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NON-HOMOGENEOUS KIRCHHOFF EQUATION ON $\mathbb{R}^3$

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Abstract. In this paper, we study the Kirchhoff equation

$$\left(1+b\int_{\mathbb{R}^3}(|\nabla u|^2+V(x)u^2)dx\right)[-\Delta u+V(x)u]=|u|^{p-2}u+g(x)\quad \text{in } \mathbb{R}^3.$$

We prove the existence of infinitely many solutions for the problem by the $\mathbb{Z}_2$-equivariant Ljusternik-Schnirelman theory for non-even functional due to Ekeland and Ghoussoub in 1998.

1. Introduction and Main Results

In this paper, we consider the following non-homogeneous Kirchhoff equation

$$\left(1+b\int_{\mathbb{R}^3}(|\nabla u|^2+V(x)u^2)dx\right)[-\Delta u+V(x)u]=|u|^{p-2}u+g(x)\quad \text{in } \mathbb{R}^3 \tag{1.1}$$

Problems related to (1.1) model several physical and biological systems, where u describes a process, which depends on the average of itself, such as the population density, see e.g. [6] and the references therein.

Let us recall some recent results in the literature on the nonlinear Kirchhoff equation (1.1) with $g(x)=0$. To our knowledge, Wu [16] is the first one who considering problem (1.1). Four existence results for nontrivial solutions and a sequence of high energy solutions for problem (1.1) are obtained by using a symmetric mountain pass theorem. Liu and He [12] studied the existence of infinitely many high energy solutions for problem (1.1) with the subcritical nonlinearity which does not need to satisfy the usual Ambrosetti-Rabinowitz-type growth conditions. Ye and Tang [19] obtained infinitely many large-energy and small-energy solutions for (1.1), which unify and sharply improve the results of Wu [16]. Cheng [5] obtained the existence of nontrivial solutions for problem (1.1) when the nonlinearity term is asymptotically linear or 4-superlinear at infinity. By some special techniques, Li and Wu [11] proved the existence and multiplicity of nontrivial solutions of problem (1.1) with a

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widely class of superlinear nonlinearities, which improves and unites Theorems 1–4 in [16]. In [10], Huang and Liu obtained some existence and nonexistence results by using variational methods and also discussed the 'energy doubling' property of nodal solutions. Ye [18] proved problem (1.1) has a least energy nodal solution with its energy exceeding twice the least energy by using constrained minimization on the sign-changing Nehari manifold.

However, if $g(x) \neq 0$, there are few results about the existence of multiple solutions of (1.1), because the forcing term g destroys the structure of $\mathbb{Z}_2$-symmetry and one can not directly apply the classical symmetric mountain-pass theorem [2] to prove the existence of infinitely many solutions. As far as we know, the only papers dealing with the case $g(x) \neq 0$ are [4, 9]. In [4], Chen and Li proved the existence of at least two solutions for (1.1). In [9], Fan and Liu obtained at least two positive solutions for a degenerate nonlocal problem on unbounded domain by using the Ekeland's variational principle combined with the mountain pass theorem.

In the present paper, following the idea of [8, 17], we consider the nonlinear Kirchhoff equation in the whole space $\mathbb{R}^3$ with $g(x) \not\equiv 0$. To overcome the lack of compactness, we assume that the nonconstant potential $V(x)$ verifies the following condition (see [14])

$$\begin{cases} V(x) \in L^2_{loc}(\mathbb{R}^3) \text{ is such that } \mathrm{infess} V(x) > 0 \text{ and} \\ \int_{B(x)} \frac{1}{V(y)} dy \to 0 \text{ if } |x| \to \infty, \end{cases} \tag{V}$$

where $B(x)$ is the unit ball in $\mathbb{R}^3$ centered at x. It is easy to see that condition (V) holds in particular if V is a strictly positive continuous function on $\mathbb{R}^3$ which goes to infinity at infinity.

Throughout this paper, we denote the norm of $H^1(\mathbb{R}^3)$ by

$$\|u\|_{H^1} = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + |u|^2 \right) dx \right)^{1/2}$$

and by $|\cdot|_s$ we denote the usual L^s-norm, C stands for different positive constants. We also introduce the space

$$H := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x) u^2 dx < \infty \right\},$$

which is a Hilbert space equipped with the inner product

$$(u, v) := \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(x) uv) dx$$

and the associated norm $\|u\|^2 = (u, u)$.

The argument in this paper is variational, i.e. the solutions of (1.1) are obtained as critical points of the action functional on H defined as follow:

$$I(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx - \int_{\mathbb{R}^3} g(x) u dx. \tag{1.2}$$

To state the main result of this paper, we will also consider the following constrained problem which is related to (1.1):

$$\begin{cases} \left(1 + b \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x) u^2) dx\right) [-\Delta u + V(x) u] = |u|^{p-2} u + \mu g(x), \quad \text{in } \mathbb{R}^3, \\ -1 \leq \mu \leq 1, \quad \int_{\mathbb{R}^3} g(x) u = 0. \end{cases} \tag{1.3}$$

Remark. *As is pointed in* [8], *it seems difficult to solve* (1.3) *by looking for critical points of I on the constrained manifold $\{u \in H : \int_{\mathbb{R}^3} g(x)u = 0\}$ with the additional restriction condition $-1 \le \mu \le 1$, when one regards μ as a Lagrange multiplier.*

The main result of this paper is the following theorem.

Theorem 1.1. *Assume that $4 < p < 6$, then for any $g(x) \in L^2(\mathbb{R}^3)$, either* (1.1) *or* (1.3) *has an unbounded sequence of solutions.*

Remark. *The nonlinearity $|u|^{p-2}u$ can be generalized to those satisfying the classical Ambrosetti–Rabinowitz condition and $g(x)$ can also be replaced by a general $g(x, u)$.*

Remark. *One can easily obtain that problem* (1.1) *admits two solutions which one has a positive energy and one has negative energy by using Ekeland's variational principle and mountain pass theorem. As mentioned above, when $g(x) = 0$, the corresponding functional is even and it is easily to get infinitely many solutions for* (1.1) *by using the symmetric mountain pass theorem. But if $g(x) \neq 0$, this situation is more complicated. This phenomenon is general called perturbations from symmetry. For the semilinear elliptic equation, the readers who are interested in it can see the following references* [13, 7, 1, 15, 3].

The paper is organized as follow. In section 2, we will recall a $\mathbb{Z}_2$-equivariant Ljusternik-Schnirelman theory for noneven functionals. In section 3, we prove the main result by using the critical point theory developed by Ekeland and Ghoussoub [8].

2. Preliminaries and basic definitions

Under condition (V), we also have the following property.

Proposition 2.1 ([14])**.** *Suppose that $V(x)$ verify assumption (V). Then, the space H is continuously embedded in $L^s(\mathbb{R}^3)$ for any $s \in [2, 6]$ and the embedding is compact for any $s \in [2, 6)$. Moreover, the spectrum of the self-adjoint operator of $-\Delta + V$ in $L^2(\mathbb{R}^3)$ is discrete, i.e. it consists of an increasing sequence λ_n of eigenvalues of finite multiplicity such that $\lambda_n \to \infty$ as $n \to \infty$ and $L^2(\mathbb{R}^3) = \sum_n M_n$, $M_n \perp M_{n'}$ for $n \neq n'$, where M_n is the eigenspace corresponding to λ_n.*

Let us recall some definitions and the symmetric mountain-pass theorem for non-even functional.

Definition 2.2. *We say that a sequence $(u_n) \subset E$ is a (PS) sequence at level c ($(PS)_c$-sequence, for short) if $I(u_n) \to c$ and $I'(u_n) \to 0$. I is said to satisfy the $(PS)_c$ condition if any $(PS)_c$ sequence contains a convergent subsequence.*

Definition 2.3 ([8])**.** *I is a C^1 functional on a Hilbert space H satisfying the symmetrized Palais–Smale condition at levels c ($(sPS)_c$ for short) if I satisfies the standard $(PS)_c$ condition and if a sequence $\{u_n\}$ in H is relatively compact in H whenever it satisfies the following conditions:*

$$\lim_{n\to\infty} I(u_n) = \lim_{n\to\infty} I(-u_n) = c$$

and $\lim_{n\to\infty} \|I'(u_n) - \lambda_n I'(-u_n)\| = 0$ for some positive sequence of reals λ_n.

As usual, we denote the set of critical points at level c by

$$K_c = \{u \in H : I(u) = c, I'(u) = 0\}.$$

Definition 2.4 ([8]). *The $\mathbb{Z}_2$-resonant points at level c are*

$$K_c^g = \{u \in H : I(u) = I(-u) = c, I'(u) = \lambda I'(-u), \lambda > 0\}.$$

And the virtual critical points at level c is defined by

$$E_c = K_c \cup K_c^g,$$

the corresponding value c is called virtual critical values.

Theorem 2.5 ([8]). *Let I be a C^1 functional satisfying (sPS)$_c$ on a Hilbert space $H = X \oplus Y$ with $dim(X) < \infty$. Assume $I(0) = 0$ as well as the following conditions:*

(i) *There is $\rho > 0$ and $\alpha \geq 0$ such that $\inf I(S_\rho(Y)) \geq \alpha$, where $S_\rho(Y)$ denote the ball with radius ρ in Y.*

(ii) *There exists an increasing sequence $\{E_n\}_n$ of finite dimensional subspace H, all containing X such that $\lim_{n\to\infty} dim(E_n) = \infty$ and for each n, $\sup I(S_{R_n}(E_n)) \leq 0$ for some $R_n > \rho$.*

Then I has an unbounded sequence of virtual critical values.

Now Theorem 1.1 can be restated as

Theorem 2.6. *Assume that $4 < p < 6$, then for any $g(x) \in L^2(\mathbb{R}^3)$,* (1.1) *has an unbounded sequence of virtual critical values.*

3. Proof of the main result

The proof of Theorem 2.6 is separated into three lemmas.

Lemma 3.1. *The functional I satisfies the symmetrized Palais–Smale condition.*

Proof. First, it is well known that I satisfies (PS)$_c$ for any c. Indeed, let $\{u_n\}$ be a (PS)$_c$ sequence, since $4 < p < 6$,

$$\begin{aligned} I(u_n) - \frac{1}{p} I'(u_n)u_n + \left(1 - \frac{1}{p}\right) \int_{\mathbb{R}^3} g(x)u_n dx &= \left(\frac{1}{2} - \frac{1}{p}\right) \|u_n\|^2 + \left(\frac{b}{4} - \frac{b}{p}\right) \|u_n\|^4 \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right) \|u_n\|^2. \end{aligned}$$

Hence there exists a constant C_1 such that for n large,

$$c + C_1\|u_n\| \geq \left(\frac{1}{2} - \frac{1}{p}\right) \|u_n\|^2,$$

and so $\{u_n\}$ is bounded. The compact embedding implies the convergence.

Now assume that $\{u_n\}$ is a sequence satisfying:

$$\lim_{n\to\infty} I(u_n) = \lim_{n\to\infty} I(-u_n) = c \tag{3.1}$$

and

$$\lim_{n\to\infty} \|I'(u_n) - \lambda_n I'(-u_n)\| = 0 \tag{3.2}$$

for some positive sequence of reals λ_n. (3.1) and (3.2) imply that

$$\int_{\mathbb{R}^3} g u_n dx \to 0, \quad I_0(u_n) \to c,$$

and

$$\langle I_0'(u_n), v\rangle - \frac{1 - \lambda_n}{1 + \lambda_n} \int_{\mathbb{R}^3} g(x)v dx \to 0, \tag{3.3}$$

for any $v \in H$, where $I_0(u) = \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx$. From (3.3), we know there exists $c_0 > 0$ such that

$$\|I_0'(u_n)\| \leq c_0.$$

Therefore

$$\begin{aligned} c + c_0\|u_n\| \geq I_0(u_n) - \frac{1}{p}\langle I_0'(u_n), u_n\rangle &= \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2 + \left(\frac{b}{4} - \frac{b}{p}\right)\|u_n\|^4 \\ &\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|u_n\|^2, \end{aligned}$$

which means $\{u_n\}$ is bounded. Then, since the Sobolev embedding $H \hookrightarrow L^r(\mathbb{R}^3)$ ($r \in [2, 6)$) is compact, we might assume that, up to subsequence, there exists $u_n \in H$ such that

$$\begin{aligned} &u_n \rightharpoonup u \quad \text{weakly in } H_r^1(\mathbb{R}^3), \\ &u_n \to u \quad \text{strongly in } L^r(\mathbb{R}^3), r \in [2, 6), \\ &u_n \to u \quad \text{a.e. in } \mathbb{R}^3. \end{aligned} \tag{3.4}$$

Note that

$$\begin{aligned} &\langle I_0'(u_n), u_n - u\rangle = \int_{\mathbb{R}^3} (\nabla u_n \cdot \nabla(u_n - u) + V(x)u_n(u_n - u))dx \\ &+ b\int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + V(x)u_n^2\right) dx \left(\int_{\mathbb{R}^3} (\nabla u_n \cdot \nabla(u_n - u) + V(x)u_n(u_n - u))dx\right) \\ &- \int_{\mathbb{R}^3} |u_n|^{p-2}u_n(u_n - u)dx \to 0. \end{aligned}$$

By using (3.4), we see that $\|u_n\|$ converges to $\|u\|$, which implies the strong convergence in H. □

Remark. *Define $\mu_n = \frac{1-\lambda_n}{1+\lambda_n}$ and let μ be a limit for the sequence μ_n. It is clear $\mu \in [-1, 1]$ and u solves* (1.3).

In the following, let e_k be the eigenfunction corresponding to the eigenvalue λ_k defined in Proposition 2.1.

Lemma 3.2. *For k_0 sufficiently large, there exists $\rho > 0$ such that $I(u) \geq 1$ for all $u \in Y := span\{e_k; k \geq k_0\}$ with $\|u\| = \rho$.*

Proof. From Proposition 2.1, we know the Sobolev embedding $H \hookrightarrow L^6(\mathbb{R}^3)$ is continuous. Setting $C_1 = |g|_2$, since $\|u\|^4 \geq 0$, by Hölder's inequality, we obtain that for $u \in Y$,

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx - \int_{\mathbb{R}^3} g(x)u dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{p}|u|_2^r |u|_6^{p-r} - C_1|u|_2 \\ &\geq \left(\frac{1}{2} - \frac{C_0}{p}\lambda_{k_0}^{-r/2}\|u\|^{p-2}\right)\|u\|^2 - C_2\|u\| \end{aligned}$$

where $r = 3 - \frac{p}{2} > 0$. Choose $\rho > 0$ be such that $\rho^2 - 4(C_2\rho + 1) = 0$ and let $k_0 \in \mathbb{N}$ be such that $\frac{C_0}{p}\lambda_{k_0}^{-r/2}\rho^{p-2} \leq \frac{1}{4}$, the conclusion follows. □

Lemma 3.3. *Let now $X = span\{e_j; j < k_0\}$ be the orthogonal complement of Y. For any finite dimensional subspace $E_n \subset H$ containing X, there exists $R_n > \rho$ such that*

$$\sup I(S_{R_n}(E_n)) \leq 0.$$

Proof. For any fixed $u \in E_n$ and any $R > 0$, we have

$$I(Ru) \leq \frac{R^2}{2}\|u\|^2 + \frac{bR^4}{4}\|u\|^4 - C\frac{R^p}{p}\|u\|_p^p + CR\|u\|,$$

since all norms on finite dimensional subspace are equivalent. This lemma is thus proved. □

Proof of Theorem 1.1. From Lemmas 3.1–3.3, we know all the assumptions of Theorem 2.5 are satisfied. Thus we obtain the existence of ubbounded sequence of virtual critical values for equation (1.1) by Theorem 2.5. □

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