t)$ hold for all $t \geq 0$. If $f_1(x) \geq 1$, then*

$$\frac{d}{dt}V_1(t) \geq Q(t)V_1(t).$$

Proof. Let $x(t) = x(t,t_0,\phi)$ be a solution of (VIDE) (1.3). In view of the assumptions of Lemma 4.1, the time derivative of the Lyapunov functional V_1 given by (4.1) along (VIDE) (1.3) implies that

$$\begin{aligned} \frac{d}{dt}V_1 &= \frac{d}{dt}\left[x(t) - \int_{t-\tau}^{t} A(t,s)f(x(s))ds\right]^2 - \lambda_1\frac{d}{dt}\int_{t-\tau}^{t} A^2(t,s)f^2(x(s))ds \\ &= 2\left[x(t) - \int_{t-\tau}^{t} A(t,s)f(x(s))ds\right] \times \left[x'(t) - \frac{d}{dt}\int_{t-\tau}^{t} A(t,s)f(x(s))ds\right] \\ &\quad -\lambda_1 A^2(t,t)f^2(x(t)) - 2\lambda_1\int_{t-\tau}^{t} A(t,s)\frac{\partial A(t,s)}{\partial t}f^2(x(s))ds \\ &\geq 2\left[x(t) - \int_{t-\tau}^{t} A(t,s)f(x(s))ds\right] \times [Q(t)f(x)] \\ &\geq 2xQ(t)f(x) - 2Q(t)f(x)\int_{t-\tau}^{t} A(t,s)f(x(s))ds - \lambda_1 A^2(t,t)f^2(x) \end{aligned}$$

$$
\begin{aligned}
&= Q(t)f_1(x)\left[x-\int_{t-\tau}^{t} A(t,s)f(x(s))ds\right]^2 \\
&\quad +Q(t)f_1(x)\left[-\left(\int_{t-\tau}^{t} A(t,s)f(x(s))ds\right)^2\right] \\
&\quad +Q(t)f_1(x)x^2-\lambda_1 A^2(t,t)f^2(x) \\
&= f_1(x)Q(t)V_1(t)+f_1(x)Q(t)\left[-\left(\int_{t-\tau}^{t} A(t,s)f(x(s))ds\right)^2\right] \\
&\quad +\lambda_1 f_1(x)Q(t)\int_{t-\tau}^{t} A^2(t,\xi)f^2(x(\xi))d\xi \\
&\quad +Q(t)f_1(x)x^2-\lambda_1 A^2(t,t)f^2(x) \\
&= f_1(x)Q(t)V_1(t)+f_1(x)Q(t)\left[-\left(\int_{t-\tau}^{t} A(t,s)f(x(s))ds\right)^2\right] \\
&\quad +\lambda_1 f_1(x)Q(t)\int_{t-\tau}^{t} A^2(t,s)f^2(x(s))ds \\
&\quad +Q(t)f_1(x)x^2-\lambda_1 A^2(t,t)f^2(x) \\
&\geq Q(t)V_1(t)+f_1(x)Q(t)\left[-\left(\int_{t-\tau}^{t} A(t,s)f(x(s))ds\right)^2\right] \\
&\quad +\lambda_1 Q(t)\int_{t-\tau}^{t} A^2(t,s)f^2(x(s))ds \\
&\quad +Q(t)x^2-\lambda_1 A^2(t,t)f^2(x)
\end{aligned}
$$

$$
\begin{aligned}
&\geq Q(t)V_1(t)-\tau f_1(x)Q(t)\int_{t-\tau}^{t} A^2(t,s)f^2(x(s))ds \\
&\quad +\lambda_1 f_1(x)Q(t)\int_{t-\tau}^{t} A^2(t,s)f^2(x(s))ds \\
&\quad +[Q(t)-\lambda_1 A^2(t,t)]f_1^2(x)x^2 \\
&= Q(t)V_1(t)+(\lambda_1-\tau)f_1(x)Q(t)\int_{t-\tau}^{t} A^2(t,s)f^2(x(s))ds \\
&\quad +[Q(t)-\lambda_1 A^2(t,t)]f_1^2(x)x^2 \\
&\geq Q(t)V_1(t).
\end{aligned}
$$

Hence, the conclusion of Lemma 4.1 follows. □

Theorem 4.2. *If assumptions of Lemma 4.1 and the assumption*

$$\int_{t_0}^{\infty} A^2(s,s)ds=\infty$$

are true, then the trivial solution of (VIDE) (1.3) is unstable.

Proof. We consider the result of Lemma 4.1, that is, the inequality

$$\frac{d}{dt}V_1(t)\geq Q(t)V_1(t).$$

Integrating this inequality from t_0 to ∞, we obtain

$$V_1(t) \geq V_1(t_0)\exp(\int_{t_0}^{t} Q(s)ds). \tag{4.2}$$

In view of the Lyapunov functional $V_1 = V_1(t)$ given by (4.1), it is clear that

$$\begin{aligned} V_1 &= x^2 - 2x\int_{t-\tau}^{t} A(t,s)f(x(s))ds + \left[\int_{t-\tau}^{t} A(t,s)f(x(s))ds\right]^2 \\ &\quad -\lambda_1\int_{t-\tau}^{t} A^2(t,s)f^2(x(s))ds, \end{aligned} \tag{4.3}$$

in which $\lambda_1 > 0$ is a constant, and we choose later.
It can be easily verified that

$$2\,|mn| \leq \tau\beta^{-1}m^2 + \beta\tau^{-1}n^2, \quad (\beta > 0, \tau > 0).$$

In the light of this inequality, the Schwartz inequality and (4.3), we obtain

$$\begin{aligned} -2x\int_{t-\tau}^{t} A(t,s)f(x(s))ds &\leq 2\,|x|\left|\int_{t-\tau}^{t} A(t,s)f(x(s))ds\right| \\ &\leq \tau\beta^{-1}x^2 + \beta\tau^{-1}\left[\int_{t-\tau}^{t} A(t,s)f(x(s))ds\right]^2 \\ &\leq \tau\beta^{-1}x^2 + \beta\int_{t-\tau}^{t} A^2(t,s)f^2(x(s))ds. \end{aligned}$$

Then, it is clear from (4.1) that

$$\begin{aligned} V_1 &\leq x^2 + \tau\beta^{-1}x^2 + \beta\int_{t-\tau}^{t} A^2(t,s)f^2(x(s))ds \\ &\quad +\tau\int_{t-\tau}^{t} A^2(t,s)f^2(x(s))ds - \lambda_1\int_{t-\tau}^{t} A^2(t,s)f^2(x(s))ds. \end{aligned}$$

Let us choose $\lambda_1 = \beta + \tau$. Hence

$$V_1 \leq x^2 + \tau\beta^{-1}x^2 \leq \lambda_1(\lambda_1 - \tau)^{-1}x^2$$

so that

$$|x(t)|^2 \geq \lambda_1^{-1}(\lambda_1 - \tau)V_1(t).$$

From the last inequality and (4.2), we obtain

$$\begin{aligned} |x(t)|^2 &\geq \lambda_1^{-1}(\lambda_1 - \tau)V_1(t) \\ &\geq \lambda_1^{-1}(\lambda_1 - \tau)V_1(t_0)\exp(\int_{t_0}^{t} Q(s)ds) \\ &\geq \lambda_1^{-1}(\lambda_1 - \tau)V_1(t_0)\exp(\int_{t_0}^{t} \lambda_1 A^2(s,s)ds). \end{aligned} \tag{4.4}$$

We note that

$$\int_{t_0}^{\infty} A^2(s,s)ds = \infty.$$

From inequality (4.4) and the above assumption, one can reach that the trivial solution of (VIDE) (1.3) is unstable. This finishes the proof of Theorem 4.2. □

Remark. *By Theorems 3.2 and 4.2 , we extend and improve (ES) and (I) results in the literature from linear (VIDEs) with time-lag to non-linear (VIDEs) with time-lag (see Adıvar and Raffoul [1, Theorems 1 and 2]). In addition, it is clear that (VIDE) (1.3) improves (VIDE) (1.2) investigated by Becker* [2]. *Our results complement to that of Becker* [2] *and they have contribution to the literature. These are newness and quality of the present paper.*

5. Conclusion

We consider a kind of first order non-linear (VIDE) with constant time-lag. We investigate (ES) and (I) of trivial solutions by two new auxiliary functionals. Our results are new and differ from those found in the literature.

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