Stepanov-like pseudo almost automorphic solutions to some stochastic differential equations

Yong-Kui Chang*, Zhuan-Xia Cheng†, and G. M. N'Guérékata‡

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Abstract

In this paper, we introduce a new concept of S^2-pseudo almost automorphy for stochastic processes. We apply the results obtained to investigate the existence and uniqueness of S^2-pseudo almost automorphic mild solutions to some stochastic differential equations in a real separable Hilbert space. Our main results extend some known ones in the sense of square-mean pseudo almost automorphy or S^2-almost automorphy for stochastic processes.

Keywords: Stepanov-like pseudo almost automorphic stochastic processes, Square-mean pseudo almost automorphy, Stochastic differential equations.

Mathematics Subject Classification(2000): 34K14, 60H10, 35B15, 34F05.

1 Introduction

The concept of almost automorphy is an important generalization of the classical almost periodicity. It was introduced by Bochner [5, 6], for more details about this topics we refer the reader to [14, 15, 17]. Diagana [11] introduced the concept of Stepanov-like pseudo almost automorphy as a natural generalization of the pseudo almost automorphy and an implement of the Stepanov-like almost automorphy due to N'Guérékata and Pankov [18]. Zhang, Chang and N'Guérékata [24, 25, 26] proved some properties and new ergodic theorems of Stepanov-like weighted pseudo almost automorphic functions and applied them to some evolution equations.

*Department of Mathematics, Lanzhou Jiaotong University, Lanzhou 730070, P. R. China. E-mail: lzchangyk@163.com

†Department of Mathematics, Lanzhou Jiaotong University, Lanzhou 730070, P. R. China. E-mail: chengzx5566@163.com

‡Department of Mathematics, Morgan State University, 1700 E. Cold Spring Lane, Baltimore, M.D. 21251, USA. E-mail: nguerekata@aol.com

Since noise or stochastic perturbation is unavoidable in real world, it is of great importance to consider the stochastic effects in the investigation of differential systems [1, 19, 21, 20, 23]. Bezandry and Diagana systematically studied the fundamental properties of almost periodic stochastic processes and investigated almost periodic solutions to different kinds of stochastic differential equations in a recent monograph [3]. Fu and Liu [12] introduced a new concept of square-mean almost automorphic stochastic processes, which is a natural generalization of almost automorphic functions in deterministic cases. The notation of square-mean pseudo almost automorhy for stochastic processes was presented by Chen and Lin [9], and was further developed by Bezandry and Diagana [4]. The concepts and properties of the square-mean weighted pseudo almost automorphy and the square-mean bi-almost automorphy for a stochastic process was also discussed in [10]. Chang, Zhao and N'Guérékata [8] presented the concept of S^2-almost automorphy for stochastic processes, which is more general than square-mean almost automorphy and can be seen as a natural generalization of Stepanov-like almost automorphic functions in deterministic cases. For more results on topic, we refer to [7, 22, 27] and references therein.

From above mentioned works, we can see that the Stepanov-like pseudo almost automorphy is a generalization of the Stepanov-like almost automorphy and pseudo almost automorphy in deterministic cases. Thus, a natural question is: *what is it in stochastic cases*? In present paper, we first introduce a new concept of S^2-pseudo almost automorphy for stochastic processes, which generalizes the notation of square-mean pseudo almost automorphy [4, 9] or S^2-almost automorphy for stochastic processes [8]. And then we present some properties of such functions and apply this new concept to investigate the existence and uniqueness of S^2-pseudo almost automorphic mild solutions to the following stochastic differential equations

$$dx(t) = Ax(t)dt + f(t)dt + g(t)dW(t), \ t \in \mathbb{R}, \tag{1.1}$$

and

$$dx(t) = Ax(t)dt + f\left(t, x(t)\right)dt + g\left(t, x(t)\right)dW(t), \ t \in \mathbb{R}, \tag{1.2}$$

where A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $L^2(\mathbb{P}, \mathbb{H})$, and $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, \mathscr{F}_t)$, where $\mathscr{F}_t = \sigma\{W(u) - W(v); u, v \leq t\}$. Here f and g are appropriate functions specified later.

The rest of this paper is organized as follows. In Section 2, we introduce the notion of S^2-pseudo almost automorphic processes and give some basic properties. In Section 3, we prove the existence and uniqueness of S^2-pseudo almost automorphic mild solutions to some linear and nonlinear stochastic differential equations, respectively.

2 Preliminaries

Throughout the paper, we assume that $(\mathbb{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ and $(\mathbb{K}, \|\cdot\|_{\mathbb{K}}, \langle\cdot,\cdot\rangle_{\mathbb{K}})$ are two real separable Hilbert spaces. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space. The notation $L^2(\mathbb{P}, \mathbb{H})$ stands for the space of all $\mathbb{H}$-valued random variables x such that

$$E\|x\|^2 = \int_\Omega \|x\|^2 d\mathbb{P} < \infty.$$

For $x \in L^2(\mathbb{P}, \mathbb{H})$, let

$$\|x\|_2 = \left(\int_\Omega \|x\|^2 d\mathbb{P}\right)^{\frac{1}{2}}.$$

Then it is routine to check that $L^2(\mathbb{P}, \mathbb{H})$ is a Hilbert space equipped with the norm $\|\cdot\|_2$. We let $C\left(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})\right)$ (respectively, $C\left(\mathbb{R}\times L^2(\mathbb{P}, \mathbb{H}), L^2(\mathbb{P}, \mathbb{H})\right)$) denote the collection of continuous stochastic processes from $\mathbb{R}$ into $L^2(\mathbb{P}, \mathbb{H})$ (respectively, the collection of continuous stochastic processes from $\mathbb{R}\times L^2(\mathbb{P}, \mathbb{H})$ into $L^2(\mathbb{P}, \mathbb{H})$). In addition, $W(t)$ is a two-sided standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, \mathscr{F}_t)$, where $\mathscr{F}_t = \sigma\{W(u) - W(v); u, v \le t\}$.

Definition 2.1 *([12]) A stochastic process $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is said to be stochastically continuous if*

$$\lim_{t\to s} E\|x(t) - x(s)\|^2 = 0.$$

Definition 2.2 *([12]) A stochastically continuous stochastic process $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is said to be square-mean almost automorphic if for every sequence of real numbers there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a stochastic process $y : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ such that*

$$\lim_{n\to\infty} E\|x(t+s_n) - y(t)\|^2 = 0 \quad \text{and} \quad \lim_{n\to\infty} E\|y(t-s_n) - x(t)\|^2 = 0$$

hold for each $t \in \mathbb{R}$. The collection of all square-mean almost automorphic stochastic processes $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is denoted by $AA\left(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\right)$.

Lemma 2.1 *([12]) $\left(AA\left(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\right), \|\cdot\|_\infty\right)$ is a Banach space when it is equipped with the norm*

$$\|x\|_\infty := \sup_{t\in\mathbb{R}} \|x(t)\|_2 = \sup_{t\in\mathbb{R}} (E\|x(t)\|^2)^{\frac{1}{2}},$$

for $x \in AA\left(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\right)$.

Let $PAP_0(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$ be the collection of all $x \in BC(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$ such that

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} E\|x(s)\|^2 ds = 0.$$

Similarly, we define $PAP_0(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}); L^2(\mathbb{P}, \mathbb{H}))$ to be the collection of all bounded jointly continuous stochastic processes $f : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$ such that

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} E\|f(s, x)\|^2 ds = 0$$

uniformly in $x \in L^2(\mathbb{P}, \mathbb{H})$.

Definition 2.3 *[4] A stochastic process $x \in BC(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$ is called square-mean pseudo almost automorphic if it can be expressed as $x = y + \varphi$, where $y \in AA(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$ and $\varphi \in PAP_0(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$. The collection of such functions will be denoted by $PAA(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$.*

Definition 2.4 *[4] A bounded continuous stochastic process $F : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H}))$ is called square-mean pseudo almost automorphic if it can be expressed as $F = G + \Phi$, where $G \in AA(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}); L^2(\mathbb{P}, \mathbb{H}))$ and $\Phi \in PAP_0(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}); L^2(\mathbb{P}, \mathbb{H}))$. The collection of such functions will be denoted by $PAA(\mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}); L^2(\mathbb{P}, \mathbb{H}))$.*

Lemma 2.2 *[4] $(PAA(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})), \|\cdot\|_\infty)$ is a Banach space with the supremum norm*

$$\|x\|_\infty = \sup_{t\in\mathbb{R}} \left(E\|x(t)\|^2\right)^{1/2}.$$

Definition 2.5 *([12]) A function $f : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$, $(t, x) \to f(t, x)$, which is jointly continuous, is said to be square-mean almost automorphic in $t \in \mathbb{R}$ for each $x \in L^2(\mathbb{P}, \mathbb{H})$ if for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a stochastic process $\widetilde{f} : f : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$ such that*

$$\lim_{n\to\infty} E\|f(t + s_n, x) - \widetilde{f}(t, x)\|^2 = 0 \quad \text{and} \quad \lim_{n\to\infty} E\|\widetilde{f}(t - s_n, x) - f(t, x)\|^2 = 0$$

for each $t \in \mathbb{R}$ and each $x \in L^2(\mathbb{P}, \mathbb{H})$.

Definition 2.6 *[2] The Bochner transform $x^b(t, s), t \in \mathbb{R}, s \in [0, 1]$, of a stochastic process $x : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H})$ is defined by*

$$x^b(t, s) := x(t + s).$$

Definition 2.7 *[2] The Bochner transform $f^b(t, s, u), t \in \mathbb{R}, s \in [0, 1], u \in L^2(\mathbb{P}, \mathbb{H})$, of a function $f : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$ is defined by*

$$f^b(t, s, u) := f(t + s, u)$$

for each $u \in L^2(\mathbb{P}, \mathbb{H})$.

Definition 2.8 *[2] The space $BS^2\left(L^2(\mathbb{P},\mathbb{H})\right)$ of all Stepanov bounded stochastic processes consists of all measurable stochastic processes $x:\mathbb{R}\to L^2(\mathbb{P},\mathbb{H})$ such that $x^b = L^\infty\left(\mathbb{R}; L^2\left(0,1;L^2(\mathbb{P},\mathbb{H})\right)\right)$. This is a Banach space with the norm*

$$\begin{aligned}\|x\|_{S^2} = \|x^b\|_{L^\infty(\mathbb{R};L^2)} &= \sup_{t\in\mathbb{R}}\left(\int_0^1 E\|x(t+s)\|^2 ds\right)^{\frac{1}{2}} \\ &= \sup_{t\in\mathbb{R}}\left(\int_t^{t+1} E\|x(\tau)\|^2 d\tau\right)^{\frac{1}{2}}.\end{aligned}$$

Definition 2.9 *[8] A stochastic process $x\in BS^2\left(L^2(\mathbb{P},\mathbb{H})\right)$ is called Stepanov-like almost automorphic (or S^2-almost automorphic) if $x^b\in AA\left(\mathbb{R};L^2\left(0,1;L^2(\mathbb{P},\mathbb{H})\right)\right)$. In other words, a stochastic process $x\in L^2_{loc}\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)$ is said to be Stepanov-like almost automorphic if its Bochner transform $x^b:\mathbb{R}\to L^2\left(0,1;L^2(\mathbb{P},\mathbb{H})\right)$ is square-mean almost automorphic in the sense that for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a stochastic process $y\in L^2_{loc}\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)$ such that*

$$\int_t^{t+1} E\|x(s+s_n)-y(s)\|^2 ds \to 0 \quad \text{and} \quad \int_t^{t+1} E\|y(s-s_n)-x(s)\|^2 ds\to 0$$

as $n\to\infty$ pointwise on $\mathbb{R}$. The collection of all such functions will be denoted by $AS^2\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)$.

Lemma 2.3 *[8] Let $(x_n(t))_{n\in\mathbb{N}}$ be a sequence of S^2-almost automorphic stochastic processes such that*

$$\int_t^{t+1} E\|x_n(s)-x(s)\|^2 ds\to 0 \quad \text{for each} \quad t\in\mathbb{R},$$

as $n\to\infty$, then $x\in AS^2\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)$.

Lemma 2.4 *[8] $AS^2\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)$ is a Banach space when it is equipped with the norm $\|\cdot\|_{S^2}$.*

Lemma 2.5 *[8] If $x:\mathbb{R}\to L^2(\mathbb{P},\mathbb{H})$ is a square-mean almost automorphic stochastic process, then x is S^2-almost automorphic, that is, $AA\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)\subseteq AS^2\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)$.*

Definition 2.10 *[8] A function $f:\mathbb{R}\times L^2(\mathbb{P},\mathbb{H})\to L^2(\mathbb{P},\mathbb{H})$, $(t,x)\to f(t,x)$ with $f(\cdot,x)\in L^2_{loc}\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)$ for each $x\in L^2(\mathbb{P},\mathbb{H})$, is said to be S^2-almost automorphic in $t\in\mathbb{R}$ uniformly in $x\in L^2(\mathbb{P},\mathbb{H})$ if $t\to f(t,x)$ is S^2-almost automorphic for each $x\in L^2(\mathbb{P},\mathbb{H})$. That means, for every sequence of real numbers $\{s'_n\}_{n\in\mathbb{N}}$, there exist a subsequence $\{s_n\}_{n\in\mathbb{N}}$ and a function $\widetilde{f}:\mathbb{R}\times L^2(\mathbb{P},\mathbb{H})\to L^2(\mathbb{P},\mathbb{H})$ with $\widetilde{f}(\cdot,x)\in$*

$L^2_{loc}\left(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\right)$ *such that*

$$\int_t^{t+1} E\|f(s+s_n,x)-\widetilde{f}(s,x)\|^2 ds \to 0 \quad and \quad \int_t^{t+1} E\|\widetilde{f}(s-s_n,x)-f(s,x)\|^2 ds \to 0$$

as $n\to\infty$ *pointwise on* $\mathbb{R}$ *and for each* $x\in L^2(\mathbb{P},\mathbb{H})$. *We denote by* $AS^2\left(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}); L^2(\mathbb{P},\mathbb{H})\right)$ *the set of all such functions.*

Let us now introduce the new concept of S^2-pseudo almost automorphy for stochastic processes.

Definition 2.11 *A stochastic process* $x\in BS^2\left(L^2(\mathbb{P},\mathbb{H})\right)$ *is called* S^2*-pseudo almost automorphic if it can be expressed as* $x=y+\varphi$, *where* $y\in AS^2\left(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\right)$ *and* $\varphi^b\in PAP_0\left(\mathbb{R}; L^2\left(0,1; L^2(\mathbb{P},\mathbb{H})\right)\right)$. *The collection of such functions will be denoted by* $PAAS^2(\mathbb{R}; L^2(\mathbb{P},\mathbb{H}))$.

We also have

Definition 2.12 *A function* $f:\mathbb{R}\times L^2(\mathbb{P},\mathbb{H})\to L^2(\mathbb{P},\mathbb{H})$, $(t,x)\to f(t,x)$ *with* $f(\cdot,x)\in L^2_{loc}\left(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\right)$ *for each* $x\in L^2(\mathbb{P},\mathbb{H})$, *is said to be* S^2*-pseudo almost automorphic in* $t\in\mathbb{R}$ *uniformly in* $x\in L^2(\mathbb{P},\mathbb{H})$ *if it can be expressed as* $f=h+\varphi$, *where* $h\in AS^2\left(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}); L^2(\mathbb{P},\mathbb{H})\right)$ *and* $\varphi^b\in PAP_0\left(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}); L^2\left(0,1; L^2(\mathbb{P},\mathbb{H})\right)\right)$. *The collection of such functions will be denoted by* $PAAS^2(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}); L^2(\mathbb{P},\mathbb{H}))$.

Lemma 2.6 *If* $\phi^b(\cdot)\in PAP_0\left(\mathbb{R}; L^2\left(0,1; L^2(\mathbb{P},\mathbb{H})\right)\right)$, *then for any* $h\in\mathbb{R}$, $\phi^b(\cdot-h)\in PAP_0\left(\mathbb{R}; L^2\left(0,1; L^2(\mathbb{P},\mathbb{H})\right)\right)$.

Proof: Since

$$\begin{aligned}\frac{1}{2r}\int_{-r}^{r}\left[\int_t^{t+1} E\|\phi(s-h)\|^2 ds\right]dt &= \frac{1}{2r}\int_{-r}^{r}\left[\int_{t-h}^{t-h+1} E\|\phi(s)\|^2 ds\right]dt\\ &= \frac{1}{2r}\int_{-r-h}^{r-h}\left[\int_t^{t+1} E\|\phi(s)\|^2 ds\right]dt.\end{aligned}$$

If $h\geq 0$,

$$\frac{1}{2r}\int_{-r}^{r}\left[\int_t^{t+1} E\|\phi(s-h)\|^2 ds\right]dt \leq \frac{2(r+h)}{2r}\frac{1}{2(r+h)}\int_{-(r+h)}^{r+h}\left[\int_t^{t+1} E\|\phi(s)\|^2 ds\right]dt,$$

which implies $\phi^b(\cdot-h)\in PAP_0\left(\mathbb{R}; L^2\left(0,1; L^2(\mathbb{P},\mathbb{H})\right)\right)$.
And if $h<0$,

$$\frac{1}{2r}\int_{-r}^{r}\left[\int_t^{t+1} E\|\phi(s-h)\|^2 ds\right]dt \leq \frac{2(r-h)}{2r}\frac{1}{2(r-h)}\int_{-(r-h)}^{r-h}\left[\int_t^{t+1} E\|\phi(s)\|^2 ds\right]dt,$$

which also implies $\phi^b(\cdot-h)\in PAP_0\left(\mathbb{R}; L^2\left(0,1; L^2(\mathbb{P},\mathbb{H})\right)\right)$. □

Note that $L^2(\mathbb{P},\mathbb{H})$ is a Banach space, we state the following lemmas (cf. [24, 11]).

Lemma 2.7 *[11] If $f \in PAA(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$, then $f \in PAAS^2(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$. In other words, $PAA(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})) \subseteq PAAS^2(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$.*

Lemma 2.8 *[11] The space $PAAS^2(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$ equipped with the norm $\| \cdot \|_{S^2}$ is a Banach space.*

Lemma 2.9 *[11] Let $F : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$ be a S^2-pseudo almost automorphic function. Suppose that $F(t, u)$ is Lipschitzian in $u \in L^2(\mathbb{P}, \mathbb{H})$ uniformly in $t \in \mathbb{R}$, that is there exists a constant $L > 0$ such that*

$$E\|F(t, u) - F(t, v)\|^2 \leq LE\|u - v\|^2$$

for all $t \in \mathbb{R}$ and $u, v \in L^2(\mathbb{P}, \mathbb{H})$. If $\Phi \in PAAS^2(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$, then the operator $\Upsilon : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H}))$ defined by $\Upsilon(\cdot) := F(\cdot, \Phi(\cdot))$ belongs to $PAAS^2(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$.

Lemma 2.10 *[24] Let $F : \mathbb{R} \times L^2(\mathbb{P}, \mathbb{H}) \to L^2(\mathbb{P}, \mathbb{H})$ be a S^2-pseudo almost automorphic function with $F = G + H$. Assume that F satisfies the following conditions:*

(i) $F(t, \cdot)$ is uniformly continuous on each bounded subset $K \subset L^2(\mathbb{P}, \mathbb{H})$ uniformly for $t \in \mathbb{R}$, that is for all $\varepsilon > 0$, there exists $\delta > 0$ such that $u, v \in K$ and $E\|u - v\|^2 < \delta$ imply that $E\|f(t, u) - f(t, v)\|^2 < \varepsilon$ for all $t \in \mathbb{R}$;

(ii) $G(t, \cdot)$ is uniformly continuous on each bounded subset $K \subset L^2(\mathbb{P}, \mathbb{H})$ uniformly for $t \in \mathbb{R}$;

(iii) For every bounded subset $K \subset L^2(\mathbb{P}, \mathbb{H})$, the set $\{F(\cdot, u) : u \in K\}$ is bounded in $PAAS^2(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$.

If $\Phi = \alpha + \beta \in PAAS^2(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$ with $\alpha \in AS^2 \left(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\right)$, $\beta^b \in PAP_0 (\mathbb{R}; L^2 \left(0, 1; L^2(\mathbb{P}, \mathbb{H})\right))$ and $\overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact, then for any $\Phi \in PAAS^2(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$, the operator $\Upsilon : \mathbb{R} \to L^2(\mathbb{P}, \mathbb{H}))$ defined by $\Upsilon(\cdot) := F(\cdot, \Phi(\cdot))$ belongs to $PAAS^2(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H}))$.

3 Main results

In this section, we investigate the existence of S^2-pseudo almost automorphic solutions for the problems (1.1) and (1.2).

Definition 3.1 *An $\mathscr{F}_t$-progressively measurable stochastic process $\{x(t)\}_{t \in \mathbb{R}}$ is called a mild solution of the problem (1.1) on $\mathbb{R}$ if it satisfies the corresponding stochastic integral equation*

$$x(t) = T(t - a)x(a) + \int_a^t T(t - s)f(s)ds + \int_a^t T(t - s)g(s)dW(s)$$

for all $t \geq a$ and for each $a \in \mathbb{R}$.

Definition 3.2 *An $\mathscr{F}_t$-progressively measurable stochastic process $\{x(t)\}_{t\in\mathbb{R}}$ is called a mild solution of the problem (1.2) on $\mathbb{R}$ if it satisfies the corresponding stochastic integral equation*

$$x(t) = T(t-a)x(a) + \int_a^t T(t-s)f\left(s, x(s)\right) ds + \int_a^t T(t-s)g\left(s, x(s)\right) dW(s)$$

for all $t \geq a$ and for each $a \in \mathbb{R}$.

We first list the following basic assumption:
(H1) The operator A is the infinitesimal generator of an exponentially stable C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $L^2(\mathbb{P},\mathbb{H})$; that is, there exist $M > 0$, $\delta > 0$ such that $\|T(t)\| \leq Me^{-\delta t}$, for all $t \geq 0$.

Theorem 3.1 *Under previous assumptions, if we assume that (H1) holds, then the problem (1.1) has a unique mild solution $x \in PAAS^2\left(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\right)$.*

Proof: Let us first prove uniqueness. It is conducted similarly as in the proof of [16, Theorem 3.1]. Assume that $x : \mathbb{R} \to L^2(\mathbb{P},\mathbb{H})$ is bounded stochastic process and satisfies the homogeneous equation

$$dx(t) = Ax(t)dt, \quad t \in \mathbb{R}. \tag{3.1}$$

Then $x(t) = T(t-s)x(s)$, for any $t \geq s$. Thus $\|x(t)\| \leq MKe^{-\delta(t-s)}$ with $\|x(s)\| \leq K$ for $s \in \mathbb{R}$ almost surely. Take a sequence of real numbers $\{s_n\}_{n\in\mathbb{N}}$ such that $s_n \to -\infty$ as $n \to \infty$. For any $t \in \mathbb{R}$ fixed, one can find a subsequence $\{s_{n_k}\}_{k\in\mathbb{N}} \subset \{s_n\}_{n\in\mathbb{N}}$ such that $s_{n_k} < t$ for all $k = 1, 2, \cdots$. By letting $k \to \infty$, we get $x(t) = 0$ almost surely.

Now, if $x_1, x_2 : \mathbb{R} \to L^2(\mathbb{P},\mathbb{H})$ are bounded solutions to Eq. (1.1), then $x = x_1 - x_2$ is a bounded solution to Eq. (3.1). In view of the above, $x = x_1 - x_2 = 0$ almost surely, that is, $x_1 = x_2$ almost surely.

Now let us investigate the existence. Since $f, g \in PAAS^2\left(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\right)$, there exist $\rho, \varpi \in AS^2\left(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\right)$ and $\phi^b, \psi^b \in PAP_0\left(\mathbb{R}; L^2\left(0,1; L^2(\mathbb{P},\mathbb{H})\right)\right)$ such that $f = \rho + \phi,\ g = \varpi + \psi$.

Consider for each $n = 1, 2, \cdots$, the integrals

$$x_n(t) = \int_{n-1}^n T(\sigma)\rho(t-\sigma)d\sigma,\ y_n(t) = \int_{n-1}^n T(\sigma)\varpi(t-\sigma)dW(\sigma),$$

and

$$z_n(t) = \int_{n-1}^n T(\sigma)\phi(t-\sigma)d\sigma,\ w_n(t) = \int_{n-1}^n T(\sigma)\psi(t-\sigma)dW(\sigma)$$

for each $t \in \mathbb{R}$. First, by using Hölder's inequality, we get

$$\int_t^{t+1} E\|x_n(s)\|^2 ds = \int_t^{t+1} E\left\|\int_{n-1}^n T(\sigma)\rho(s-\sigma)d\sigma\right\|^2 ds$$

$$\begin{aligned}
&\leq \int_t^{t+1}\int_{n-1}^{n}\|T(\sigma)\|^2 E\|\rho(s-\sigma)\|^2 d\sigma ds\\
&\leq M^2\int_t^{t+1}\int_{n-1}^{n} e^{-2\delta\sigma} E\|\rho(s-\sigma)\|^2 d\sigma ds\\
&\leq M^2\int_{n-1}^{n} e^{-2\delta\sigma}\left(\int_t^{t+1} E\|\rho(s-\sigma)\|^2 ds\right) d\sigma\\
&\leq M^2\|\rho\|^2_{S^2}\int_{n-1}^{n} e^{-2\delta\sigma} d\sigma\\
&\leq \frac{M^2}{2\delta}\|\rho\|^2_{S^2} e^{-2\delta n}(e^{2\delta}-1).
\end{aligned}$$

Since $\frac{M^2}{2\delta}\|\rho\|^2_{S^2}(e^{2\delta}-1)\sum_{n=1}^{\infty} e^{-2\delta n} < \infty$, we deduce from the well-known Weierstrass test that the series $\sum_{n=1}^{\infty} x_n(t)$ is convergent in the sense of the norm $\|\cdot\|_{S^2}$ uniformly on $\mathbb{R}$.

Now let

$$\Phi(t) := \sum_{n=1}^{\infty} x_n(t) \qquad \text{for each} \quad t\in\mathbb{R}.$$

Observe that

$$\Phi(t) = \int_{-\infty}^{t} T(t-s)\rho(s)ds \qquad \text{for each} \quad t\in\mathbb{R}.$$

Clearly, $\Phi(t)\in C\left(\mathbb{R}, L^2(\mathbb{P},\mathbb{H})\right)$.

Now, by using an estimate on Itô integral established in Ichikawa [13], we obtain that

$$\begin{aligned}
\int_t^{t+1} E\|y_n(s)\|^2 ds &= \int_t^{t+1} E\left\|\int_{n-1}^{n} T(\sigma)\varpi(s-\sigma)dW(\sigma)\right\|^2 ds\\
&\leq \int_t^{t+1}\int_{n-1}^{n}\|T(\sigma)\|^2 E\|\varpi(s-\sigma)\|^2 d\sigma ds\\
&\leq M^2\int_t^{t+1}\int_{n-1}^{n} e^{-2\delta\sigma} E\|\varpi(s-\sigma)\|^2 d\sigma ds\\
&\leq M^2\int_{n-1}^{n} e^{-2\delta\sigma}\left(\int_t^{t+1} E\|\varpi(s-\sigma)\|^2 ds\right) d\sigma\\
&\leq M^2\|\varpi\|^2_{S^2}\int_{n-1}^{n} e^{-2\delta\sigma} d\sigma\\
&\leq \frac{M^2}{2\delta}\|\varpi\|^2_{S^2} e^{-2\delta n}(e^{2\delta}-1).
\end{aligned}$$

Since $\frac{M^2}{2\delta}\|\varpi\|^2_{S^2}(e^{2\delta}-1)\sum_{n=1}^{\infty} e^{-2\delta n} < \infty$, we deduce from the Weierstrass test that the series $\sum_{n=1}^{\infty} y_n(t)$ is convergent in the sense of the norm $\|\cdot\|_{S^2}$ uniformly on $\mathbb{R}$. Furthermore,

$$\Psi(t) := \int_{-\infty}^{t} T(t-s)\varpi(s)dW(s) = \sum_{n=1}^{\infty} y_n(t), \quad t\in\mathbb{R},$$

and clearly $\Psi(t) \in C\left(\mathbb{R}, L^2(\mathbb{P}, \mathbb{H})\right)$.

Now let us show that each $x_n, y_n \in AS^2\left(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\right)$. First, we prove that $x_n \in AS^2\left(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\right)$. Indeed, let $\{s'_m\}_{m\in\mathbb{N}}$ be a sequence of real numbers. Since $\rho \in AS^2\left(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\right)$, there exist a subsequence $\{s_m\}_{m\in\mathbb{N}}$ of $\{s'_m\}_{m\in\mathbb{N}}$ and a stochastic process $\widetilde{\rho} \in L^2_{loc}\left(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\right)$

$$\int_t^{t+1} E\|\rho(s+s_m) - \widetilde{\rho}(s)\|^2 ds \to 0 \quad \text{and} \quad \int_t^{t+1} E\|\widetilde{\rho}(s-s_m) - \rho(s)\|^2 ds \to 0$$

as $m \to \infty$ pointwise on $\mathbb{R}$. Moreover, if we let $\widetilde{x}_n(t) = \int_{n-1}^n T(\sigma)\widetilde{\rho}(t-\sigma)d\sigma$, we have

$$\begin{aligned}
&\int_t^{t+1} E\|x_n(s+s_m) - \widetilde{x}_n(s)\|^2 ds \\
= &\int_t^{t+1} E\left\|\int_{n-1}^n T(\sigma)\rho(s+s_m-\sigma)d\sigma - \int_{n-1}^n T(\sigma)\widetilde{\rho}(s-\sigma)d\sigma\right\|^2 ds \\
= &\int_t^{t+1} E\left\|\int_{n-1}^n T(\sigma)[\rho(s+s_m-\sigma) - \widetilde{\rho}(s-\sigma)]d\sigma\right\|^2 ds \\
\leq &\int_t^{t+1}\int_{n-1}^n \|T(\sigma)\|^2 E\|\rho(s+s_m-\sigma) - \widetilde{\rho}(s-\sigma)\|^2 d\sigma ds \\
\leq &\; M^2\int_t^{t+1}\int_{n-1}^n e^{-2\delta\sigma} E\|\rho(s+s_m-\sigma) - \widetilde{\rho}(s-\sigma)\|^2 d\sigma ds \\
\leq &\; M^2\int_{n-1}^n e^{-2\delta\sigma}\left(\int_t^{t+1} E\|\rho(s+s_m-\sigma) - \widetilde{\rho}(s-\sigma)\|^2 ds\right) d\sigma.
\end{aligned}$$

Obviously, the last inequality goes to 0 as $m \to \infty$ pointwise on $\mathbb{R}$. Similarly we can prove that

$$\int_t^{t+1} E\|\widetilde{x}_n(s-s_m) - x_n(s)\|^2 ds \to 0$$

as $m \to \infty$ pointwise on $\mathbb{R}$. Thus we conclude that each $x_n \in AS^2\left(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\right)$ and consequently their uniform limit $\Phi(t) \in AS^2\left(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\right)$, by using Lemma 2.3.

Next, we show that each $y_n \in AS^2\left(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\right)$. Since $\varpi \in AS^2\left(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\right)$, then for every sequence of real numbers $\{s'_m\}_{m\in\mathbb{N}}$ there exists a subsequence $\{s_m\}_{m\in\mathbb{N}} \subset \{s'_m\}_{m\in\mathbb{N}}$ and a stochastic process $\widetilde{\varpi} \in L^2_{loc}\left(\mathbb{R}; L^2(\mathbb{P}, \mathbb{H})\right)$ such that

$$\int_t^{t+1} E\|\varpi(s+s_m) - \widetilde{\varpi}(s)\|^2 ds \to 0 \quad \text{and} \quad \int_t^{t+1} E\|\widetilde{\varpi}(s-s_m) - \varpi(s)\|^2 ds \to 0$$

as $m \to \infty$ pointwise on $\mathbb{R}$. Moreover, if we let $\widetilde{y}_n(t) = \int_{n-1}^n T(\sigma)\widetilde{\varpi}(t-\sigma)dW(\sigma)$, by using the Ito integral, we get

$$\int_t^{t+1} E\|y_n(s+s_m) - \widetilde{y}_n(s)\|^2 ds$$

$$\begin{aligned}
&= \int_t^{t+1} E\left\| \int_{n-1}^{n} T(\sigma)\varpi(s+s_m-\sigma)dW(\sigma) - \int_{n-1}^{n} T(\sigma)\widetilde{\varpi}(s-\sigma)dW(\sigma)\right\|^2 ds \\
&= \int_t^{t+1} E\left\| \int_{n-1}^{n} T(\sigma)[\varpi(s+s_m-\sigma) - \widetilde{\varpi}(s-\sigma)]dW(\sigma)\right\|^2 ds \\
&\le \int_t^{t+1} \int_{n-1}^{n} \|T(\sigma)\|^2 E\|\varpi(s+s_m-\sigma) - \widetilde{\varpi}(s-\sigma)\|^2 d\sigma ds \\
&\le M^2 \int_t^{t+1} \int_{n-1}^{n} e^{-2\delta\sigma} E\|\varpi(s+s_m-\sigma) - \widetilde{\varpi}(s-\sigma)\|^2 d\sigma ds \\
&\le M^2 \int_{n-1}^{n} e^{-2\delta\sigma} \left(\int_t^{t+1} E\|\varpi(s+s_m-\sigma) - \widetilde{\varpi}(s-\sigma)\|^2 ds\right) d\sigma.
\end{aligned}$$

Obviously, the last inequality goes to 0 as $m \to \infty$ pointwise on $\mathbb{R}$. Arguing in a similar way, we infer that

$$\int_t^{t+1} E\|\widetilde{y}_n(s-s_m) - y_n(s)\|^2 ds \to 0$$

as $m \to \infty$ pointwise on $\mathbb{R}$. Thus we conclude that each $y_n \in AS^2\left(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\right)$ and consequently their uniform limit $\Psi(t) \in AS^2\left(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\right)$, by using Lemma 2.3.

In the following, we intend to verify that each $z_n^b, w_n^b \in PAP_0\left(\mathbb{R}; L^2\left(0,1; L^2(\mathbb{P},\mathbb{H})\right)\right)$. We first prove that $z_n^b \in PAP_0\left(\mathbb{R}; L^2\left(0,1; L^2(\mathbb{P},\mathbb{H})\right)\right)$. Now by the fact $\{T(t)\}_{t\ge 0}$ is exponentially stable, we have

$$\begin{aligned}
\int_t^{t+1} E\|z_n(s)\|^2 ds &= \int_t^{t+1} E\left\| \int_{n-1}^{n} T(\sigma)\phi(s-\sigma)d\sigma\right\|^2 ds \\
&\le \int_t^{t+1} \int_{n-1}^{n} \|T(\sigma)\|^2 E\|\phi(s-\sigma)\|^2 d\sigma ds \\
&\le M^2 \int_t^{t+1} \int_{n-1}^{n} e^{-2\delta\sigma} E\|\phi(s-\sigma)\|^2 d\sigma ds \\
&\le M^2 \int_{n-1}^{n} e^{-2\delta\sigma} \left(\int_t^{t+1} E\|\phi(s-\sigma)\|^2 ds\right) d\sigma,
\end{aligned}$$

and hence for $r > 0$,

$$\frac{1}{2r}\int_{-r}^{r}\left(\int_t^{t+1} E\|z_n(s)\|^2 ds\right) dt \le M^2 \int_{n-1}^{n} e^{-2\delta\sigma}\left[\frac{1}{2r}\int_{-r}^{r}\left(\int_t^{t+1} E\|\phi(s-\sigma)\|^2 ds\right) dt\right] d\sigma.$$

It follows from Lemma 2.6 that

$$\lim_{r\to\infty} \frac{1}{2r}\int_{-r}^{r}\left(\int_t^{t+1} E\|\phi(s-\sigma)\|^2 ds\right) dt = 0$$

as $s \to \phi^b(s-\sigma) \in PAP_0\left(\mathbb{R}; L^2\left(0,1; L^2(\mathbb{P},\mathbb{H})\right)\right)$.

Applying Lebesgue dominated convergence theorem it follows that

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\left(\int_t^{t+1}E\|z_n(s)\|^2ds\right)dt=0,$$

and therefore $z_n^b\in PAP_0\left(\mathbb{R};L^2\left(0,1;L^2(\mathbb{P},\mathbb{H})\right)\right)$.

Next, we prove that $w_n^b\in PAP_0\left(\mathbb{R};L^2\left(0,1;L^2(\mathbb{P},\mathbb{H})\right)\right)$. By the fact $\{T(t)\}_{t\geq 0}$ is exponentially stable and Itô isometry, we get

$$\begin{aligned}\int_t^{t+1}E\|w_n(s)\|^2ds &= \int_t^{t+1}E\left\|\int_{n-1}^{n}T(\sigma)\psi(s-\sigma)dW(\sigma)\right\|^2ds\\ &\leq \int_t^{t+1}\int_{n-1}^{n}\|T(\sigma)\|^2E\|\psi(s-\sigma)\|^2d\sigma ds\\ &\leq M^2\int_t^{t+1}\int_{n-1}^{n}e^{-2\delta\sigma}E\|\psi(s-\sigma)\|^2d\sigma ds\\ &\leq M^2\int_{n-1}^{n}e^{-2\delta\sigma}\left(\int_t^{t+1}E\|\psi(s-\sigma)\|^2ds\right)d\sigma,\end{aligned}$$

and hence for $r>0$,

$$\frac{1}{2r}\int_{-r}^{r}\left(\int_t^{t+1}E\|w_n(s)\|^2ds\right)dt\leq M^2\int_{n-1}^{n}e^{-2\delta\sigma}\left[\frac{1}{2r}\int_{-r}^{r}\left(\int_t^{t+1}E\|\psi(s-\sigma)\|^2ds\right)dt\right]d\sigma.$$

It again from Lemma 2.6 follows that

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\left(\int_t^{t+1}E\|\psi(s-\sigma)\|^2ds\right)dt=0$$

as $s\to\psi^b(s-\sigma)\in PAP_0\left(\mathbb{R};L^2\left(0,1;L^2(\mathbb{P},\mathbb{H})\right)\right)$.
By Lebesgue dominated convergence theorem it follows that

$$\lim_{r\to\infty}\frac{1}{2r}\int_{-r}^{r}\left(\int_t^{t+1}E\|w_n(s)\|^2ds\right)dt=0,$$

and hence $w_n^b\in PAP_0\left(\mathbb{R};L^2\left(0,1;L^2(\mathbb{P},\mathbb{H})\right)\right)$.

Arguing in the same way as previous, we can conclude from the Weierstrass test that

$$\begin{aligned}\mathbb{Z}(t) &:= \int_{-\infty}^{t}T(t-s)\phi(s)ds=\sum_{n=1}^{\infty}z_n(t),\quad t\in\mathbb{R},\\ \mathbb{W}(t) &:= \int_{-\infty}^{t}T(t-s)\psi(s)dW(s)=\sum_{n=1}^{\infty}w_n(t),\quad t\in\mathbb{R}.\end{aligned}$$

Applying $z_n^b,w_n^b\in PAP_0\left(\mathbb{R};L^2\left(0,1;L^2(\mathbb{P},\mathbb{H})\right)\right)$ and by a similar estimation of the remainder term of uniformly convergent series as [24], we can also deduce that the uniformly limit $\mathbb{Z}^b(t)=\sum_{n=1}^{\infty}z_n(t)\in PAP_0\left(\mathbb{R};L^2\left(0,1;L^2(\mathbb{P},\mathbb{H})\right)\right)$ and $\mathbb{W}^b(t)=\sum_{n=1}^{\infty}w_n(t)\in$

$PAP_0\left(\mathbb{R}; L^2\left(0,1; L^2(\mathbb{P},\mathbb{H})\right)\right)$. Therefore, $x(t) := \Phi(t)+\Psi(t)+\mathbb{Z}(t)+\mathbb{W}(t)$ is a S^2-pseudo almost automorphic mild solution of Eq.(1.1).

Define

$$x(t) = \int_{-\infty}^{t} T(t-s)f(s)ds + \int_{-\infty}^{t} T(t-s)g(s)dW(s), \quad t \in \mathbb{R}.$$

Obviously x is a bounded solution to Eq. (1.1). Let us prove that $x(t)$ is a mild solution of the Eq. (1.1). Indeed, if we let $x(a) = \int_{-\infty}^{a} T(a-s)f(s)ds + \int_{-\infty}^{a} T(a-s)g(s)dW(s)$, then

$$T(t-a)x(a) = \int_{-\infty}^{a} T(t-s)f(s)ds + \int_{-\infty}^{a} T(t-s)g(s)dW(s).$$

But for $t \geq a$,

$$\begin{aligned}
\int_{a}^{t} T(t-s)g(s)dW(s) &= \int_{-\infty}^{t} T(t-s)g(s)dW(s) - \int_{-\infty}^{a} T(t-s)g(s)dW(s) \\
&= x(t) - \int_{-\infty}^{t} T(t-s)f(s)ds + \int_{-\infty}^{a} T(t-s)f(s)ds - T(t-a)x(a) \\
&= x(t) - T(t-a)x(a) - \int_{a}^{t} T(t-s)f(s)ds.
\end{aligned}$$

It follows that

$$x(t) = T(t-a)x(a) + \int_{a}^{t} T(t-s)f(s)ds + \int_{a}^{t} T(t-s)g(s)dW(s).$$

In view of the above, it follows that x is the only bounded S^2-pseudo almost automorphic mild solution to the equation (1.1). The proof is now complete. □

In order to investigate the solutions to the problem (1.2), we need the following additional assumptions:

(H2) The function $f \in PAAS^2\left(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}); L^2(\mathbb{P},\mathbb{H})\right) \cap C\left(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}), L^2(\mathbb{P},\mathbb{H})\right)$ and there exists a constant $L_f > 0$ such that

$$E\|f(t,x) - f(t,y)\|^2 \leq L_f E\|x-y\|^2$$

for all $t \in \mathbb{R}$ and each $x, y \in L^2(\mathbb{P},\mathbb{H})$.

(H3) The function $g \in PAAS^2\left(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}); L^2(\mathbb{P},\mathbb{H})\right) \cap C\left(\mathbb{R}\times L^2(\mathbb{P},\mathbb{H}), L^2(\mathbb{P},\mathbb{H})\right)$ and there exists a positive number L_g such that

$$E\|g(t,x) - g(t,y)\|^2 \leq L_g E\|x-y\|^2$$

for all $t \in \mathbb{R}$ and each $x, y \in L^2(\mathbb{P},\mathbb{H})$.

Theorem 3.2 *Assume the conditions (H1)-(H3) are satisfied, then the problem (1.2) admits a unique S^2-pseudo almost automorphic mild solution on $\mathbb{R}$ provided that*

$$L_0 = \left[\frac{2}{\delta^2}M^2 L_f + \frac{M^2}{\delta}L_g\right] < 1. \tag{3.2}$$

Proof: Let $\Lambda : PAAS^2\left(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\right) \to PAAS^2\left(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\right)$ be the operator defined by

$$\Lambda x(t) = \int_{-\infty}^{t} T(t-s) f\left(s, x(s)\right) ds + \int_{-\infty}^{t} T(t-s) g\left(s, x(s)\right) dW(s), \quad t \in \mathbb{R}.$$

From previous assumptions and the properties of $\{T(t)\}_{t\geq 0}$, one can easily see that Λx is well defined and continuous. By Lemma 2.9, we infer that both $F(\cdot) = f\left(\cdot, x(\cdot)\right)$ and $G(\cdot) = g\left(\cdot, x(\cdot)\right) \in PAAS^2\left(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\right)$. Then by using the proof of Theorem 3.1, we have that $\Lambda x \in PAAS^2\left(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\right)$ whenever $x \in PAAS^2\left(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\right)$. Thus Λ maps $PAAS^2\left(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\right)$ into itself.

Now we prove that Λ is a contraction mapping on $PAAS^2\left(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\right)$. Indeed, for each $t \in \mathbb{R}$, $x, y \in PAAS^2\left(\mathbb{R}; L^2(\mathbb{P},\mathbb{H})\right)$, we see that

$$\begin{aligned}
& \int_t^{t+1} E\|(\Lambda x)(s) - (\Lambda y)(s)\|^2 ds \\
= & \int_t^{t+1} E\left\| \int_{-\infty}^{s} T(s-\sigma)[f\left(\sigma, x(\sigma)\right) - f\left(\sigma, y(\sigma)\right)] d\sigma \right. \\
& \left. + \int_{-\infty}^{s} T(s-\sigma)[g\left(\sigma, x(\sigma)\right) - g\left(\sigma, y(\sigma)\right)] dW(\sigma) \right\|^2 ds \\
\leq & \ 2\int_t^{t+1} E\left\| \int_{-\infty}^{s} T(s-\sigma)[f\left(\sigma, x(\sigma)\right) - f\left(\sigma, y(\sigma)\right)] d\sigma \right\|^2 ds \\
& + 2\int_t^{t+1} E\left\| \int_{-\infty}^{s} T(s-\sigma)[g\left(\sigma, x(\sigma)\right) - g\left(\sigma, y(\sigma)\right)] dW(\sigma) \right\|^2 ds \\
\leq & \ 2M^2 \int_t^{t+1} \left[\left(\int_{-\infty}^{s} e^{-\delta(s-\sigma)} d\sigma\right)\left(\int_{-\infty}^{s} e^{-\delta(s-\sigma)} E\|f\left(\sigma, x(\sigma)\right) - f\left(\sigma, y(\sigma)\right)\|^2 d\sigma\right)\right] ds \\
& + 2\int_t^{t+1} \int_{-\infty}^{s} \|T(s-\sigma)\|^2 E\|g\left(\sigma, x(\sigma)\right) - g\left(\sigma, y(\sigma)\right)\|^2 d\sigma ds \\
\leq & \ \frac{2}{\delta}M^2 \int_t^{t+1} \int_{-\infty}^{s} e^{-\delta(s-\sigma)} E\|f\left(\sigma, x(\sigma)\right) - f\left(\sigma, y(\sigma)\right)\|^2 d\sigma ds \\
& + 2M^2 \int_t^{t+1} \int_{-\infty}^{s} e^{-2\delta(s-\sigma)} E\|g\left(\sigma, x(\sigma)\right) - g\left(\sigma, y(\sigma)\right)\|^2 d\sigma ds \\
\leq & \ \frac{2}{\delta}M^2 L_f \int_t^{t+1} \int_{-\infty}^{s} e^{-\delta(s-\sigma)} E\|x(\sigma) - y(\sigma)\|^2 d\sigma ds \\
& + 2M^2 L_g \int_t^{t+1} \int_{-\infty}^{s} e^{-2\delta(s-\sigma)} E\|x(\sigma) - y(\sigma)\|^2 d\sigma ds
\end{aligned}$$

$$\begin{aligned}
&\le \frac{2}{\delta}M^2L_f\int_0^\infty e^{-\delta\xi}\int_{t-\xi}^{t-\xi+1}E\|x(s)-y(s)\|^2dsd\xi \\
&\quad +2M^2L_g\int_0^\infty e^{-2\delta\xi}\int_{t-\xi}^{t-\xi+1}E\|x(s)-y(s)\|^2dsd\xi \\
&\le \left[\frac{2}{\delta^2}M^2L_f+\frac{M^2}{\delta}L_g\right]\|x-y\|_{S^2}^2.
\end{aligned}$$

Hence

$$\|\Lambda x-\Lambda y\|_{S^2}=\sup_{t\in\mathbb{R}}\left(\int_t^{t+1}E\|(\Lambda x)(s)-(\Lambda y)(s)\|^2ds\right)^{\frac{1}{2}}\le\sqrt{L_0}\|x-y\|_{S^2},$$

which implies that Λ is a contraction by (3.2). So by the Banach contraction principle, we draw a conclusion that there exists a unique fixed point $x(\cdot)$ for Λ in $PAAS^2\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)$, such that $\Lambda x=x$. Moreover, using the same proof as in Theorem 3.1, we can see that $x(t)=T(t-a)x(a)+\int_a^t T(t-s)f\left(s,x(s)\right)ds+\int_a^t T(t-s)g\left(s,x(s)\right)dW(s)$ is a mild solution of the equation (1.2) and $x(\cdot)\in PAAS^2\left(\mathbb{R};L^2(\mathbb{P},\mathbb{H})\right)$. This finishes the proof. □

Remark 3.1 *Our main results can be applied to investigate the existence and uniqueness of S^2-pseudo almost automorphic mild solutions for the example in [9, Section 6].*

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