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INTEGRAL REPRESENTATION OF THE GENERALIZED BESSEL LINEAR FUNCTIONAL.

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Abstract. In this paper, we are interested in the integral representation problem of the generalized Bessel linear functional $B[\nu]$, well-known by the Pearson equation that it satisfies: $\big(x^3B[\nu]\big)' - \big(2(\nu+1)x^2+\frac{1}{2}\big)B[\nu]=0$. By means of some integral estimation and technical results including important inequalities of the incomplete gamma function and the exponential integral, we obtain an integral representation of $B[\nu]$, for every real number $\nu \neq -n$, $n \geq 0$. The connection formula between $B[\nu]$ and the classical Bessel linear functional $B(\alpha)$ allows us to obtain an integral representation of $B(\alpha)$, for all real number $\alpha \neq -(n/2)$, $n \geq 0$.

1. Introduction

Let $\mathcal{P}$ be the linear space of polynomials in one variable with complex coefficients and let $\mathcal{P}'$ be its algebraic dual. We denote by $\langle \mathcal{U}, f\rangle$ the action of $\mathcal{U}$ in $\mathcal{P}'$ on f in $\mathcal{P}$ and by $(\mathcal{U})_n := \langle \mathcal{U}, x^n\rangle$, $n \geq 0$, the moments of $\mathcal{U}$ with respect to the monomial sequence $\{x^n\}_{n\geq 0}$. When $(\mathcal{U})_0 = 1$, the linear functional $\mathcal{U}$ is said to be normalized. Let us define some operations in $\mathcal{P}'$, (see [1, 6, 9]). For any $\mathcal{U}$ in $\mathcal{P}'$, any q in $\mathcal{P}$ and any complex numbers a, b, c with $a \neq 0$, let $D\mathcal{U} = \mathcal{U}'$, $q\mathcal{U}$, $h_a\mathcal{U}$, $\tau_b\mathcal{U}$ and $\sigma\mathcal{U}$ be respectively the derivative, the left multiplication, the translation, the homothetic and the pair part of the linear functionals defined by duality:

$$\begin{aligned}
&\langle \mathcal{U}', f\rangle := -\langle \mathcal{U}, f'\rangle,\\
&\langle q\mathcal{U}, f\rangle := \langle \mathcal{U}, qf\rangle,\\
&\langle h_a\mathcal{U}, f\rangle := \langle \mathcal{U}, h_a f\rangle = \langle \mathcal{U}, f(ax)\rangle,\\
&\langle \tau_b\mathcal{U}, f\rangle := \langle \mathcal{U}, \tau_{-b} f\rangle = \langle \mathcal{U}, f(x+b)\rangle,\\
&\langle \sigma\mathcal{U}, f\rangle := \langle \mathcal{U}, \sigma f\rangle = \langle \mathcal{U}, f(x^2)\rangle,\ f \in \mathcal{P}.
\end{aligned}$$

Consider the symmetric generalized Bessel linear functional $B[\nu]$ given by its moments [2]:

$$\begin{aligned}
&(B[\nu])_{2n} = \frac{(-1)^n\Gamma(\nu+1)}{2^{2n}\Gamma(n+\nu+1)},\ n \geq 0,\\
&(B[\nu])_{2n+1} = 0,\ n \geq 0,
\end{aligned} \tag{1}$$

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where $\nu \neq -(n+1)\,, n \geq 0$ and here Γ is the gamma function.
The linear functional $B[\nu]$ is symmetric, *i.e.*, $(B[\nu])_{2n+1} = 0,\ n \geq 0$, and semiclassical (see [4, 6]) of class one satisfying the Pearson equation [2]:

$$\big(x^3 B[\nu]\big)' - \big(2(\nu+1)x^2 + \frac{1}{2}\big)B[\nu] = 0.$$

By referring to [2], the linear functional $B[\nu]$ for $\nu \geq (1/2)$, has the following integral representation:

$$\langle B[\nu], p\,(x)\rangle = S_\nu^{-1} \int_{-\infty}^{+\infty} U_\nu(x) p\,(x)\,dx,\ p \in \mathcal{P},$$

with $S_\nu = \int_{-\infty}^{+\infty} U_\nu\,(x)\,dx$ and where the function U_ν is in $L^1(\mathbb{R})$ and has the following expression:

$$U_\nu(x) = \begin{cases} 0, & x = 0, \\ \dfrac{1}{x^2} \displaystyle\int_{|x|}^{+\infty} \left(\frac{|x|}{t}\right)^{2\nu+1} e^{\frac{1}{4t^2} - \frac{1}{4x^2}} s(t^2) dt, & x \neq 0, \end{cases} \tag{2}$$

where s is the Stieltjes function given by $s\,(x) = \begin{cases} 0, & x \leq 0, \\ e^{-x^{\frac{1}{4}}} \sin x^{\frac{1}{4}}, & x > 0. \end{cases}$
By Fubini's theorem, S_ν can be written as follows:

$$S_\nu = 4 \int_0^{+\infty} G_\nu\,(t) \sin t\; dt, \tag{3}$$

with $G_\nu\,(t) = f_\nu\,(t)\, e^{-t},\ f_\nu\,(t) = t^{-4\nu-1} e^{\frac{1}{4t^4}} \varphi_{\nu-\frac{3}{2}}(t^2),\ \varphi_\nu\,(t) = \displaystyle\int_0^t x^{2\nu+2} e^{-\frac{1}{4x^2}}\,dx.$
Notice that $y = U_\nu(x)$ is the solution of the first-order linear differential equation:

$$\begin{cases} (x^3 y)' - \big(2(\nu+1)x^2 + \frac{1}{2}\big) y = g(x), \\ y(0) = 0, \end{cases} \tag{4}$$

where the function $g(x) = -\,|x|\, s(x^2)$ represents the null linear functional.

The main purpose of this paper is to give an integral representation of $B[\nu]$, for all real number $\nu \neq -n$, $n \geq 0$. To reach our goal, we need to treat two cases separately, the first one is $\nu \geq 0$ and the second is $\nu < 0$. In the first case, our approach is based essentially on the use of the fundamental Lemma 9 . The connection formulas that we highlight between the function φ_ν and the incomplete gamma function (resp. the exponential integral), as well as some double-inequalities established thereafter, will be important in the success of this approach. In the second cases, we use another approach based on a new connection formula between the linear functional $B[\nu]$ and $B[\nu+1]$. Finally, thanks to the connection formula between $B[\nu]$ and the classical Bessel linear functional $\mathcal{B}(\alpha)$, (see [2, 6, 9]), we obtain an integral representation of $\mathcal{B}(\alpha)$, for all real number $\alpha \neq -(n/2)$, $n \geq 0$.

The rest of this paper is organized as follows. In section 2, we develop some basic results and technical lemmas for future use. Section 3 is devoted to the integral representation problem of the generalized as well as the classical Bessel linear functional.

2. Preliminaries results.

2.1. **Some properties of the functions φ_ν, f_ν and G_ν.** For each real number ν, recall that the function φ_ν is given by

$$\varphi_\nu(x)=\int_0^x t^{2\nu+2}e^{-\frac{1}{4t^2}}\,dt,\ x\geq 0.$$

Upon the change of variable $y=\dfrac{1}{4t^2}$, we get

$$\varphi_\nu\left(x\right)=\frac{1}{2^{2\nu+4}}\Gamma\left(-\nu-\frac{3}{2},\frac{1}{4x^2}\right),\ x>0, \tag{5}$$

where for every $x>0$ and $a\in\mathbb{R}$, $\Gamma(a,x)=\displaystyle\int_x^{+\infty}t^{a-1}e^{-t}dt$, is the incomplete gamma function, known by the following useful properties (see [3, 9]),

$$\Gamma(a,x)=(a-1)\Gamma(a-1,x)+x^{a-1}e^{-x},\quad \Gamma(1,x)=e^{-x}. \tag{6}$$

$$\frac{x^a}{x+1-a}e^{-x}\leq\Gamma(a,x)\leq\frac{(1+x)x^{a-1}}{x+2-a}e^{-x},\quad a\leq 1. \tag{7}$$

$$\frac{1}{2}e^{-x}\ln(1+\frac{2}{x})\leq E_1(x)\leq e^{-x}\ln(1+\frac{1}{x}), \tag{8}$$

and $E_1(x)=\Gamma(0,x)$ is the exponential integral.
By substituting of (6) into (7) and then replacing a by $a+1$, we obtain

$$\frac{x^a}{x+1-a}e^{-x}\leq\Gamma(a,x)\leq\frac{x^a}{x-a}e^{-x},\quad a\leq 0. \tag{9}$$

Lemma 1. *For every $\nu\geq-(3/2)$, we have*

$$\frac{2x^{2\nu+5}}{1+2(2\nu+5)x^2}e^{-\frac{1}{4x^2}}\leq\varphi_\nu(x)\leq\frac{2x^{2\nu+5}}{1+2(2\nu+3)x^2}e^{-\frac{1}{4x^2}},\ x>0. \tag{10}$$

Proof. Immediate from (5) and (9). ∎

Using (3) and (10), the following double-inequalities hold,

$$\frac{2x^3}{1+4(\nu+1)x^4}\leq f_\nu(x)\leq\frac{2x^3}{1+4\nu x^4}, \tag{11}$$

$$\frac{2x^3}{1+4(\nu+1)x^4}e^{-x}\leq G_\nu(x)\leq\frac{2x^3}{1+4\nu x^4}e^{-x},\ \text{for all } x\geq 0 \text{ and } \nu\geq 0. \tag{12}$$

In view of (12), it is clear that $G_\nu(0)=0$, $G_\nu(x)>0$ for all $x>0$, and $\lim\limits_{x\to+\infty}G_\nu\left(x\right)=0$, for every $\nu\geq 0$. Thus, G_ν has a maximum for $x=\overline{x}$ satisfying $G'_\nu(\overline{x})=0$, *i.e.*, $f'_\nu(\overline{x})=f_\nu(\overline{x})$.

Lemma 2. *For every $\nu\geq 0$, the function G_ν is decreasing on $[2\pi,+\infty[$.*

Proof. Let $t>0$, be an extremum of the function G_ν. Then, $G'_\nu(t)=0$. Equivalently, $f'_\nu(t)=f_\nu(t)$. By (11), we get $f_\nu(t)=\frac{2t^3}{1+(4\nu+1)t^4+t^5}\geq\frac{2t^3}{1+4(\nu+1)t^4}$. An easy computation leads to $t\leq 3<2\pi$. This finishes the proof of the lemma. ∎

The following double-inequality will be useful for the sequel.

Lemma 3. *For every $\nu \geq -(3/2)$, the following double-inequality holds,*

$$\frac{x^{2\nu+3}e^{\frac{-1}{4x^2}}\ln(1+8x^2)}{2\big(2+(2\nu+3)\ln(1+4x^2)\big)} \leq \varphi_\nu(x) \leq \frac{1}{2}x^{2\nu+3}e^{\frac{-1}{4x^2}}\ln(1+4x^2),\ x>0. \tag{13}$$

Proof. Let $\nu \geq -(3/2)$. We can write $\varphi_\nu(x) = \frac{1}{2}\int_0^x t^{2\nu+3}\frac{d}{dt}\big(E_1(\frac{1}{4t^2})\big)dt$.
Upon integration by parts, we get

$$\varphi_\nu(x) = \frac{1}{2}x^{2\nu+3}E_1(\frac{1}{4x^2}) - \frac{1}{2}(2\nu+3)\int_0^x t^{2\nu+2}E_1(\frac{1}{4t^2})dt,\ x>0.$$

Using (8), (3) and the fact that the function $x \mapsto \ln(1+x)$ is increasing on the interval $]-1,+\infty[$, it follows that

$$\varphi_\nu(x) \geq \frac{1}{4}x^{2\nu+3}e^{-\frac{1}{4x^2}}\ln(1+8x^2) - \frac{1}{2}(2\nu+3)\ln(1+4x^2)\varphi_\nu(x),\ x>0.$$

This implies, $\varphi_\nu(x) \geq \frac{1}{2}\frac{x^{2\nu+3}e^{-\frac{1}{4x^2}})\ln(1+8x^2)}{2+(2\nu+3)\ln(1+4x^2)},\ x>0.$
For the same reason, we find the right-hand inequality of (13), as follows:

$$\varphi_\nu(x) \leq \frac{1}{2}x^{2\nu+3}E_1(\frac{1}{4x^2}) \leq \frac{1}{2}x^{2\nu+3}e^{-\frac{1}{4x^2}}\ln(1+4x^2), \quad x>0.$$

The proof of lemma 3, is complete. ∎

According to lemma 3 and by (5), we can establish the following new double-inequality for the incomplete gamma function:

$$\frac{1}{2}x^a e^{-x}\frac{\ln(1+\dfrac{2}{x})}{1-a\ln(1+\dfrac{1}{x})} \leq \Gamma(a,x) \leq x^a e^{-x}\ln(1+\frac{1}{x}),\ x>0,\ a\leq 0.$$

Again by lemma 3 and by (3), the following double-inequalities is achieved,

$$\frac{1}{4}\frac{\ln(1+8x^4)}{x[1+\nu\ln(1+4x^4)]} \leq f_\nu(x) \leq \frac{1}{2}\frac{\ln(1+4x^4)}{x}, \tag{14}$$

$$\frac{1}{4}\frac{\ln(1+8x^4)}{x[1+\nu\ln(1+4x^4)]}e^{-x} \leq G_\nu(x) \leq \frac{1}{2}\frac{\ln(1+4x^4)}{x}e^{-x},\ x>0,\ \nu\geq 0. \tag{15}$$

2.2. **Some technical results on integral estimation.**

Proposition 4. *Let $g:[0,+\infty[\,\to[0,+\infty[$ be a decreasing function, continuous on $[0,+\infty[$ and differentiable on $]0,+\infty[$. Then, for every $\rho\geq 0$, we have*

$$\int_0^x t^\rho g(t)e^{-t}dt \geq \Omega_\rho(x)g(x),\ x\geq 0, \tag{16}$$

where $\Omega_\rho(x) = \Gamma(\rho+1)e^{-x}\sum_{n\geq 0}\frac{x^{n+\rho+1}}{\Gamma(n+2+\rho)}$.
In particular, $\Omega_p(x) = p!\big[1-e^{-x}\sum_{k=0}^{p}\frac{x^k}{k!}\big]$, for all integer $p\geq 0$.

Proof. When $\rho=0$, the assumption g is decreasing on $[0,+\infty[$ implies,

$$\int_0^x g(t)e^{-t}dt \geq g(x)\int_0^x e^{-t}dt = g(x)\Omega_0(x), \quad x>0,$$

where $\Omega_0(x) = 1-e^{-x} = e^{-x}\sum_{n\geq 0}\frac{x^{n+1}}{(n+1)!}$.
Hence, (16) is valid for $\rho = 0$.

If $\rho > 0$, setting $R_\rho(t) = t^\rho g(t)$, $t > 0$ and $R_\rho(0) = 0$. By assumption g is decreasing on $[0, +\infty[$ and differentiable on $]0, +\infty[$, we can write $tR'_\rho(t) - \rho R_\rho(t) = t^{\rho+1} g'(t) \le 0,\ t > 0$. This yields,

$$\rho t^n R_\rho(t) e^{-t} \ge t^{n+1} R'_\rho(t) e^{-t}, \quad t > 0. \tag{17}$$

For every $x > 0$, let $\{J_n(x)\}_{n\ge 0}$ be the sequence of nonnegative real numbers,

$$J_n(x) := \frac{1}{\Gamma(n+1+\rho)} \int_0^x t^n R_\rho(t) e^{-t} dt,\ n \ge 0.$$

Using (17), it is easy to see that $\Gamma(n+1+\rho) J_n(x) \ge \frac{1}{\rho} \int_0^x t^{n+1} e^{-t} R'_\rho(t) dt,\ n \ge 0$. Upon integration by parts, we obtain

$$\Gamma(n+1+\rho) J_n(x) \ge \frac{1}{\rho} \big[x^{n+1} e^{-x} R_\rho(x) - \int_0^x t^n e^{-t} (n+1-t) R_\rho(t) dt \big],\ n \ge 0.$$

Equivalently, we have $J_n(x) - J_{n+1}(x) \ge \frac{x^{n+1}}{\Gamma(n+2+\rho)} e^{-x} R_\rho(x),\ n \ge 0$. This implies, $J_0(x) \ge J_n(x) + e^{-x} R_\rho(x) \sum_{l=0}^{n-1} \frac{x^{l+1}}{\Gamma(l+2+\rho)},\ n \ge 1$. Since $J_n(x) \ge 0$ for all $n \ge 0$ and $x > 0$, we get $J_0(x) \ge e^{-x} R_\rho(x) \sum_{l=0}^{n-1} \frac{x^{l+1}}{\Gamma(l+2+\rho)},\ n \ge 1$. If n tends to $+\infty$, then $\int_0^x t^\rho g(t) e^{-t} dt \ge \Omega_\rho(x) g(x)$, where $\Omega_\rho(x) = \Gamma(\rho+1) e^{-x} \sum_{n\ge 0} \frac{x^{n+\rho+1}}{\Gamma(n+2+\rho)}$.
Hence, the inequality (16) is valid for all $\rho \ge 0$ and all $x > 0$.

If $\rho = p$: an nonnegative integer, we have

$$\Omega_p(x) = p! e^{-x} \sum_{n\ge 0} \frac{x^{n+p+1}}{(n+1+p)!} = p! \big(1 - e^{-x} \sum_{k=0}^{p} \frac{x^k}{k!}\big),\ x > 0.$$

This finishes the proof of the proposition. ∎

Furthermore, the following inequalities are needed for what comes next.

Proposition 5. *For every $x > 0$, we have*

$$\int_0^x \frac{t^4 e^{-t}}{1 + 4(\nu+1) t^4} dt \ge \Omega_4(x) \frac{1}{1 + 4(\nu+1) x^4}, \quad \nu \ge -1, \tag{18}$$

$$\int_0^x e^{-t} \ln(1 + 8t^4) dt \ge \Omega_4(x) \frac{\ln(1 + 8x^4)}{x^4}, \tag{19}$$

where $\Omega_4(x) = 24\big(1 - e^{-x} \sum_{k=0}^{4} \frac{x^k}{k!}\big)$.

Proof. To establish (18), we use proposition 4, with $\rho = p = 4$ and $g(t) = \frac{1}{1+4(\nu+1)t^4}$, for $t \ge 0$. Clearly, $g'(t) = \frac{-16(\nu+1)t^3}{(1+4(\nu+1)t^4)^2} \le 0$, for $t \ge 0$ and $\nu \ge -1$.

To establish (19), we use proposition 4, with $\rho = p = 4$ and $g(t) = 8h(8t^4)$, for all $t \ge 0$, where $h(t) = \frac{\ln(1+t)}{t}$, for all $t > 0$ and $h(0) = 1$. We can show that, $g'(t) = 256 t^3 h'(8t^4) \le 0,\ t > 0$. Indeed, it suffices to show that $h'(t) \le 0$, for all $t \ge 0$. Clearly, $h'(t) = \frac{l(t)}{t^2}$, for all $t > 0$, where $l(t) = \frac{t}{t+1} - \ln(t+1)$, for all $t \ge 0$. Since $l'(t) = -\frac{t}{(t+1)^2} \le 0$, for all $t \ge 0$, then $l(t) \le l(0) = 0$, for all $t \ge 0$.
Hence, $h'(t) \le 0$, for all $t \ge 0$ and then $g'(t) \le 0$, for all $t \ge 0$. ∎

2.3. **Some asymptotic behavior results.** Let h be a function defined on $\mathbb{R}$, and having the following properties:

- $\mathbf{P}_1$. $h(x) = f(\sqrt{|x|})$, for every x in $\mathbb{R}$, where $f(x) = \sum_{n\geq 0} a_n x^n$ is an entire function.
- $\mathbf{P}_2$. The function h and all its derivatives, are with rapid decay at $\pm\infty$, i.e., for every integers $k \geq 0$ and $n \geq 0$,

$$\sup_{x\in\mathbb{R}} \mid x^k h^{(n)}(x) \mid < +\infty.$$

Now, for any complex number ν and any function h satisfying the properties $\mathbf{P}_i$, $i = 1, 2$, let us consider the following first-order linear differential equation:

$$E_\nu(h) : \begin{cases} \left(x^3 y\right)' - \left[2\,(\nu+1)\,x^2 + \frac{1}{2}\right] y = h\,(x)\,, \\ y(0) = -2h(0). \end{cases} \tag{20}$$

The solution of (20) is defined on the real line $\mathbb{R}$ and given by

$$y(x) = \begin{cases} -\,|x|^{2\nu-1}\, e^{\frac{-1}{4x^2}} \displaystyle\int_{|x|}^{+\infty} t^{-2(\nu+1)} e^{\frac{1}{4t^2}}\, h\,(t)\, dt, & x \neq 0, \\ -2h(0), \ x = 0. \end{cases} \tag{21}$$

Lemma 6. *The function y given by (21) is even, infinitely differentiable on $\mathbb{R} - \{0\}$ and fulfills the following properties:*

(i) *When $\mid x \mid \to +\infty$, we have $\left|x^n y^{(n)}(x)\right| = \mathrm{O}(\frac{1}{|x|^{k+2}})$, for each positive integer k such that $k > k_\nu = \max\{0, -2\Re(\nu) - 1\}$, and each integer $n \geq 0$.*

(ii) *The function $y \in L^1(\mathbb{R}) \cap C^0(\mathbb{R})$.*

Proof. By assumption $\mathbf{P}_2$, for every integers $k \geq 0$ and $n \geq 0$, there exist $M_{k,n} > 0$ and $\eta_{k,n} > 0$, such that

$$\left|h^{(n)}(x)\right| \leq \frac{M_{k,n}}{\mid x \mid^k}, \quad \mid x \mid > \eta_{k,n}. \tag{22}$$

i.e., for every integers $k \geq 0$ and $n \geq 0$, $\left|h^{(n)}(x)\right| = \mathrm{O}\big(\frac{1}{|x|^k}\big)$, $\mid x \mid \to +\infty$.
By (21), (22) with $n = 0$, and since $e^{\frac{1}{4t^2}} \leq e^{\frac{1}{4x^2}}$, for all $t \geq |x|$, we obtain

$$|y(x)| \leq |x|^{2\Re(\nu)-1}\, e^{\frac{-1}{4x^2}} \int_{|x|}^{+\infty} t^{-2(\Re(\nu)+1)} e^{\frac{1}{4t^2}}\, |h(t)|\, dt \leq \frac{M_{k,0}}{(k + 2\Re\,(\nu) + 1)} \frac{1}{|x|^{k+2}},$$

for all $x \in \mathbb{R}$ such that $|x| > \eta_{k,0}$ and all integer $k > k_\nu = \max\{0, -2\Re(\nu) - 1\}$. So, for every integer $k > k_\nu$,

$$|y\,(x)| = \mathrm{O}\left(\frac{1}{|x|^{k+2}}\right), \ \mid x \mid \to +\infty. \tag{23}$$

By (20), it is clear that $\left|x^{k+3} y'\right| \leq \left((2\,|\nu| + 1)x^2 + \frac{1}{2}\right) \left|x^k y(x)\right| + \left|x^k h(x)\right|$, and on account of (22) and (23), it follows that for every integer $k > k_\nu$,

$$\mid x y'(x) \mid = \mathrm{O}\left(\frac{1}{|x|^{k+2}}\right), \ \mid x \mid \to +\infty. \tag{24}$$

By induction on the integer $n \geq 0$, let's show that for every integer $k > k_\nu$, when $\mid x \mid \to +\infty$, we have

$$\left| x^n y^{(n)}(x) \right| = \mathrm{O}\left(\frac{1}{|x|^{k+2}} \right), \mid x \mid \to +\infty.$$

For $n = 0$, and $n = 1$, the recurrence property is true by (23) and (24) respectively. Suppose that the recurrence property is valid up to the order m ($m \geq 1$), and let's show that it remains valid to the order $m+1$.
By (20), after differentiating $m-$times and by using the Leibnitz's formula,

$$x^3 y^{(m+1)}(x) = \big((2\nu - 3m - 1)x^2 + \frac{1}{2}\big) y^{(m)}(x) + m(4\nu - 3m + 1) x y^{(m-1)}(x) + \\ m(m-1)(2\nu - m + 1) y^{(m-2)}(x) + h^{(m)}(x),$$

then

$$\mid x^{3+k+m} y^{(m+1)}(x) \mid \leq (2\,|\nu| + 3m + 1)x^2 + \frac{1}{2} \left| x^{k+m} y^{(m)}(x) \right| \\ + m(4 \mid \nu \mid + 3m + 1) \mid x^{1+k+m} y^{(m-1)}(x) \mid \\ + m(m-1)(2\,|\nu| + m + 1) \left| x^{k+m} y^{(m-2)}(x) \right| + \left| x^{k+m} h^{(m)}(x) \right|, \tag{25}$$

By induction hypothesis and (22), each one of the quantities $\left| x^{2+k+m} y^{(m)}(x) \right|$, $\left| x^{1+k+m} y^{(m-1)}(x) \right|$, $\left| x^{k+m} y^{(m-2)}(x) \right|$ and $\left| x^{k+m} h^{(m)}(x) \right|$, is equal to $\mathrm{O}(1)$. So, by (25), $\left| x^{m+1} y^{(m+1)}(x) \right| = \mathrm{O}\left(\frac{1}{|x|^{k+2}} \right)$, $\mid x \mid \to +\infty$, for every integer $k > k_\nu$.
Hence, (i) holds.

When $\mid x \mid < 1$, we start by noting that

$$y(x) = w(|x|) - |x|^{2\nu - 1} e^{\frac{-1}{4x^2}} \int_1^{+\infty} t^{-2(\nu+1)} e^{\frac{1}{4t^2}} h(t)\, dt, \tag{26}$$

where $w(x) = \begin{cases} -x^{2\nu-1} e^{\frac{-1}{4x^2}} \displaystyle\int_x^1 t^{-2(\nu+1)} e^{\frac{1}{4t^2}} h(t)\, dt, \ x > 0, \\ -2h(0), \ x = 0. \end{cases}$

Clearly, $y(x) = w(|x|) + \mathrm{o}(e^{\frac{-1}{8x^2}})$. By applying the Hospital's rule to the ratio,

$$\lim_{x \to 0^+} w(x) = \lim_{x \to 0^+} \frac{-\displaystyle\int_x^1 t^{-2(\nu+1)} e^{\frac{1}{4t^2}} h(t)\, dt}{x^{-2\nu+1} e^{\frac{1}{4x^2}}} \\ = \lim_{x \to 0^+} \frac{h(x)}{(-2\nu + 1)x^2 - \frac{1}{2}} = -2h(0) = w(0).$$

So, $\lim\limits_{x \to 0^+} w(x) = -2h(0)$. Hence, $\lim\limits_{x \to 0} y(x) = -2h(0)$. Thus, y is continuous on $\mathbb{R}$. Accordingly, the function y given by (21), is in $L^1(\mathbb{R}) \cap C^0(\mathbb{R})$.
Hence, (ii) holds. ∎

Lemma 7. *For x small enough the function y given by (21) has the following expansion:*

$$y(x) = \sum_{l=0}^{N} \alpha_l \,|x|^{\frac{l}{2}} + \mathrm{o}\left(|x|^{\frac{N}{2}} \right), \textit{ for every integer } N \geq 4,$$

where the $\alpha_l' s$ are the coefficients of the series representation of f appearing in property $\boldsymbol{P}_1$ and given by

$$\begin{cases} a_l - (\frac{l}{2} - 1 - 2\nu)\alpha_{l-4} + \frac{1}{2}\alpha_l = 0,\ 0 \leq l \leq N, \\ \alpha_{-l} = 0,\ l \geq 1, \quad and \quad \alpha_l = 0,\ l \geq N+1. \end{cases}$$

Proof. Recall that the function w given by (26) is continuous on $[0, +\infty[$ and infinitely differentiable on $]0, +\infty[$. It is easy to show that w satisfies

$$\begin{cases} \left(x^3 w\right)' - \left[2\left(\nu+1\right)x^2 + \frac{1}{2}\right] w = h(x), \\ w(0) = -2h(0), \quad w(1) = 0, \end{cases} \tag{27}$$

where the function h satisfies the properties $\mathbf{P}_i$, $i = 1, 2$.
For any integer $N \geq 4$, let $(\alpha_l)_{l\geq 0}$ be the sequence of complex numbers given by

$$\begin{cases} a_l - (\frac{l}{2} - 1 - 2\nu)\alpha_{l-4} + \frac{1}{2}\alpha_l = 0,\ 0 \leq l \leq N, \\ \alpha_{-l} = 0,\ l \geq 1, \quad \alpha_l = 0,\ l \geq N+1, \end{cases} \tag{28}$$

and C_N be the function defined on $[0, +\infty[$ by

$$C_N\left(x\right) = \begin{cases} \left(w(x) - \sum_{l=0}^{N} a_l x^{\frac{l}{2}}\right) x^{-\frac{N+1}{2}}, & x > 0, \\ -2[a_{N+1} + (2\nu_N + 1)\alpha_{N-3}], & x = 0, \end{cases} \tag{29}$$

where $\nu_N = \nu - \frac{N+1}{4}$.
Substituting (29) into (27) and taking (28) into account, the function v defined on $[0, +\infty[$ by $v(x) = C_N(x)$, satisfies

$$\begin{cases} (x^3 v)' - [2(\nu_N+1)x^2 + \frac{1}{2}]v = f_N(\sqrt{x}), \\ v(0) = -2[a_{N+1} + (2\nu_N+1)\alpha_{N-3}], \quad v(1) = -\sum_{l=0}^{N}\alpha_l, \end{cases}$$

where the function f_N is defined on the interval $[0, +\infty[$ by

$$f_N(t) = \begin{cases} \sum_{l=0}^{\infty}\left[a_{l+N+1} + (-\frac{l}{2} + 2\nu_N + 1)\alpha_{l+N-3}\right] t^l, & t > 0, \\ a_{N+1} + \alpha_{N-3}(2\nu_N+1), & t = 0. \end{cases}$$

The resolution of the last first-order differential equation gives us

$$v\left(x\right) = C_N(x) = \Big[- e^{\frac{1}{4}} \sum_{l=0}^{N} \alpha_l - \int_x^1 t^{-2(\nu_N+1)} f_N(\sqrt{t}) e^{\frac{1}{4t^2}}\, dt \Big] x^{2\nu_N - 1} e^{\frac{-1}{4x^2}}, \quad x > 0.$$

If we apply the Hospital's rule to the ratio, we get

$$\begin{aligned} \lim_{x\to 0^+} C_N(x) &= \lim_{x\to 0^+} \frac{-e^{\frac{1}{4}}\sum_{l=0}^{N}\alpha_l - \int_x^1 t^{-2(\nu_N+1)} f_N\left(\sqrt{t}\right) e^{\frac{1}{4t^2}}\, dt}{x^{-2\nu_N+1} e^{\frac{1}{4x^2}}} \\ &= \lim_{x\to 0^+} \frac{f_N\left(\sqrt{x}\right)}{\left(-2\nu_N+1\right)x^2 - \frac{1}{2}} = -2f_N(0) = C_N(0). \end{aligned}$$

Thus, the function $x \mapsto C_N(x)$ is continuous on $[0, +\infty[$.
Accordingly, for x small enough, the function given by (21) has the following expansion:

$$y(x) = \sum_{l=0}^{N} \alpha_l \left|x\right|^{\frac{l}{2}} + \left|x\right|^{\frac{N+1}{2}} R_N(|x|),$$

where the function R_N is continuous on $[0, +\infty[$ and given by

$$R_N(x) = \begin{cases} x^{2\nu_N - 1} V_N(x) e^{\frac{-1}{4x^2}}, & x > 0, \\ -2f_N\left(0\right), & x = 0, \end{cases}$$

$V_N(x) = -e^{\frac{1}{4}} \sum_{l=0}^{N} \alpha_l - \int_x^1 t^{-2(\nu_N+1)} f_N(\sqrt{t}) e^{\frac{1}{4t^2}}\, dt - \int_1^{+\infty} t^{-2(\nu+1)} f(\sqrt{t}) e^{\frac{1}{4t^2}}\, dt.$ ∎

Lemma 8. *The function y given by (21) satisfies:*

$$\lim_{x\to 0} x^n y^{(n)}(x) = \begin{cases} -2h(0), & n = 0, \\ 0, & n \geq 1. \end{cases}$$

Proof. Let y_1 be the function defined on $\mathbb{R}$ and given by

$$y_1(x) = \begin{cases} xy'(x) - (2\nu - 1)y(x), & x \neq 0, \\ -2(1-2\nu)h(0), & x = 0, \end{cases} \tag{30}$$

where y is the function given by (21).

From (20) and (30), we get $x^2 y_1(x) = \frac{1}{2}y(x) + h(x),\ x \in \mathbb{R}$. Besides, the function y_1 satisfies

$$\begin{cases} (x^3 y_1)' - (2\nu x^2 + \frac{1}{2})y_1 = h_1(x), \\ y_1(0) = -2(1-2\nu)h(0) = -2h_1(0), \end{cases} \tag{31}$$

where $h_1(x) = xh'(x) + (1-2\nu)h(x),\ x \in \mathbb{R}$. Notice that the function h_1 satisfies the properties $\mathbf{P}_i$, $i = 1, 2$. So, by lemma 6, where ν is replaced by $\nu - 1$ and h by h_1, the solution y_1 of (31), is in $L^1(\mathbb{R}) \cap C^0(\mathbb{R})$.

Clearly, $y_1(0) = \lim_{x\to 0} y_1(x) = \lim_{x\to 0} xy'(x) - (2\nu - 1)y(x)$, on account of (30).

So, $-2(1-2\nu)h(0) = \lim_{x\to 0} xy'(x) + 2(2\nu - 1)h(0)$. Hence, $\lim_{x\to 0} xy'(x) = 0$.

Let $\{h_n\}_{n\geq 0}$ be the sequence of functions defined on $\mathbb{R}$ and entirely given by

$$\begin{cases} h_0(x) = h(x), \\ h_{n+1}(x) = xh_n'(x) + \big[1 - 2(\nu - n)\big]h_n(x), \quad n \geq 0. \end{cases} \tag{32}$$

For every integer $n \geq 0$, we can see that h_n satisfies the properties $\mathbf{P}_i$, $i = 1, 2$.

Let $(y_n)_{n\geq 0}$ be the sequence of functions given by

$$\begin{cases} y_{n+1}(x) = \begin{cases} xy_n'(x) - \big(2(\nu - n) - 1\big)y_n(x), & x \neq 0, \\ -2h_{n+1}(0), & x = 0, \end{cases} \\ y_0 = y, \text{ where } y \text{ is given by (21).} \end{cases} \tag{33}$$

By induction on the integer n, it is easy to show that the functions y_n, $n \geq 0$, satisfying

$$\begin{cases} (x^3 y_n)' - \big(2(\nu - n + 1)x^2 + \frac{1}{2}\big)y_n = h_n(x), \\ y_n(0) = -2h_n(0). \end{cases} \tag{34}$$

In addition, we have $x^2 y_{n+1}(x) = \frac{1}{2}y_n(x) + h_n(x),\ x \in \mathbb{R}$. From (34) and by lemma 6, the functions y_n, $n \geq 0$, are continuous on $\mathbb{R}$, and satisfying

$$\lim_{x\to 0} y_n(x) = -2h_n(0), \quad n \geq 0.$$

From (32) and (33), it comes that

$$\begin{aligned} -2\big(1 - 2(\nu - n)\big)h_n(0) = -2h_{n+1}(0) &= \lim_{x\to 0} y_{n+1}(x) \\ &= \lim_{x\to 0} \big[xy_n'(x) - \big(2(\nu - n) - 1\big)\big]y_n(x) \\ &= \lim_{x\to 0} xy_n'(x) - 2\big(1 - 2(\nu - n)\big)h_n(0). \end{aligned}$$

This implies, $\lim_{x\to 0} xy_n'(x) = 0$, $n \geq 0$. By induction on the integer $k \geq 1$, let's show that $\lim_{x\to 0} x^k y_n^{(k)}(x) = 0$, for every integer $n \geq 0$.

For $k = 1$, we have already seen that $\lim_{x\to 0} xy_n'(x) = 0$, for every integer $n \geq 0$.

Suppose that the recurrence property is valid until the order m and let's show that

it remains valid to the order $m+1$.
By induction hypothesis and from (33), we obtain

$$\begin{aligned} 0 &= \lim_{x\to 0} x^m y_{n+1}^{(m)}(x) = \lim_{x\to 0} x^m \Big(x y_n'(x) - \big(2(\nu-n)-1\big) y_n(x)\Big)^{(m)} \\ &= \lim_{x\to 0} x^{m+1} y_n^{(m+1)}(x) + \big(m - 2(\nu-n)+1\big) x^m y_n^{(m)}(x) \\ &= \lim_{x\to 0} x^{m+1} y_n^{(m+1)}(x). \end{aligned}$$

Hence, the recurrence property holds.
Accordingly, $\lim_{x\to 0} x^k y^{(k)}(x) = 0$, for every integer $k \geq 1$, since $y_0 = y$, where y is given by (21). ∎

3. Integral representations of $B[\nu]$ and $\mathcal{B}(\alpha)$.

3.1. An integral representation of $B[\nu]$.

Case $\nu \geq 0$.
Let us show that the integral representation of the linear functional $B[\nu]$ given by the authors in [2] remains valid for all $\nu \geq 0$. To do so, we need the following fundamental lemma.

Lemma 9. *Consider the following integral:* $S = \int_0^{+\infty} G(x) \sin x \, dx$, *where* $G : [0, +\infty[\to \mathbb{R}$ *is a nonnegative, continuous and decreasing function on* $[2\pi, +\infty[$, *satisfying the following condition:*

$$\int_0^{\pi} [G(x) - G(x+\pi)] \sin x \, dx > 0. \tag{35}$$

Then, $S > 0$.

Proof. Let $S_n = \int_0^{\pi} [G(x+2n\pi) - G(x+(2n+1)\pi)] \sin x \, dx, \ n \geq 0$. Clearly, $\int_0^{2n\pi} G(x) \sin x \, dx = \sum_{k=0}^{n-1} S_k, \ n \geq 1$. Since $\sin x \geq 0$ on $[0, \pi]$, and by assumption G is decreasing on $[2\pi, +\infty[$, we get $S_n \geq 0, \ n \geq 1$. Therefore, $\int_0^{2n\pi} G(x) \sin x \, dx \geq S_0, \ n \geq 1$. While n tends to $+\infty$ and by taking (35) into account, we obtain $S \geq S_0 > 0$. ∎

Theorem 10. *For any* $\nu \geq 0$, *we have* $S_\nu > 0$ *and then the generalized Bessel linear functional* $B[\nu]$ *has the following integral representation:*

$$\langle B[\nu], p\rangle = S_\nu^{-1} \int_{-\infty}^{+\infty} \frac{1}{x^2} \int_{|x|}^{+\infty} \left(\frac{|x|}{t}\right)^{2\nu+1} e^{\frac{1}{4t^2} - \frac{1}{4x^2}} s(t^2) dt \, p(x) \, dx, \ p \in \mathcal{P}. \tag{36}$$

Proof. By lemmas 2 and 9, where $G = G_\nu$, in order to show that $S_\nu > 0$, just check the condition (35). To achieve this goal, we need to distinguish three cases.
$\mathbf{C_1}$**.** $\nu = 0$.
For $\nu = 0$ in (15), we get $\frac{1}{4} \frac{\ln(1+8x^4)}{x} e^{-x} \leq G_0(x) \leq \frac{1}{2} \frac{\ln(1+4x^4)}{x} e^{-x}$, for all $x > 0$. So, the inequality (35) is fulfilled if the following condition is verified,

$$\int_0^{\pi} \frac{\ln(1+8x^4)}{x} e^{-x} \sin x \, dx > 2 \int_0^{\pi} \frac{\ln(1+4(x+\pi)^4)}{x+\pi} e^{-x-\pi} \sin x \, dx.$$

A lower bound for $\displaystyle\int_0^{\pi} \frac{\ln(1+8x^4)}{x} e^{-x} \sin(x)\, dx$:
We always have

$$\int_0^{\pi} \frac{\ln(1+8x^4)}{x} e^{-x} \sin x\, dx = \int_0^{\frac{\pi}{2}} \frac{\ln(1+8x^4)}{x} e^{-x} \sin x\, dx + \int_{\frac{\pi}{2}}^{\pi} \frac{\ln(1+8x^4)}{x} e^{-x} \sin x\, dx.$$

Since

$$\sin x \geq \frac{2}{\pi} x, \quad x \in [0, \frac{\pi}{2}], \tag{37}$$

then $\int_0^{\frac{\pi}{2}} \frac{\ln(1+8x^4)}{x} e^{-x} \sin x\, dx \geq \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \ln(1+8x^4) e^{-x} dx$.
From (19) taking with $x = \frac{\pi}{2}$, it follows that

$$\int_0^{\frac{\pi}{2}} \frac{\ln(1+8x^4)}{x} e^{-x} \sin x\, dx \geq \Delta_1, \tag{38}$$

where $\Delta_1 = \frac{32}{\pi^5} \Omega_4(\frac{\pi}{2}) \ln(1 + \frac{\pi^4}{2}) \simeq 0,216716$.
On the other hand, since $\ln(1+x) \geq \frac{\ln(1+\alpha)}{\alpha} x$, for all $x \in [0, \alpha]$ and $\alpha > 0$, we get

$$\int_{\frac{\pi}{2}}^{\pi} \frac{\ln(1+8x^4)}{x} e^{-x} \sin x\, dx \geq \Delta_2, \tag{39}$$

where $\Delta_2 = \frac{\ln(1+8\pi^4)}{\pi^4} \int_{\frac{\pi}{2}}^{\pi} x^3 e^{-x} \sin x\, dx$, and after integration by parts, we obtain

$$\Delta_2 = \frac{e^{-\frac{\pi}{2}} \ln\left(1+8\pi^4\right)}{2\pi^4} \left\{ e^{-\frac{\pi}{2}} \left(\pi^3 + 3\pi^2 + 3\pi\right) + \frac{\pi^3}{8} - \frac{3}{2}\pi - 3 \right\} \simeq 0,07620.$$

From (38) and (39), we get

$$\int_0^{\pi} \frac{\ln\left(1+8x^4\right)}{x} e^{-x} \sin x\, dx \geq \Delta_3, \tag{40}$$

where $\Delta_3 = \Delta_1 + \Delta_2 \simeq 0,292916$.

An upper bound for $\displaystyle 2\int_0^{\pi} \frac{\ln(1+4(x+\pi)^4)}{x+\pi} e^{-x-\pi} \sin x\, dx$:
The fact that the function $x \mapsto \dfrac{\ln(1+4x^4)}{x}$ is decreasing on $[\pi, +\infty[$, yields

$$2\int_0^{\pi} \frac{\ln(1+4(x+\pi)^4)}{x+\pi} e^{-x-\pi} \sin x\, dx \leq \Delta_4, \tag{41}$$

where $\Delta_4 = \frac{2e^{-\pi} \ln(1+4\pi^4)}{\pi} \int_0^{\pi} e^{-t} \sin t\, dt$ and after integrations by parts, we obtain $\Delta_4 = \frac{\ln(1+4\pi^4)}{\pi} e^{-\pi}(e^{-\pi}+1) \simeq 0,08563$. Clearly, $\Delta_3 > \Delta_4$ and then (35) is fulfilled.
$\mathbf{C}_2$. $0 < \nu \leq \mu$, with $\mu \simeq 0,405589$.
Using (15), the inequality (35) is fulfilled if we have

$$\int_0^{\pi} \frac{e^{-x} \sin x \ln(1+8x^4)}{x\big(1+\nu\ln(1+4x^4)\big)} dx > 2\int_0^{\pi} \frac{e^{-x-\pi} \sin x \ln(1+4(x+\pi)^4)}{x+\pi} dx. \tag{42}$$

For any $\nu > 0$, since the function $x \mapsto \frac{1}{1+\nu\ln(1+4x^4)}$ is decreasing on $[0, \pi]$, then

$$\int_0^{\pi} \frac{e^{-x} \sin x \ln(1+8x^4)}{x\big(1+\nu\ln(1+4x^4)\big)} dx \geq \int_0^{\pi} \frac{e^{-x} \sin x \ln(1+8x^4)}{x\big(1+\nu\ln(1+4\pi^4)\big)} dx.$$

Clearly, the inequality (42) is satisfied if ν is positive and such that

$$\int_0^\pi \frac{e^{-x}\sin x\ln(1+8x^4)}{x\big(1+\nu\ln(1+4\pi^4)\big)}dx > 2\int_0^\pi \frac{e^{-x-\pi}\sin x\ln(1+4(x+\pi)^4)}{x+\pi}dx. \tag{43}$$

By the fact that $2\int_0^\pi \frac{e^{-x-\pi}\sin x\ln(1+4(x+\pi)^4)}{x+\pi}dx > 0$, the inequality (43) is equivalent to $0<\nu<\Delta_5$, where

$$\Delta_5 = \frac{\int_0^\pi \frac{e^{-x}\sin x\ \ln(1+8x^4)}{x}\ dx - 2\int_0^\pi \frac{e^{-x-\pi}\sin x\ln(1+4(x+\pi)^4)}{x+\pi}dx}{2\ln(1+4\pi^4)\int_0^\pi \frac{e^{-x-\pi}\sin x\ \ln(1+4(x+\pi)^4)}{x+\pi}dx}.$$

From (40) and (41) and the fact that $\Delta_3-\Delta_4>0$, then

$$\int_0^\pi \frac{e^{-x}\sin(x)\ln(1+8x^4)}{x}\ dx - 2\int_0^\pi \frac{e^{-x-\pi}\sin(x)\ln(1+4(x+\pi)^4)}{x+\pi}dx > 0,$$

and $\Delta_5 \geq \mu$, where $\mu = \frac{\Delta_3-\Delta_4}{\Delta_4\ln(1+4\pi^4)} \simeq 0,405589$.
Accordingly, the inequality (43) is satisfied for all ν on $]0,\mu]$ and hence (42) is satisfied for all ν on the interval $]0,\mu]$.
C$_3$. $\nu>\mu$.
For $\nu\geq 0$, if we take (12) into account, we infer that (35) is fulfilled if

$$\int_0^\pi \frac{x^3e^{-x}\sin x}{1+4(\nu+1)x^4}\ dx > \int_0^\pi \frac{(\pi+x)^3e^{-x-\pi}\sin x}{1+4\nu(\pi+x)^4}\ dx. \tag{44}$$

A lower bound for $\displaystyle\int_0^\pi \frac{x^3e^{-x}\sin x}{1+4(\nu+1)x^4}\ dx$:
We can write $\int_0^\pi \frac{x^3e^{-x}\sin x}{1+4(\nu+1)x^4}\ dx = \int_0^{\frac{\pi}{2}} \frac{x^3e^{-x}\sin x}{1+4(\nu+1)x^4}\ dx + \int_{\frac{\pi}{2}}^\pi \frac{x^3e^{-x}\sin x}{1+4(\nu+1)x^4}\ dx$. By (37), $\int_0^{\frac{\pi}{2}} \frac{x^3e^{-x}\sin x}{1+4(\nu+1)x^4}\ dx \geq \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{x^4e^{-x}}{1+4(\nu+1)x^4}dx$. From (18) with $x=(\pi/2)$, we obtain

$$\int_0^{\frac{\pi}{2}} \frac{x^3e^{-x}\sin(x)}{1+4(\nu+1)x^4}\ dx \geq \Theta_1(\nu), \quad \nu\geq 0, \tag{45}$$

where $\Theta_1(\nu) = \frac{\frac{8}{\pi}\Omega_4(\frac{\pi}{2})}{4+(\nu+1)\pi^4} \simeq \frac{1,35110}{4+(\nu+1)\pi^4}$.
The fact that the function $x\mapsto \frac{x^4}{1+4(\nu+1)x^4}$ is increasing on $[\frac{\pi}{2},\pi]$, leads to

$$\int_{\frac{\pi}{2}}^\pi \frac{x^3}{1+4(\nu+1)x^4}e^{-x}\sin x\ dx \geq \Theta_2(\nu), \quad \nu\geq 0, \tag{46}$$

where $\Theta_2(\nu) = \frac{\frac{\pi^3}{2}\int_{\frac{\pi}{2}}^\pi e^{-x}\sin x\ dx}{4+(\nu+1)\pi^4} = \frac{\frac{\pi^3}{4}(1+e^{\frac{\pi}{2}})e^{-\pi}}{4+(\nu+1)\pi^4} \simeq \frac{1.94637}{4+(\nu+1)\pi^4}$.
From (45) and (46), $\int_0^\pi \frac{x^3}{1+4(\nu+1)x^4}e^{-x}\sin x\ dx \geq \Theta_3(\nu)$, for every $\nu\geq 0$, where $\Theta_3(\nu)=\Theta_1(\nu)+\Theta_2(\nu) = \frac{\omega_1}{4+(\nu+1)\pi^4}$, with $\omega_1 = \frac{8}{\pi}\Omega_4(\frac{\pi}{2})+\frac{\pi^3}{4}(1+e^{\frac{\pi}{2}})e^{-\pi} \simeq 3,29747$.

An upper bound for $\displaystyle\int_0^\pi \frac{(\pi+x)^3}{1+4\nu(\pi+x)^4}e^{-x-\pi}\sin x\ dx$.
Since the function $x\mapsto \frac{x^4}{1+4(\nu+1)x^4}$ is increasing on $[\pi,2\pi]$, then

$$\begin{aligned}\int_0^\pi \frac{(\pi+x)^3}{1+4\nu(\pi+x)^4}e^{-x-\pi}\sin x\ dx &= \int_0^\pi \frac{1}{\pi+x}\frac{(\pi+x)^4}{1+4\nu(\pi+x)^4}e^{-x-\pi}\sin x\ dx \\ &\leq \frac{1}{\pi}\frac{(2\pi)^4}{1+4\nu(2\pi)^4}e^{-\pi}\int_0^\pi e^{-x}\sin x\ dx.\end{aligned}$$

So, $\int_0^\pi \frac{(\pi+x)^3 e^{-x-\pi}\sin x}{1+4\nu(\pi+x)^4}dx \le \Theta_4(\nu)$, where $\Theta_4(\nu) = \frac{16\pi^3 e^{-\pi}\int_0^\pi e^{-x}\sin x\, dx}{1+64\pi^4\nu} = \frac{\omega_2}{1+64\pi^4\nu}$ and $\omega_2 = 8\pi^3 e^{-\pi}(e^{-\pi}+1) \simeq 11,18244$. Thus, (44) is fulfilled if $\Theta_3(\nu) > \Theta_4(\nu)$, *i.e.*, if $\nu > \frac{\omega_2(\pi^4+4)-\omega_1}{\pi^4(64\omega_1-\omega_2)} \simeq 0,05808$, and so, if $\nu > \mu \simeq 0,405589$.
Hence, the desired result of the theorem is an immediate consequence of the three cases already treated. ∎

Case $\nu < 0$.
Using (1), the linear function $B[\nu]$ where $\nu \in \mathbb{C}$, $\nu \neq -n,\ n \geq 1$, satisfies

$$\begin{aligned} &B[\nu+1] = -4(\nu+1)x^2 B[\nu],\\ &-2(\nu+1)B[\nu] = x(B[\nu+1])' - (1+2\nu)B[\nu+1]. \end{aligned}$$

More general, by an easy induction we can show that

$$(-2)^m \frac{\Gamma(\nu+m+1)}{\Gamma(\nu+1)} B[\nu] = \sum_{l=0}^{m} \alpha_{m,l} x^l B^{(l)}[\nu+m],\ m \geq 0, \tag{47}$$

where $(\alpha_{m,l})_{l=0}^m$, $m \geq 0$, are given by

$$\begin{cases} \alpha_{m,m} = 1,\ m \geq 0,\\ \alpha_{m,l-1} + \big(l-1-2(\nu+m)\big)\alpha_{m,l} = \alpha_{m+1,l},\ \ 1 \leq l \leq m,\ m \geq 1,\\ \alpha_{m+1,0} = -\big(1+2(\nu+m)\big)\alpha_{m,0},\ \ m \geq 0. \end{cases} \tag{48}$$

Theorem 11. *Let $\nu < 0$, with $\nu \neq -n$, $n \geq 1$. For each integer $m \geq 1$, such that $\nu > -m$, the generalized Bessel linear functional $B[\nu]$ has the following integral representation:*

$$\langle B[\nu], p\rangle = \int_{-\infty}^{+\infty} V_{\nu+m}(x)p(x)dx, \quad p \in \mathcal{P},\ \text{and where} \tag{49}$$

$$V_{\nu+m}(x) = \frac{\Gamma(\nu+1)}{(-2)^m S_{\nu+m}\Gamma(\nu+m+1)} \sum_{l=0}^{m} \alpha_{m,l} x^l U_{\nu+m}^{(l)}(x). \tag{50}$$

The sequence $(\alpha_{m,l})_{l=0}^m$ is given by (48), and

$$U_{\nu+m}(x) = \begin{cases} 0, \quad x = 0,\\ \dfrac{1}{x^2}\displaystyle\int_{|x|}^{+\infty} \left(\frac{|x|}{t}\right)^{2(\nu+m)+1} e^{\frac{1}{4t^2}-\frac{1}{4x^2}} s(t^2)dt, \quad x \neq 0. \end{cases} \tag{51}$$

Proof. Let $\nu < 0$, with $\nu \neq -n$, $n \geq 1$. Now, let $m \geq 1$ be an integer such that $\nu > -m$. From (4), the function $U_{\nu+m}$ satisfies

$$(x^3y)' - \big(2(\nu+m+1)x^2 + \frac{1}{2}\big)y = g(x), \quad y(0) = 0,$$

where $g(x) = -|x|\, s(x^2) = -|x|\, e^{-\sqrt{|x|}}\sin(\sqrt{|x|})$ for all $x \in \mathbb{R}$. Clearly, $g(x) = f(\sqrt{|x|})$, where f is an entire function, $f(t) = -t^2 e^{-t}\sin t = \sum_{n=0}^{+\infty} a_n t^n$ with $a_0 = a_1 = 0$ and $a_n = -\frac{2^{\frac{n-2}{2}}}{(n-2)!}\cos(\frac{3n\pi}{4})$, $n \geq 2$. Besides, f satisfies $\mathbf{P}_2$. In concordance of (20), $U_{\nu+m}$ is a solution of the first-order differential equation $E_{\nu+m}(g)$. In view of lemmas 6 and 8 and by using theorem 10, $U_{\nu+m}$ is even, infinitely differentiable on $\mathbb{R} - \{0\}$, in $L^1(\mathbb{R}) \cap C^0(\mathbb{R})$ and when $|\, x \,| \to +\infty$, we have

$$\left|x^n U_{\nu+m}^{(n)}(x)\right| = \mathrm{O}\left(\frac{1}{|x|^{k+2}}\right), \text{ for every integers } k \geq k_\nu \text{ and } n \geq 0. \tag{52}$$

Moreover, for every integer $n \geq 0$, we have $\lim_{x\to 0} x^n U^{(n)}_{\nu+m}(x) = 0$. Since $S_{\nu+m} > 0$, then $B[\nu+m]$ has the following integral representation:

$$\langle B[\nu+m], p\rangle = S^{-1}_{\nu+m}\int_{-\infty}^{+\infty} U_{\nu+m}(x)p(x)dx, \quad p \in \mathcal{P}, \tag{53}$$

where

$$U_{\nu+m}(x) = \begin{cases} 0, \ x = 0, \\ \dfrac{1}{x^2}\displaystyle\int_{|x|}^{+\infty}\left(\frac{|x|}{t}\right)^{2(\nu+m)+1} e^{\frac{1}{4t^2}-\frac{1}{4x^2}} s(t^2)dt, \ x \neq 0, \end{cases}$$

By (47), (52) and (53), we get after finite number of integrations by parts,

$$\begin{aligned}(-2)^m\frac{\Gamma(\nu+m+1)}{\Gamma(\nu+1)}\langle B[\nu], p\rangle &= \sum_{l=0}^{m}(-1)^l a_{m,l}\langle B[\nu+m], (x^l p)^{(l)}\rangle \\ &= S^{-1}_{\nu+m}\sum_{l=0}^{m}(-1)^l a_{m,l}\int_{-\infty}^{+\infty} U_{\nu+m}(t)(t^l p)^{(l)}(t)dt \\ &= S^{-1}_{\nu+m}\int_{-\infty}^{+\infty}\sum_{l=0}^{m} a_{m,l}t^l U^{(l)}_{\nu+m}(t)\,p(t)\,dt.\end{aligned}$$

This archived the proof of the theorem. ∎

3.2. **An integral representation of $\mathcal{B}(\alpha)$.** Recall that the Bessel linear functional $\mathcal{B}(\alpha)$, where α is a complex number such that $\alpha \neq -(n/2)$, $n \geq 0$, is D-classical satisfying [7] :

$$\big(x^2\mathcal{B}(\alpha)\big)' - 2(\alpha x + 1)\mathcal{B}(\alpha) = 0.$$

By referring to [2], there is a connection formula between the two linear functionals $B[\nu]$ and $\mathcal{B}(\alpha)$,

$$\sigma B[\nu] = h_{\frac{1}{8}}\mathcal{B}(\frac{\nu+1}{2}), \ \text{ for all } \nu \neq -n, \ n \geq 1.$$

Equivalently,

$$\mathcal{B}(\alpha) = h_8\sigma B[2\alpha-1], \ \text{ for all } \alpha \neq -(n/2), \ n \geq 0. \tag{54}$$

As a straightforward consequence of (54) and by theorems 10 and 11, we obtain an integral representation of $\mathcal{B}(\alpha)$, for all $\alpha \in \mathbb{R}$ such that $\alpha \neq -(n/2)$, $n \geq 0$.
For $\alpha \geq (1/2)$, we have

$$\langle \mathcal{B}(\alpha), p\rangle = \int_0^{+\infty}\frac{U_{2\alpha-1}(\sqrt{\frac{t}{8}})}{S_{2\alpha-1}\sqrt{8t}}\, p(t)\, dt, \ p \in \mathcal{P}, \tag{55}$$

where $S_{2\alpha-1} > 0$ and the function $U_{2\alpha-1}$ is given by (2).
For $\alpha < \frac{1}{2}$ and $\alpha \neq -(n/2)$, $n \geq 0$, we have for each integer $m \geq 1$ such that $\alpha > \frac{-m+1}{2}$,

$$\langle \mathcal{B}(\alpha), p\rangle = \int_0^{+\infty}\frac{V_{2\alpha-1+m}(\sqrt{\frac{t}{8}})}{\sqrt{8t}}\, p(t)dt, \quad p \in \mathcal{P}, \tag{56}$$

where the function $V_{2\alpha-1+m}$ is given by (50).

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