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ON THE GLOBAL STABILITY OF A NEUTRAL DIFFERENTIAL EQUATION WITH VARIABLE TIME-LAGS

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Abstract. In this work, we get assumptions that guaranteeing the global exponential stability (GES) of the zero solution of a neutral differential equation (NDE) with time-lags. By help of the Liapunov-Krasovskii functional (LKF) approach, we obtain a new result related (GES) of the zero solution of the studied (NDE). An example is given to illustrate the applicability and correctness of the obtained result by MATLAB-Simulink. The obtained result includes and improves the results found in the literature.

1. Introduction

In 2014, Keadnarmol and Rojsiraphisal [10] considered the first order neutral differential equation (NDE) with two variable time-lags,

$$\frac{d}{dt}[x + px(t-\tau(t))] = -ax + b\tan hx(t-\sigma(t)). \tag{1.1}$$

Using Lyapunov functionals, the authors established some sufficient conditions for the (GES) of solutions of (NDE) (1.1). By this work, the authors [10] established an improved criterion for the (GES) of solutions of (NDE) (1.1). In this paper, instead of (NDE) (1.1), we consider the first order non-linear (NDE) with two variable time-lags:

$$\begin{aligned}\frac{d}{dt}[x + p(t)x(t-\tau(t)] = &- a(t)h(x) - b(t)g(x(t-\tau(t))) \\ &+ c(t)\tan hx(t-\sigma(t)), t \geq 0,\end{aligned} \tag{1.2}$$

where "′" represents $\frac{d}{dt}, a, b, c, p : [t_0, \infty) \to [0, \infty), t_0 \geq 0$, and $g, h : \Re \to \Re$ are continuous functions with $g(0) = 0, h(0) = 0$; the function c is continuous and differentiable, and $|p(t)| \leq p_0 < 1$ (p_0-constant). The variable time-lags τ and σ are continuous and differentiable, defined by $\tau(t) : [0, \infty) \to [0, \tau_0]$ and $\sigma(t) : [0, \infty) \to [0, \sigma_0]$ satisfying

$$\begin{aligned}&0 \leq \tau(t) \leq \tau_0, \ 0 \leq \sigma(t) \leq \sigma_0, \\ &\tau'(t) \leq \delta_1, \ \sigma'(t) \leq \delta_2 < 1,\end{aligned} \tag{1.3}$$

where $\tau_0 > 0, \sigma_0 > 0, \delta_1 > 0, \delta_2(> 0) \in \Re$.

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Throughout the paper, we assume that assumptions given by (1.3) hold when we need x shows $x(t)$.

For (NDE) (1.2), we assume the existence initial condition

$$x_0(\theta) = \phi(\theta), \theta \in [-r, 0],$$

where $r = \max\{\tau_0, \sigma_0\}$, $\phi \in C([-r, 0]; \Re)$.

Define

$$h_1(x) = \begin{cases} \frac{h(x)}{x}, x \neq 0 \\ \frac{dh(0)}{dt}, x = 0 \end{cases} \tag{1.4}$$

and

$$g_1(x) = \begin{cases} \frac{g(x)}{x}, x \neq 0 \\ \frac{dg(0)}{dt}, x = 0. \end{cases} \tag{1.5}$$

It is seen from (NDE) (1.2) and (1.4), (1.5) that

$$\begin{aligned} \frac{d}{dt}[x + p(t)x(t - \tau(t))] =& - a(t)h_1(x)x - b(t)g_1(x(t - \tau(t)))x(t - \tau(t)) \\ &+ c(t) \tan hx(t - \sigma(t)). \end{aligned} \tag{1.6}$$

In this paper, we discuss the (GES) of the zero solution of (NDE) (1.2). Meanwhile, it is well known that (NDEs) without or with time-lags often occur in many scientific areas such as engineering techniques fields, physics, medicine and etc. (see [1-29] and the references therein). Therefore, it is worth investigating the (GES) of (NDE) (1.2). In the relevant literature, the most of researchers have focused on the qualitative properties of special case of (NDEs) (1.1) and (1.2) with constant coefficients and constant time-lags like

$$\frac{d}{dt}[x + px(t - \tau)] = -ax + b \tan hx(t - \sigma) \tag{1.7}$$

or its different models. During the investigations, the authors benefited from different methods such as the Liapunovs function (direct) method, Liapunov-Krasovskii functional (LKF) method, integral inequalities, LMI, perturbation techniques, model transformations, etc., to obtain specific conditions on the various qualitative properties of (NDEs) (see [1-29]). It is also worth mentioning that this paper especially motivated by the results of Keadnarmol and Rojsiraphisal [10] and those can be found in the references of this paper. When we consider (NDEs) (1.1), (1.2) and (1.7) and compare our equation, (NDE) (1.2) with that discussed by Keadnarmol and Rojsiraphisal [10], (NDE) (1.1), it follows that (NDE) (1.2) includes and improves (NDE) (1.1). In fact, if we choose $p(t) = p$ is a constant, $a(t)h(x) = ax, a > 0$, $a \in \Re$ is a positive constant, $b(t) = 0$, $g(.) = 0$ and $c(t) = 1$, then (NDE) (1.2) reduces to (NDE) (1.1) and includes (NDE) (1.7). This fact clearly shows how this work improves the results of [10] and do a contribution to the relevant literature. In addition, giving an example and using MATLAB-Simulink show the other novelty of this paper. These are the originality of this paper.

2. Preliminaries

For convenience, let $D_1(t) = x + p(t)x(t-\tau(t))$. Hence, (NDE) (1.6) can be written as

$$\begin{aligned} D_1' =& \frac{d}{dt}[x + p(t)x(t-\tau(t))] = -a(t)h_1(x)x \\ &- b(t)g_1(x(t-\tau(t)))x(t-\tau(t)) + c(t)\tan hx(t-\sigma(t)). \end{aligned}$$

Therefore, we have

$$\begin{cases} D_1' = -a(t)h_1(x)x - b(t)g_1(x(t-\tau(t)))x(t-\tau(t)) + c(t)\tan hx(t-\sigma(t)) \\ 0 = -D_1 + x + p(t)x(t-\tau(t)). \end{cases} \tag{2.1}$$

Definition 1. The solution $x = 0$ of (NDE)(1.6) is (ES) if

$$\|x\| \le K\exp(-\lambda t) \sup_{-r\le s\le 0} \|x(s)\| = K\exp(-\lambda t)\|x_0\|_s, \tag{2.2}$$

where $K(>0) \in \Re$, $\lambda(>0) \in \Re$, and $\|x_t\|_s = \sup_{-r\le s\le 0} \|x(t+s)\|$.

Lemma 2. Let $N \in \Re^{n\times n}$ be any symmetric and positive definite matrix and $x, y \in \Re^n$. Then

$$\pm 2x^T y \le x^T N x + y^T N^{-1} y.$$

Proposition 3. Let $M > 0, \mu > 0$, $|p(t)| \le p_0 < 1$, and $0 \le \tau(t) \le \tau_0$. If $x : [-\tau_0, \infty) \to \Re$ satisfies

$$\|x\| \le \sup_{s\in[-\tau_0,0]} \|x(s)\| = \|x_0\|_s, t \in [-\tau_0, 0]$$

and

$$\|x\| \le p_0\|x(t-\tau(t))\| + M\exp(-\mu t),$$

then there are positive constants $\varepsilon, m \in [0, \frac{-lnp_0}{\tau_0}]$ such that

$$p_0\exp(\varepsilon\tau_0) < 1$$

and

$$\|x\| \le \|x_0\|_s\exp(-mt) + \frac{M}{1-p_0\exp(\varepsilon\tau_0)}\exp(-\varepsilon t) \le N\exp(-\vartheta t), \tag{2.3}$$

where $t \ge 0, N = \|x_0\|_s + \frac{M}{1-p_0\exp(\varepsilon\tau_0)}$ and $\vartheta = \min\{m, \varepsilon\}$.

Proof. In view of the assumptions $|p(t)| \le p_0 < 1$ and $0 \le \tau(t) \le \tau_0$ one can find sufficient small positive constant $\varepsilon, m \in [0, \frac{-lnp_0}{\tau_0}]$ such that $p_0\exp(\varepsilon\tau_0) < 1$ and $p_0\exp(m\tau_0) < 1$. We verify that inequality (2.3) is true. If $\mu \le \varepsilon$, we can choose $\mu = \varepsilon$; else if $\mu > \varepsilon$, we have $\exp(-\mu t) \le \exp(-\varepsilon t)$.

Let $t = 0$. Hence we have

$$\begin{aligned} \|x(0)\| &\le p_0\|x(-\tau(0))\| + M \le p_0 \sup_{-\tau_0\le s\le 0}\|x(s)\| + M \\ &< \|x_0\|_s + \frac{M}{1-p_0\exp(\varepsilon\tau_0)} \equiv N. \end{aligned}$$

Therefore, estimate (2.3) is true when $t = 0$.

Now, let $t > 0$. Assume that inequality (2.3) fails. Then, there is $t^* > 0$ such that

$$\|x(t^*)\| > \|x_0\|_s\exp(-mt^*) + \frac{M}{1-p_0\exp(\varepsilon\tau_0)}\exp(-\varepsilon t^*) \equiv N\exp(-\vartheta t^*) \tag{2.4}$$

and

$$\|x(t)\| \le \|x_0\|_s \exp(-mt) + \frac{M}{1-p_0\exp(\varepsilon\tau_0)}\exp(-\varepsilon t) \equiv N\exp(-\vartheta t) \; for\ all\ t \in [0,t^*).$$

I. Let $t^* > \tau(t^*) > 0$. Then,

$$\begin{aligned}\|x(t^*)\| &\le p_0\|x(t^*-\tau(t^*))\| + M\exp(-\mu t^*)\\ &\le p_0\{\|x_0\|_s\exp(-m(t^*-\tau(t^*))) + \frac{M}{1-p_0\exp(\varepsilon\tau_0)}\exp(-\varepsilon(t^*-\tau(t^*)))\}\\ &+ M\exp(-\varepsilon t^*)\\ &\le p_0\exp(m\tau_0)\|x_0\|_s\exp(-mt^*) + \frac{Mp_0\exp(\varepsilon\tau_0)}{1-p_0\exp(\varepsilon\tau_0)}\exp(-\varepsilon t^*)\\ &+ M\exp(-\varepsilon t^*)\\ &\le \|x_0\|_s\exp(-mt^*) + \frac{M}{1-p_0\exp(\varepsilon\tau_0)}\exp(-\varepsilon t^*) \equiv N\exp(-\vartheta t^*).\end{aligned}$$

II. Let $-\tau_0 < 0 < t^* < \tau(t^*)$. Then

$$\|x(t^*-\tau(t^*))\| \le \|x_0\|_s = \sup_{s\in[-\tau_0,0]}\|x(s)\|,$$

and hence, it follows that

$$\begin{aligned}\|x(t^*)\| &\le p_0\|x(t^*-\tau(t^*)\| + M\exp(-\mu t^*)\\ &\le \|x_0\|_s\exp(-mt^*) + \frac{M}{1-p_0\exp(\varepsilon\tau_0)}\exp(-\varepsilon t^*)\\ &\equiv N\exp(-\vartheta t^*).\end{aligned}$$

Thus, for both the cases I and II, we have a contradiction to inequality (2.4). Therefore, inequality (2.3) is true for all $t \ge 0$.

3. Exponential stability

We assume that there exist nonnegative a_i, b_i, m_i, n_i and positive constants $c_i, (i = 1, 2)$, such that for $t \ge t_0$,

$$a_1 \le a(t) \le a_2,\ b_1 \le b(t) \le b_2,\ c_1 \le c(t) \le c_2,\ c'(t) \le 0, \tag{3.1}$$

$$m_1 \le g_1(x) \le m_2,\ n_1 \le h_1(x) \le n_2. \tag{3.2}$$

Theorem 4. Let a_i, b_i, m_i, n_i be nonnegative constants and estimate (1.3) holds. Then trivial solution of (NDE) (1.6) is (GES) if the operator D_1 is stable ($i.e.|p(t)| \le p_0 < 1$) and there exist positive constants c_1, c_2, q_1, α, k and constants $q_i, (i = 2, 3, ..., 6)$, such that

$$\Omega = \begin{bmatrix} 2kq_1-2q_2 & (1,2) & (1,3) & q_1c_2 & q_3\\ * & (2,2) & (2,3) & 0 & q_5+2q_6\\ * & * & (3,3) & 0 & -2q_6\\ * & * & * & -c_1(1-\delta_2) & 0\\ * & * & * & * & -2q_6 \end{bmatrix} < 0, \tag{3.3}$$

where $(1,2) = -q_1a_1n_1 + q_2 - q_3 - q_4$, $(1,3) = -q_1b_1m_1 + q_2p_0 + q_3$, $(2,2) = 2q_4 - 2q_5 - 2q_6 + \alpha e^{2k\tau_0} + c_2e^{2k\sigma_0}$, $(2,3) = q_4p_0 + q_5 + 2q_6$, $(3,3) = -2q_6 - \alpha(1-\delta_1)$ and the symbols $*$ indicates the elements below the main diagonal of the symmetric matrix Ω.

Proof. We define a (LKF) $V = V_1 + V_2 + V_3 = V_1(t) + V_2(t) + V_3(t)$ by

$$V_1(t) = e^{2kt}[D_1, x, 0]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} q_1 & 0 & 0 \\ q_2 & q_3 & 0 \\ q_4 & q_5 & q_6 \end{bmatrix}\begin{bmatrix} D_1 \\ x \\ 0 \end{bmatrix} = e^{2kt}q_1D_1^2,$$

$$V_2(t) = \alpha\int_{t-\tau(t)}^{t} e^{2k(s+\tau_0)}x^2(s)ds + c(t)\int_{t-\sigma(t)}^{t} e^{2k(s+\sigma_0)}\tan h^2x(s)ds,$$

$$V_3(t) = \eta e^{2kt}D_1^2,$$

where $D_1 = x + p(t)x(t-\tau(t)), q_1 > 0, \alpha > 0, c(t) > 0, q_i \in \Re, (i = 2, ..., 6)$, and $\eta(> 0) \in \Re$, we determine it later.

Differentiating V_1 and V_2 along system (2.1), we get

$$\begin{aligned} V_1'(t) &= e^{2kt}(2kq_1D_1^2 + 2q_1D_1D_1') \\ &= 2e^{2kt}kq_1D_1^2 + 2e^{2kt}[D_1, x, 0]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} q_1 & q_2 & q_3 \\ 0 & q_4 & q_5 \\ 0 & 0 & q_6 \end{bmatrix}\begin{bmatrix} D_1' \\ 0 \\ 0 \end{bmatrix}. \end{aligned}$$

Benefited from the formula, $x - x(t-\tau(t)) = \int_{t-\tau(t)}^{t} x'(s)ds$, we obtain

$$\begin{aligned} V_1'(t) &= 2e^{2kt}kq_1D_1^2 \\ &+ 2e^{2kt}[D_1, x, -\int_{t-\tau(t)}^{t} x'(s)ds + x - x(t-\tau(t))]\begin{bmatrix} q_1 & q_2 & q_3 \\ 0 & q_4 & q_5 \\ 0 & 0 & q_6 \end{bmatrix} \\ &\times \begin{bmatrix} -a(t)h_1(x)x - b(t)g_1(x(t-\tau(t)))x(t-\tau(t)) + c(t)\tan hx(t-\sigma(t)) \\ -D_1 + x + p(t)x(t-\tau(t)) \\ \int_{t-\tau(t)}^{t} x'(s)ds - x + x(t-\tau(t)) \end{bmatrix} \\ &= 2e^{2kt}kq_1D_1^2 \\ &+ 2e^{2kt}[D_1q_1, D_1q_2 + q_4x, D_1q_3 + q_5x - q_6\int_{t-\tau(t)}^{t} x'(s)ds + q_6x - q_6x(t-\tau(t))] \\ &\times \begin{bmatrix} -a(t)h_1(x)x - b(t)g_1(x(t-\tau(t)))x(t-\tau(t)) + c(t)\tan hx(t-\sigma(t)) \\ -D_1 + x + p(t)x(t-\tau(t)) \\ \int_{t-\tau(t)}^{t} x'(s)ds - x + x(t-\tau(t)) \end{bmatrix} \\ &= 2e^{2kt}kq_1D_1^2 + 2e^{2kt}\{-D_1q_1a(t)h_1(x)x \\ &- D_1q_1b(t)g_1(x(t-\tau(t)))x(t-\tau(t)) + D_1q_1c(t)\tan hx(t-\sigma(t)) \\ &- D_1^2q_2 + D_1q_2x + D_1q_2p(t)x(t-\tau(t)) \\ &- D_1q_4x + q_4x^2 + q_4p(t)xx(t-\tau(t)) \\ &+ D_1q_3\int_{t-\tau(t)}^{t} x'(s)ds - D_1q_3x + D_1q_3x(t-\tau(t)) \\ &+ q_5x\int_{t-\tau(t)}^{t} x'(s)ds - q_5x^2 + q_5xx(t-\tau(t)) \\ &- q_6[\int_{t-\tau(t)}^{t} x'(s)ds]^2 + q_6x\int_{t-\tau(t)}^{t} x'(s)ds - q_6x(t-\tau(t))\int_{t-\tau(t)}^{t} x'(s)ds \end{aligned}$$

$$
\begin{aligned}
&+ q_6 x \int_{t-\tau(t)}^{t} x'(s)ds - q_6 x^2 + q_6 x x(t-\tau(t)) \\
&- q_6 x(t-\tau(t)) \int_{t-\tau(t)}^{t} x'(s)ds + q_6 x x(t-\tau(t)) - q_6 x^2(t-\tau(t))\} \\
&= e^{2kt}\{(2kq_1 - 2q_2)D_1^2 + 2[-q_1 a(t) h_1(x) + q_2 - q_3 - q_4] x D_1 \\
&+ (2q_4 - 2q_5 - 2q_6)x^2 + 2D_1 q_1 c(t) \tan h x(t-\sigma(t)) \\
&+ 2[-q_1 b(t) g_1(x(t-\tau(t))) + q_2 p(t) + q_3] x(t-\tau(t)) D_1 \\
&+ 2(q_4 p(t) + q_5 + 2q_6) x x(t-\tau(t)) + 2D_1 q_3 \int_{t-\tau(t)}^{t} x'(s)ds \\
&- 2q_6 x^2(t-\tau(t)) - 4q_6 x(t-\tau(t)) \int_{t-\tau(t)}^{t} x'(s)ds \\
&+ 2(q_5 + 2q_6) x \int_{t-\tau(t)}^{t} x'(s)ds - 2q_6 [\int_{t-\tau(t)}^{t} x'(s)ds]^2\}.
\end{aligned}
$$

Using conditions (3.1), (3.2) and $|p(t)| \le p_0 < 1$, we have

$$
\begin{aligned}
V_1'(t) \le\ & e^{2kt}\{(2kq_1 - 2q_2)D_1^2 + 2(-q_1 a_1 n_1 + q_2 - q_3 - q_4) x D_1 \\
&+ 2(q_4 - q_5 - q_6)x^2 + 2q_1 c_2 D_1 \tan h x(t-\sigma(t)) \\
&+ 2(-q_1 b_1 m_1 + q_2 p_0 + q_3) D_1 x(t-\tau(t)) \\
&+ 2(q_4 p_0 + q_5 + 2q_6) x x(t-\tau(t)) \\
&+ 2q_3 D_1 \int_{t-\tau(t)}^{t} x'(s)ds - 2q_6 x^2(t-\tau(t)) - 4q_6 x(t-\tau(t)) \int_{t-\tau(t)}^{t} x'(s)ds \\
&+ 2(q_5 + 2q_6) x \int_{t-\tau(t)}^{t} x'(s)ds - 2q_6 [\int_{t-\tau(t)}^{t} x'(s)ds]^2\}, \qquad (3.4) \\
V_2'(t) \le\ & \alpha e^{2k(t+\tau_0)} x^2 - \alpha e^{2k(t+\tau_0-\tau(t))}(1-\tau'(t)) x^2((t-\tau(t)) \\
&+ c'(t) \int_{t-\sigma(t)}^{t} e^{2k(s+\sigma_0)} \tan h^2 x(s)ds + c(t) e^{2k(t+\sigma_0)} \tan h^2 x \\
&- c(t) e^{2k(t+\sigma_0-\sigma(t))}(1-\sigma'(t)) \tan h^2 x(t-\sigma(t)).
\end{aligned}
$$

Using conditions (1.3), (3.1) and applying the estimate $\tan h^2 x \le x^2$, we obtain

$$
\begin{aligned}
V_2'(t) \le\ & e^{2kt}\{(\alpha e^{2k\tau_0} + c_2 e^{2k\sigma_0})x^2 - \alpha(1-\delta_1)x^2(t-\tau(t)) \\
&- c_1(1-\delta_2) \tan h^2 x(t-\sigma(t))\}. \qquad (3.5)
\end{aligned}
$$

Combining equations (3.4) and (3.5), we have

$$
\begin{aligned}
V_1'(t) + V_2'(t) \le\ & e^{2kt}\{(2kq_1 - 2q_2)D_1^2 \\
&+ 2(-q_1 a_1 n_1 + q_2 - q_3 - q_4) x D_1 \\
&+ (2q_4 - 2q_5 - 2q_6 + \alpha e^{2k\tau_0} + c_2 e^{2k\sigma_0})x^2 \\
&+ 2q_1 c_2 D_1 \tan h x(t-\sigma(t))
\end{aligned}
$$

$$
\begin{aligned}
&+2(-q_1 b_1 m_1 + q_2 p_0 + q_3) D_1 x(t-\tau(t)) \\
&+2(q_4 p_0 + q_5 + 2q_6) x x(t-\tau(t)) \\
&+2q_3 D_1 \int_{t-\tau(t)}^{t} x'(s)ds + 2(q_5 + 2q_6) x \int_{t-\tau(t)}^{t} x'(s)ds \\
&+[-2q_6 - \alpha(1-\delta_1)] x^2(t-\tau(t)) - 4q_6 x(t-\tau(t)) \int_{t-\tau(t)}^{t} x'(s)ds \\
&- c_1(1-\delta_2) \tan h^2 x(t-\sigma(t)) - 2q_6 [\int_{t-\tau(t)}^{t} x'(s)ds]^2 \} \\
&= e^{2kt} l^T(t) \Omega l(t)
\end{aligned}
$$

where $l(t) = [D_1, x, x(t-\tau(t)), \tan hx(t-\sigma(t)), \int_{t-\tau(t)}^{t} x'(s)ds]^T$ and Ω is defined by (3.3). Making use of assumption $\Omega < 0$, we have

$$V_1'(t) + V_2'(t) \leq e^{2kt} l^T(t) \Omega l(t) < 0.$$

Therefore, there is a constant $\lambda, \lambda > 0$, such that

$$
\begin{aligned}
V_1'(t) + V_2'(t) &\leq -\lambda e^{2kt} \Big(\|D_1\|^2 + \|x\|^2 + \|x(t-\tau(t)\|^2 \\
&\quad + \|tanhx(t-\sigma(t))\|^2 + \| \int_{t-\tau(t)}^{t} x'(s)ds\|^2 \Big) \\
&\leq -\lambda e^{2kt} \|x(t)\|^2.
\end{aligned}
$$

Calculating the derivate of V_3 along system (2.1), we have

$$
\begin{aligned}
V_3'(t) &= 2e^{2kt} \eta (D_1 D_1' + k D_1^2) \\
&= 2e^{2kt} \eta \{ [x + p(t) x(t-\tau(t))] \times [-a(t) h_1(x) x \\
&- b(t) g_1(x(t-\tau(t))) x(t-\tau(t)) + c(t) \tan hx(t-\sigma(t))] \\
&+ k[x + p(t) x(t-\tau(t))]^2 \} \\
&= 2e^{2kt} \eta \{ -a(t) h_1(x) x^2 - b(t) g_1(x(t-\tau(t))) x x(t-\tau(t)) \\
&+ c(t) x \tan hx(t-\sigma(t)) - a(t) h_1(x) p(t) x x(t-\tau(t)) \\
&- b(t) g_1(x(t-\tau(t))) p(t) x^2(t-\tau(t)) \\
&+ c(t) p(t) x(t-\tau(t)) \tan hx(t-\sigma(t)) \\
&+ kx^2 + 2kp(t) x x(t-\tau(t)) + kp^2(t) x^2(t-\tau(t)) \}.
\end{aligned}
$$

Utilizing conditions (3.1), (3.2) and $|p(t)| \leq p_0 < 1$, we have

$$
\begin{aligned}
V_3'(t) \leq e^{2kt} \eta \{ &-2a_1 n_1 x^2 - 2b_1 m_1 x x(t-\tau(t)) \\
&+ 2c_2 x \tan hx(t-\sigma(t)) - 2a_1 n_1 p_0 x x(t-\tau(t)) \\
&- 2b_1 m_1 p_0 x^2(t-\tau(t)) + 2c_2 p_0 x(t-\tau(t)) \tan hx(t-\sigma(t)) \\
&+ 2kx^2 + 4kp_0 x x(t-\tau(t)) + 2p_0^2 k x^2(t-\tau(t)) \}.
\end{aligned}
$$

By means of Lemma 2, we find

$$
\begin{aligned}
V_3'(t) \leq e^{2kt} \eta \{ &(2k - 2a_1 n_1 + |b_1 m_1|^2 + 4|p_0 k|^2 + |c_2|^2 + |a_1 n_1 p_0|^2) x^2 \\
&+ (-2b_1 m_1 p_0 + 2p_0^2 k + |c_2 p_0|^2 + 3) x^2(t-\tau(t)) + 2 \tan h^2 x(t-\sigma(t)) \}.
\end{aligned}
$$

Let us choose the constant η as

$$\eta = \begin{cases} \frac{\lambda}{2}\min\{\frac{1}{\xi}, \frac{1}{2}\} \; if \; \psi \leq 0, \\ \frac{\lambda}{2}\min\{\frac{1}{\xi}, \frac{1}{\psi}\frac{1}{2}\} \; if \; \psi > 0, \end{cases}$$

where
$\xi = -2b_1m_1p_0 + 2p_0^2k + |c_2p_0|^2 + 3$ and $\psi = 2k - 2a_1n_1 + |b_1m_1|^2 + 4|p_0k|^2 + |c_2|^2 + |a_1n_1p_0|^2$.

Hence, we can obtain

$$V_1'(t) + V_2'(t) + V_3'(t) \leq -\frac{\lambda}{2}e^{2kt}\|x(t)\|^2 < 0.$$

Since $V'(t)$ is negative definite and $0 \leq \tau(t) \leq \tau_0$, $0 \leq \sigma(t) \leq \sigma_0$ then, $V(x) \leq V(x(0))$ for all $t \geq 0$, with

$$\begin{aligned} V(x(0)) &= V_1(x(0)) + V_2(x(0)) + V_3(x(0)) \\ &= q_1[x(0) + p(0)x(-\tau(0))]^2 \\ &+ \alpha\int_{-\tau(0)}^{0} e^{2k(s+\tau_0)}x^2(s)ds + c(0)\int_{-\sigma(0)}^{0} e^{2k(s+\sigma_0)}\tan h^2x(s)ds \\ &+ \eta[x(0) + p(0)x(-\tau(0))]^2 \\ &\leq q_1(1+p_0)^2\|x_0\|_s^2 + \alpha\int_{-\tau(0)}^{0} e^{2k(s+\tau_0)}(\sup_{-r\leq s\leq 0}\|x(s)\|)^2ds \\ &+ c_2\int_{-\sigma(0)}^{0} e^{2k(s+\sigma_0)}(\sup_{-r\leq s\leq 0}\|x(s)\|)^2ds + \eta(1+p_0)^2\|x_0\|_s^2 \\ &\leq q_1(1+p_0)^2\|x_0\|_s^2 + \alpha e^{2k\tau_0}\tau_0\|x_0\|_s^2 + c_2e^{2k\sigma_0}\sigma_0\|x_0\|_s^2 + \eta((1+p_0)^2\|x_0\|_s^2 \\ &= \Delta\|x_0\|_s^2 \end{aligned}$$

where $\Delta = q_1(1+p_0)^2 + \alpha e^{2k\tau_0}\tau_0 + c_2e^{2k\sigma_0}\sigma_0 + \eta((1+p_0)^2$.

From $\eta e^{2kt}\|D_1\|^2 \leq V(x) \leq \Delta\|x_0\|_s^2$, we obtain $\|D_1\| \leq Me^{-kt}$, where $M = \sqrt{\frac{\Delta}{\eta}}\|x_0\|_s$. Because of $D_1 = x + p(t)x(t-\tau(t))$, we have

$$\|x\| = \|D_1 - p(t)x(t-\tau(t))\| \leq \|D_1\| + \|p(t)x(t-\tau(t))\| \leq Me^{-kt} + p_0\|x(t-\tau(t))\|.$$

Since $|p(t)| \leq p_0 < 1$ and $0 \leq \tau(t) \leq \tau_0$, we can choose sufficiently small positive constant $\vartheta = k < \frac{-lnp_0}{\tau_0}$ so that $p_0e^{\vartheta\tau_0} < 1$. Utilizing Proposition 3, we have

$$\|x\| \leq (\|x_0\|_s + \frac{M}{1-p_0e^{\vartheta\tau_0}})e^{-\vartheta t}, t \geq 0.$$

Choosing $\gamma = \max\{\|x_0\|_s, \frac{M}{1-p_0e^{\vartheta\tau_0}}\}$, we obtain

$$\|x\| \leq 2\gamma e^{-\vartheta t}.$$

This implies that the zero solution of (NDE) (1.6) is (ES). By radially unboundedness, it is also (GES) with rate of convergence $k = \vartheta > 0$.

Remark 5 If $k = 0$ one can easily see that the zero solution of (NDE) (1.6) is uniformly asymptotically stable when the following criterion holds:

$$\Omega = \begin{bmatrix} -2q_2 & (1,2) & (1,3) & q_1 c_2 & q_3 \\ * & (2,2) & (2,3) & 0 & q_5 + 2q_6 \\ * & * & (3,3) & 0 & -2q_6 \\ * & * & * & -c_1(1-\delta_2) & 0 \\ * & * & * & * & -2q_6 \end{bmatrix} < 0, \tag{3.6}$$

where $(1,2) = -q_1 a_1 n_1 + q_2 - q_3 - q_4$, $(1,3) = -q_1 b_1 m_1 + q_2 p_0 + q_3$, $(2,2) = 2q_4 - 2q_5 - 2q_6 + \alpha + c_2$, $(2,3) = q_4 p_0 + q_5 + 2q_6$ and $(3,3) = -2q_6 - \alpha(1-\delta_1)$.

Example 1 As a special case of (NDE) (1.2), we consider the following nonlinear (NDE) with variable time- lags,

$$\begin{aligned} \frac{d}{dt}\left[x + \frac{1}{8+t^2}x(t-\tau(t))\right] &= -(2+\exp(-t))\left[x + \frac{x}{1+x^2}\right] \\ &\quad - \left(\frac{1}{2} + \exp(-t)\right)\left[x(t-\tau(t)) + \frac{x(t-\tau(t))}{1+x^2(t-\tau(t))}\right] \\ &\quad + \frac{1}{3}\tan hx(t-\sigma(t)), t \geq 0. \end{aligned} \tag{3.7}$$

Here,

$$\begin{aligned} & D_1(t) = x + \frac{1}{8+t^2}x(t-\tau(t)), p(t) = \frac{1}{8+t^2} \\ & a(t) = 2 + \exp(-t), b(t) = \frac{1}{2} + \exp(-t), c(t) = \frac{1}{3} \\ & \tau(t) = \sigma(t) = \frac{\sin^2 t}{20}, \tau'(t) = \frac{\sin 2t}{20} < 1 \\ & h(x) = x + \frac{x}{1+x^2}, h_1(x) = \begin{cases} 1 + \frac{1}{1+x^2}, x \neq 0 \\ h'(0), x = 0 \end{cases} \\ & g(x) = x + \frac{x}{1+x^2}, g_1(x) = \begin{cases} 1 + \frac{1}{1+x^2}, x \neq 0 \\ g'(0), x = 0 \end{cases} . \end{aligned}$$

Then, we have

$$\begin{aligned} & h(0) = 0, n_1 = 1 \leq h_1(x) \leq 2 = n_2 \\ & g(0) = 0, m_1 = 1 \leq g_1(x) \leq 2 = m_2 \\ & a_1 = 2 \leq a(t) = 2 + \exp(-t) \leq 3 = a_2 \\ & b_1 = \frac{1}{2} \leq b(t) = \frac{1}{2} + \exp(-t) \leq \frac{3}{2} = b_2 \\ & p(t) = \frac{1}{8+t^2} \leq \frac{1}{8} = p_0 < 1 \\ & c_1 = c_2 = \frac{1}{3}, \delta_1 = \frac{1}{5}, \delta_2 = \frac{1}{10}, \alpha = \frac{5}{8}, k = \frac{1}{4} \\ & q_1 = q_2 = \frac{3}{2}, q_3 = q_5 = q_6 = 0, q_4 = -1, \end{aligned}$$

and

$$\Omega = \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Omega_{14} \\ * & \Omega_{22} & \Omega_{23} & \Omega_{24} \\ * & * & \Omega_{33} & \Omega_{34} \\ * & * & * & \Omega_{44} \end{bmatrix} < 0,$$

where $\Omega_{11} = -2,25, \Omega_{12} = -0,5, \Omega_{13} = -0,5625, \Omega_{14} = 0,5, \Omega_{22} = -1,0174, \Omega_{23} = -0,125, \Omega_{24} = 0, \Omega_{33} = -0,5, \Omega_{34} = 0, \Omega_{44} = -0,3$. The eigenvalues of this matrix, -2,6729, -0,8803, -0,4154 and -0,0988. Clearly, all the assumptions of Theorem 4 hold. This discussion implies that (NDE) (3.7) is (GES) if the operator D_1 is stable.

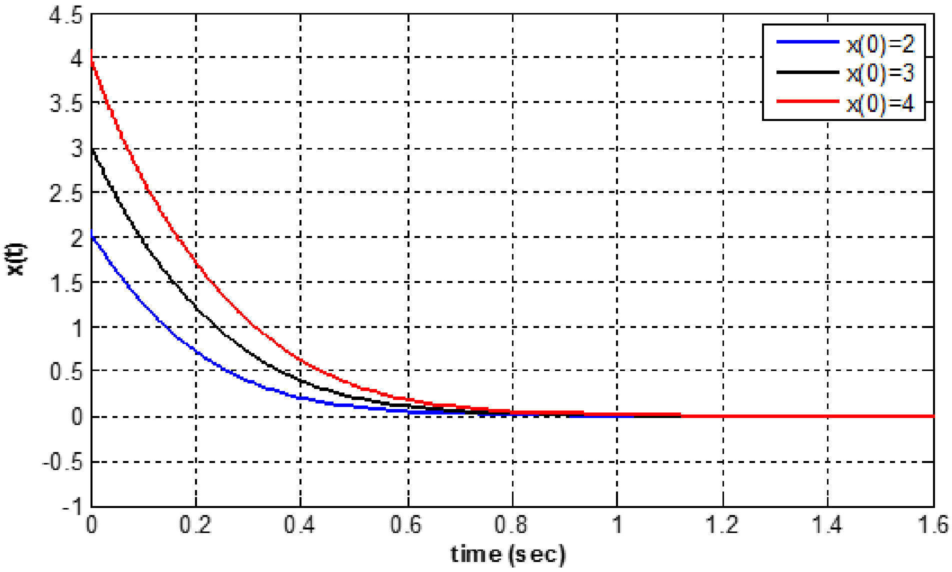

FIGURE 1. Trajectory of $x(t)$ of Eq. of (3.7) in Example, when

$$\tau(t) = \sigma(t) = \frac{\sin^2(t)}{20}, t \geq 0.$$

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