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REPRESENTING TREES AS RELATIVELY COMPACT SUBSETS OF THE FIRST BAIRE CLASS

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A b s t r a c t. We show that there is a scattered compact subset K of the first Baire class, a Baire space X and a separately continuous mapping $f : X \times K \longrightarrow \mathbb{R}$ which is not continuous on any set of the form $G \times K$, where G is a comeager subset of X. We also show that it is possible to have a scattered compact subset K of the first Baire class which does have the Namioka property though its function space $\mathcal{C}(K)$ fails to have an equivalent Fréchet-differentiable norm and its weak topology fails to be σ-fragmented by the norm.

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Recall the well-known classical theorem of R.Baire which says that for every separately continuous real function defined on the unit square $[0,1]\times[0,1]$ is continuous at every point of a set of the form $G\times[0,1]$ where G is a comeager subset of $[0,1]$. This result was considerably extended by I.Namioka [12] who showed that the same conclusion can be reached for separately continuous functions on products of the form $X \times K$ where X is a Cech-complete space and where K is a compact space. A typical application of this result in functional analysis would say that a separately continuous group operation on a locally compact space is actually jointly continuous, so we in fact have a topological group. Another application would say that a weakly compact subset of a Banach space contains a comeager subset on which the weak and norm topologies coincide. Following [1], let us say that a compact space K has the *Namioka property* if for every Baire space X and every separately continuous mapping $f : X \times K \longrightarrow \mathbb{R}$ there is a comeager set $G \subseteq X$ such that f is continuous at every point of the product $G \times K$. Note that this is equivalent to saying that for every continuous map f from a Baire space X into $(\mathcal{C}(K), \tau_{\mathrm{p}})$, where τ_{p} denotes the topology of pointwise convergence on K, there is a comeager subset G of X such that $f : X \longrightarrow (\mathcal{C}(K), \mathrm{norm})$ is continuous at every point of G. In this reformulation, the Namioka property enters into the theory of smoothness and renormings of Banach spaces [2] where it has been shown to be a useful principle of distinguishing various classes of spaces. For example, Deville and Godefroy [1] have shown that if $\mathcal{C}(K)$ admits a locally uniformly convex[1] equivalent norm that is pointwise lower semicontinuous, then K has the Namioka property. The Namioka property is particularly interesting in the class of scattered compacta K which in terms of the function space $\mathcal{C}(K)$ is equivalent to saying that every continuous convex real-valued function defined on a convex open subset of $\mathcal{C}(K)$ is Frechet-differentiable at every point of a comeager subset of its domain[2]. The class of strong differentiability function spaces $\mathcal{C}(K)$ is rich enough to distinguish many more smoothness requirements that one can have on a given Banach space. This has been shown by R. Haydon [3] by analyzing a particular kind of scattered locally compact space given by set-theoretic trees T with their interval topologies, i.e. topologies τ_{in} generated by basic open sets of the form $(s,t] = \{x \in T : s <_T x \leq_T t\}$, where $s, t \in T \cup \{-\infty\}$ (see [17]). The purpose of this note is to show that there exist interesting trees T for which the corresponding locally compact space is homeomorphic to a relatively compact subset of the first Baire-class. Our first result in this direction was motivated by a conjecture of Haydon [4] and a question of A. Moltó [11] regarding a result from [5] which shows that if K is a separable compact set of Baire-class-1 functions, each of which have only countably many discontinuities, then $\mathcal{C}(K)$ admits an equivalent norm that is locally uniformly convex. Our example shows that this is no longer

[1] A norm $\| \cdot \|$ is *locally uniformly convex* if $\| x_n \| \longrightarrow \| x \|$ and $\| x + x_n \| \longrightarrow 2 \| x \|$ imply that $\| x - x_n \| \longrightarrow 0$.

[2] A Banach space with this property is usually called a *strong differentiability space* or an *Asplund space.*

true for an arbitrary (scattered) compact subset of the first Baire class. We use the K's failure to satisfy the Namioka property in order to prevent its function space $\mathcal{C}(K)$ to have locally uniformly convex renorming. Our second example, however, shows that there might exist a compact scattered subset K of the first Baire class that has the Namioka property but still its function space $\mathcal{C}(K)$ failing to have an equivalent locally uniformly convex norm. This example should be compared with the example of Namioka and Pol[13] of a compact scattered space K whose function space $\mathcal{C}(K)$ distinguishes the same properties but which is far from being representable inside the first Baire class. Our results are obtained via a general procedure which represents countably branching trees admitting a strictly increasing map into the reals as relatively compact subsets of the first Baire class. So our results also bare on the possible structure theory of compact subsets of the first Baire class, a theory that already has isolated some of its critical examples [19].

1. *Trees as relatively compact subsets of the first Baire class*

The Helly space[3] H, the split interval[4] $S(I)$, and the one-point compactification $A(D)$ of a discrete space of cardinality at most continuum are some of the standard examples of pointwise compact sets of Baire-class-1 functions. Given a point x of some Polish space X, let $\delta_X : X \longrightarrow 2$ be the corresponding Dirac-function $\delta_X(y) = 0$ iff $x \neq y$, then

$$A(X) = \{\delta_X : x \in X\} \cup \{\bar{0}\}$$

is one of the representations of $A(D)$ inside the first Baire class over X. The structure theory of compact subsets of the first Baire class developed in [19] unravels another critical example, the *Alexandroff duplicate* $D(M)$ of a compact metric space M. It is the space on $M \times 2$ where the points of $M \times \{1\}$ are taken to be isolated and where a typical open neighborhood of some $(x, 0)$ has the form $(U \times 2) \setminus F$ for U an open neighborhood of x in M and F a finite subset of $M \times \{1\}$. To see that $D(M)$ is representable inside

[3]Recall that the *Helly space* is the set $H = \{f : [0,1] \longrightarrow [0,1] : f \text{ is monotonic}\}$ equipped with the topology τ_p of pointwise convergence on $[0,1]$. The result of [5] shows that its function space $\mathcal{C}(H)$ admits a locally uniformly convex renorming.

[4]The *split interval* $S(I)$ is the lexicographic product $I \times 2$ of the unit interval $I = [0,1]$ and the 2-element ordering $2 = \{0,1\}$. Note that $S(I)$ is homeomorphic to a subspace of H so its function space also admits a locally uniformly convex renorming.

the first Baire class, note that the subspace

$$\{(x,0) : x \in M\} \cup \{(x,\delta_X) : x \in M\}$$

of $M \times A(M)$ is homeomorphic to $D(M)$.

Let us identify the power set of the rationals with the Cantor cube $2^{\mathbb{Q}}$. For $s,t \in 2^{\mathbb{Q}}$, let $s \sqsubseteq t$ denote the fact that $s \subseteq t$ and

$$(\forall x \in s)(\forall y \in t \setminus s) x <_{\mathbb{Q}} y.$$

For $t \in 2^{\mathbb{Q}}$, let

$$[t] = \{x \in 2^{\mathbb{Q}} : t \sqsubseteq x\}.$$

Note that $[t]$ is a compact subset of $2^{\mathbb{Q}}$ which reduces to a singleton if $\sup(t) = \infty$; otherwise if $\sup(t) < \infty$, the set $[t]$ is homeomorphic to $2^{\mathbb{Q}}$ via a natural homeomorphism induced by an order-isomorphism of $\mathbb{Q}$ and $\mathbb{Q} \cap [\sup(t), \infty)$. Let $1_{[t]} : 2^{\mathbb{Q}} \longrightarrow 2$ be the characteristic function of the subset $[t]$ of $2^{\mathbb{Q}}$, i.e.

$$1_{[t]}(x) = 1 \text{ iff } t \sqsubseteq x.$$

Note that $1_{[t]}$ is a Baire-class-1 function on $2^{\mathbb{Q}}$ and that $1_{[t]} = \delta_t$ iff $\sup(t) = \infty$.

Let $\mathrm{w}\mathbb{Q}$ be the collection of all subsets of $\mathbb{Q}$ that are well-ordered under the induced ordering from $\mathbb{Q}$. We consider $\mathrm{w}\mathbb{Q}$ as a tree under the ordering $\sqsubseteq$. The tree $(\mathrm{w}\mathbb{Q}, \sqsubseteq)$ has been indentified long time ago by Kurepa [9] who showed that, while there is a strictly increasing mapping from $(\mathrm{w}\mathbb{Q}, \sqsubseteq)$ into the reals, there is no strictly increasing mapping from $(\mathrm{w}\mathbb{Q}, \sqsubseteq)$ into the rationals. Since we are going to consider several different topologies some living on the same set, let us fix some notation for them. We have already reserved the notation τ_{in} for the locally compact topology on $\mathrm{w}\mathbb{Q}$ and its subtrees that is generated by subbasic clopen sets of the form

$$(-\infty, t] = \{x : x \sqsubseteq t\} \ (t \in \mathrm{w}\mathbb{Q}).$$

When we consider the first Baire class $\mathcal{B}_1(X)$ over some separable space X we usually consider it equipped with the topology τ_p of pointwise convergence on X. The notation for few other topologies living typically on $\mathrm{w}\mathbb{Q}$ or its subtrees will be fixed as we go on.

Consider the following subset of the first Baire-class $\mathcal{B}_1(2^{\mathbb{Q}})$ over the Cantor set $2^{\mathbb{Q}}$:

$$K_{\mathrm{w}\mathbb{Q}} = \{1_{[t]} : t \in \mathrm{w}\mathbb{Q}\}.$$

Lemma 1. *The set $K_{\mathrm{w}\mathbb{Q}}$ is a relatively compact*[5] *subset of $\mathcal{B}_1(2^{\mathbb{Q}})$ with only the constantly equal to 0 mapping $\bar{0}$ as its proper accumulation point.*

P r o o f. By Rosenthal's theorem [16] characterizing the relative compactness of subsets of $\mathcal{B}_1(2^{\mathbb{Q}})$, it suffices to show that every sequence (t_n) of elements of $\mathrm{w}\mathbb{Q}$ has a subsequence (t_{n_k}) such that the sequence $(1_{[t_{n_k}]})$ converges pointwise to an element of $K_{\mathrm{w}\mathbb{Q}} \cup \{\bar{0}\}$. To see this, we apply Ramsey's theorem and get a subsequence (t_{n_k}) of (t_n) such that either

1. t_{n_k} and t_{n_l} are incomparable in $\mathrm{w}\mathbb{Q}$ whenever $k \neq l$ or
2. $t_{n_k} \sqsubseteq t_{n_l}$ whenever $k < l$.

If (1) holds, $(1_{[t_{n_k}]})$ converges pointwise to $\bar{0}$, while if (2) holds then $(1_{[t_{n_k}]})$ converges pointwise to $1_{[t]}$ where $t = \bigcup_{k=0}^{\infty} t_{n_k}$. □

Lemma 2. *The map $t \mapsto 1_{[t]}$ is a homeomorphism between $(\mathrm{w}\mathbb{Q}, \tau_{\mathrm{in}})$ and $(K_{\mathrm{w}\mathbb{Q}}, \tau_{\mathrm{p}})$.*

P r o o f. Consider a subbasic clopen set

$$\{1_{[t]} : t \in \mathrm{w}\mathbb{Q}, x \in [t]\}$$

of $K_{\mathrm{w}\mathbb{Q}}$, where $x \in 2^{\mathbb{Q}}$. Its preimage is equal to

$$\{t \in \mathrm{w}\mathbb{Q} : t \sqsubseteq x\},$$

a typical subbasic clopen set of the interval topology of $\mathrm{w}\mathbb{Q}$. □

Combining Lemmas 1 and 2 we obtain the following.

Theorem 3. *The one-point compactification of the tree $\mathrm{w}\mathbb{Q}$ is homeomorphic to a compact subset of the first Baire class.* □

Our interest in the tree $\mathrm{w}\mathbb{Q}$ is partly based on its universality in the following sense.

Theorem 4. *Suppose T is a Hausdorff*[6] *tree of cardinality at most continuum which admits a strictly increasing mapping into the reals. Then T is homeomorphic to a subspace of $\mathrm{w}\mathbb{Q}$. If T is moreover countably branching then T is homeomorphic to an open subspace of $\mathrm{w}\mathbb{Q}$.*

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[5] Recall that we take $\mathcal{B}_1(2^{\mathbb{Q}})$ equipped with the topology τ_{p} of pointwise convergence on $2^{\mathbb{Q}}$ and that a subset S of $\mathcal{B}_1(2^{\mathbb{Q}})$ is *relatively compact* if its τ_{p}-closure is compact.

[6] A tree T is *Hausdorff* if two different nodes on the same level of T have different sets of predecessors.

P r o o f. Note that every tree T satisfying the hypothesis of the first part of Theorem 4 is isomorphic to a restriction to the form $S \upharpoonright \Lambda$, where S is some countably branching tree and where Λ denotes the set of all countable limit ordinals. It follows that it suffices to prove that every countably branching Hausdorff tree T which admits a strictly increasing map $f : T \longrightarrow \mathbb{R}$ is isomorphic to a downwards closed subtree of w$\mathbb{Q}$ and therefore homeomorphic to an open subspace of w$\mathbb{Q}$ when we view T and w$\mathbb{Q}$ as locally compact spaces with their interval topologies. Let T^0 denote the set of all nodes of T whose length is a successor ordinal, or topologically the set of all isolated points of (T, τ_{in}). Changing f if necessary, we may assume that $f[T^0] \subseteq \mathbb{Q}$. In fact, we may assume that for every $t \in T$, if $t_0 \neq t_1$ are two of its immediate successors, then $f(t_0)$ and $f(t_1)$ are two distinct rationals. To see that this can be arranged, let us assume as we may, that actually $f[T_0] \subseteq \mathbb{Q}_{\text{d}}$, where $\mathbb{Q}_{\text{d}}$ denotes the set of all right-hand points of the complementary intervals to the Cantor ternary set. Since T is a well-founded ordering we can recursively change f to an $\bar{f} : T \longrightarrow \mathbb{R}$ in such a way that for all $t \in T$ and $q \in \mathbb{Q}_{\text{d}}$, the new mapping $\bar{f}$ maps the subset

$$\{x \in \text{ImSuc}(t) : f(x) = q\}$$

of the set ImSuc(t) of immediate successors of t in T in a one-to-one fashion to the rationals of the complementary interval of the Cantor ternary set just left of q. Finally define $\varphi : T \longrightarrow \text{w}\mathbb{Q}$ by

$$\varphi(t) = \{f(x) : x \in T^0, x \leq_T t\}.$$

Then φ is an isomorphic embedding of T into a downwards closed[7] subtree of w$\mathbb{Q}$. □

Corollary 5. *Every Hausdorff tree T of cardinality at most continuum admitting a strictly increasing real-valued function has a scattered compactification αT representable as a compact subset of the first Baire class.* □

Remark 6. Note that in this representation the remainder $\alpha T \setminus T$ is either a singleton or homeomorphic to a one-point compactification of some discrete space. If one is willing to dispense with the remainder being scattered, one can easily have a compactification γT of T representable inside the first Baire class for which the remainder is homeomorphic to a subspace

[7]Recall that a subset T of w$\mathbb{Q}$ is *downwards closed* if from $s \sqsubseteq t$ and $t \in T$ we can conclude $s \in T$. Note that a downwards closed subset of w$\mathbb{Q}$ is a τ_{in}-open subset of w$\mathbb{Q}$.

of the Alexandroff duplicate $D(2^{\mathbb{N}})$ of the Cantor set $2^{\mathbb{N}}$. Hence one can have a first countable compactification of any tree satisfying the hypothesis of Corollary 5 that is still representable inside the first Baire class. To see this, referring to the above representation of $D(2^{\mathbb{N}})$ inside the first Baire class, it suffices to check that the subspace

$$\{(x, \bar{0}) : x \in 2^{\mathbb{Q}}\} \cup \{(t, 1_{[t]}) : t \in \mathrm{w}\mathbb{Q}\}$$

of $2^{\mathbb{Q}} \times A(2^{\mathbb{Q}})$ is a compactification of $\mathrm{w}\mathbb{Q}$ whose remainder is homeomorphic to a subspace of the duplicate $D(2^{\mathbb{Q}})$. While this sort of Baire class-1 compactification γT fail to give us a strong differentiability space $\mathcal{C}(\gamma T)$, their interest come from a recent work of W. Kubis and others about classes of metrizably fibered compacta (see [6]).

2. *Separate versus joint continuity*

Let

$$\sigma\mathbb{Q} = \{t \in \mathrm{w}\mathbb{Q} : \sup(t) < \infty\}.$$

We consider $\sigma\mathbb{Q}$ as a subtree of $\mathrm{w}\mathbb{Q}$ and besides its locally compact interval topology τ_{in} we also consider the topology τ_{bc} generated by the following family of subbasic clopen sets

$$\{x \in \sigma\mathbb{Q} : t \sqsubseteq x\} \text{ and } \{x \in \sigma\mathbb{Q} : \sup(x) < q\},$$

where $t \in \sigma\mathbb{Q}$ and $q \in \mathbb{Q}$. In his numerous papers about trees of the form $\sigma\mathbb{Q}$ and $\mathrm{w}\mathbb{Q}$, Kurepa has presented at least two different proofs that the two trees admit no strictly increasing mapping into the rationals. The argument from one of these two proofs, which is exposed in the author's [17], is sufficient for proving the following property of the space $(\sigma\mathbb{Q}, \tau_{\mathrm{bc}})$ though its present formulation was inspired by a similar fact about a closely related tree (also known to Kurepa) appearing in Haydon's paper [3].

Lemma 7. $(\sigma\mathbb{Q}, \tau_{\mathrm{bc}})$ *is a Baire space.*

P r o o f. It suffices to show that the player Nonempty has a winning strategy σ in the Banach-Mazur game (see [14]). Given a nonempty open set $U \in \tau_{\mathrm{bc}}$, we let $\sigma(U)$ be any basic open subset of U of the form

$$[t, \infty)_q = \{x \in \sigma\mathbb{Q} : t \sqsubseteq x, \sup(x) < q\},$$

where $t \in \sigma\mathbb{Q}$ is such that $\sup(t) \notin \mathbb{Q}$ and where q is a rational such that $q > \sup(t)$. Thus in any infinite run

$$U_0 \supseteq [t_0, \infty)_{q_0} \supseteq U_1 \supseteq [t_1, \infty)_{q_1} \supseteq \dots$$

of the Banach-Mazur game

$$t \in \bigcap_{n=0}^{\infty} U_n = \bigcap_{n=0}^{\infty} [t_n, \infty)_{q_n},$$

where $t = \bigcup_{n=0}^{\infty} t_n$. □

Now define $f : \sigma\mathbb{Q} \times (\mathrm{w}\mathbb{Q} \cup \{\infty\}) \longrightarrow \{0, 1\}$ by

$$f(s,t) = 1 \text{ iff } s \sqsupseteq t.$$

Note that for each $s \in \sigma\mathbb{Q}$, the corresponding fiber-mapping $f_s : \mathrm{w}\mathbb{Q} \cup \{\infty\} \longrightarrow \{0,1\}$ is continuous since

$$\{t \in \mathrm{w}\mathbb{Q} \cup \{\infty\} : f_s(t) = f(s,t) = 1\} = \{t \in \mathrm{w}\mathbb{Q} : t \sqsubseteq s\}$$

is a τ_{in}-clopen subset of $\mathrm{w}\mathbb{Q} \cup \{\infty\}$. Similarly for each $t \in \mathrm{w}\mathbb{Q} \cup \{\infty\}$ the corresponding fiber-mapping $f^t : \sigma\mathbb{Q} \longrightarrow \{0,1\}$ is continuous since

$$\{s \in \sigma\mathbb{Q} : f^t(s) = f(s,t) = 1\} = \{s \in \sigma\mathbb{Q} : s \sqsupseteq t\}$$

is by definition a τ_{bc}-clopen subset of $\sigma\mathbb{Q}$.

Lemma 8. *The mapping f is not continuous on any set of the form $G \times (\mathrm{w}\mathbb{Q} \cup \{\infty\})$, where G is a comeager subset of $\sigma\mathbb{Q}$.*

P r o o f. Note that every comeager $G \subseteq \sigma\mathbb{Q}$ relative to the topology τ_{bc} contains a point s and two sequences (s_n) and (t_n) such that $s \sqsubseteq t_n \sqsubseteq s_n$ for all n, such that $s_n \longrightarrow s$ relative to τ_{bc}, and such that t_n is incomparable with t_m whenever $n \neq m$. It follows that $t_m \longrightarrow \infty$ in the one-point compactification $\mathrm{w}\mathbb{Q} \cup \{\infty\}$ of the interval topology of $\mathrm{w}\mathbb{Q}$. Thus, $(s_n, t_n) \longrightarrow (s, \infty)$ in the product space $\sigma\mathbb{Q} \times (\mathrm{w}\mathbb{Q} \cup \{\infty\})$. However, $f(s_n, t_n) = 1$ for all n while $f(s, \infty) = 0$. □

The following result summarizes Lemmas 7 and 8 modulo the representation $K_{\mathrm{w}\mathbb{Q}} \cup \{\bar{0}\}$ of $\mathrm{w}\mathbb{Q} \cup \{\infty\}$ given above in Section 1.

Theorem 9. *The compact scattered set $K_{\mathrm{w}\mathbb{Q}} \cup \{\bar{0}\}$ of Baire-class-1 functions on the Cantor cube $2^{\mathbb{Q}}$ does not have the Namioka property about continuity of separately continuous functions on its products with Baire spaces.* □

Remark 10. In [5], Haydon, Moltó, and Orihuala have shown that if a compactum K can be represented as a compact set of Baire class-1 functions that have only countably many discontinuities, then the topology of $\mathcal{C}(K)$ of pointwise convergence on K is σ-fragmented by the norm, so in particular K has the Namioka property. In our representation of $K_{\mathrm{w}\mathbb{Q}} \cup \{\bar{0}\}$ inside the first Baire class over the Cantor cube $2^{\mathbb{Q}}$ we use characteristic functions of closed subsets of $2^{\mathbb{Q}}$ which most of the time have uncountably many discontinuities. Theorem 9 and the result of [5] show that the amount of discontinuities is necessary.

3. *A tree that has the Namioka property*

Recall that a subset A of a topological space X is a *universally Baire* subset of X if for every topological space Y, or equivalently for every Baire space Y, and every continuous mapping $f : Y \longrightarrow X$ the preimage $f^{-1}[A]$ has the property of Baire as a subset of Y. We say that A is a *universally meager* subset of X if for every Baire space Y and every continuous $f : Y \longrightarrow X$, the preimage $f^{-1}[A]$ is a meager subset of Y, unless f is constant on some nonempty open subset of Y. We transfer these notions to subtrees of $\mathrm{w}\mathbb{Q}$ viewed as subsets of the Cantor set $2^{\mathbb{Q}}$. Recall that a subtree T of $\mathrm{w}\mathbb{Q}$ is an *Aronszajn-subtree* of $\mathrm{w}\mathbb{Q}$ (in short, *A-subtree* of $\mathrm{w}\mathbb{Q}$) if

$$\{t \in T : \mathrm{otp}(t) = \alpha\}$$

is countable for every countable ordinal α. The first known A-subtree of $\mathrm{w}\mathbb{Q}$ was constructed by Kurepa [8]. In the next Section we shall however rely on a different construction of A-subtrees of $\mathrm{w}\mathbb{Q}$ discovered by the author in [18]. Our interest in these trees is based on the following observation.

Lemma 11. *The one-point compactification $T \cup \{\infty\}$ of a universally Baire A-subtree T of $\mathrm{w}\mathbb{Q}$ has the Namioka property about continuity of separately continuous functions defined on its products with Baire spaces.*

P r o o f. Let X be a given Baire space and let $f : X \times (T \cup \{\infty\}) \longrightarrow \mathbb{R}$ be a given separately continuous mapping. Note that f can be identified with the mapping $F : X \longrightarrow \mathcal{C}(T \cup \{\infty\})$ sending $x \in X$ into the fiber $f_x : T \cup \{\infty\} \longrightarrow \mathbb{R}$. The mapping F is continuous when we take $\mathcal{C}(T \cup \{\infty\})$ equipped with the topology τ_{p} of pointwise convergence on $T \cup \{\infty\}$. The conclusion that there is a comeager subset G of X such that f is continuous at every point of the product $G \times (T \cup \{\infty\})$ is equivalent to the existence of a $G \subseteq X$ such that F viewed as a mapping from X into $\mathcal{C}(T \cup \{\infty\})$ with the norm topology is continuous at every point of G. To obtain such a G, it suffices to show that for every $\varepsilon > 0$ every nonempty open set $U \subseteq X$ contains a nonempty open subset V such that the image $F[U]$ has norm-diameter $\leq \varepsilon$. Working towards a contradiction, let us assume that for some U and ε such a $V \subseteq X$ cannot be found. Replacing X by U, we may assume that $U = X$. Since $T \cup \{\infty\}$ is a scattered space, standard arguments (see [3], [13]) show that by changing X and F we may assume that

$$F : X \longrightarrow \mathcal{C}_0(T, 2),$$

where $\mathcal{C}_0(T, 2)$ is the family of all continuous $\{0, 1\}$-valued functions on T that vanish at ∞. Thus, $\mathcal{C}_0(T, 2)$ can be identified with the family of all compact clopen subsets of T. Since every compact clopen subset of T is a finite union of intervals of the form $(s, t]$ $(s, t \in T)$. So changing X and F again, we may assume that for every $x \in X$ the image $F(x)$ is an interval of the form $(-\infty, t]$ for $t \in T$. Thus we can replace $(\mathcal{C}_0(T, 2), \tau_{\mathrm{p}})$ with (T, τ_{c}) as the range of F, where τ_{c} is the topology on T generated by subbasic open sets of the form

$$[t, \infty) = \{x \in T : t \sqsubseteq x\} \ (t \in T).$$

Our assumption about F-images of open subsets of X amount to the assumption that

$$F : X \longrightarrow (T, \tau_{\mathrm{c}})$$

is not constant on any nonempty open subset of X. Note that for every $q \in \mathbb{Q}$,

$$\{t \in T : q \in t\}$$

is a τ_c-open subset of T, so the function F viewed as a function from X into T with the separable metrizable topology τ_{m} of T induced by the Cantor cube $2^{\mathbb{Q}}$ is a Borel map. So going to a comeager subset of X, we may assume that

$$F : X \longrightarrow (T, \tau_{\mathrm{m}})$$

is continuous. Let $\beta F : \beta X \longrightarrow 2^{\mathbb{Q}}$ be its extension to the Cech-Stone compactification βX of X. Our assumption that T is a universally Baire subset of $2^{\mathbb{Q}}$ gives us that the preimage $\beta F^{-1}[T]$ has property of Baire in βX. So pick an open set U and a meager set M such that

$$\beta F^{-1}[T] = U \bigtriangleup M.$$

Note that since $X \subseteq \beta F^{-1}[T]$ is a Baire space, the open set $U \subseteq \beta X$ is not empty. Let $M = \bigcup_{n=0}^{\infty} M_n$, where each M_n is nowhere dense in βX. Using the assumption that F is nowhere constant, which amounts to the assumption that βF is nowhere constant, it is straightforward to produce a Cantor scheme U_σ ($\sigma \in 2^{<\mathbb{N}}$) of nonempty open subsets of U such that:

(i) $\overline{U}_\sigma \subseteq U \setminus M_{|\sigma|}$,

(ii) $\overline{U}_{\sigma^\frown i} \subseteq U_\sigma$ for all $i < 2$ and

(iii) $\beta F[\overline{U}_{\sigma^\frown 0}] \cap \beta F[\overline{U}_{\sigma^\frown 1}] = \emptyset$.

Let $Z = \bigcap_{n=0}^{\infty} \bigcup_{\sigma \in 2^n} \overline{U}_\sigma$. Then Z is a compact subset of $U \setminus M$, so the image $P = \beta F[Z]$ is a compact subset of T of size continuum. This contradicts the fact that an A-subtree T of w$\mathbb{Q}$, viewed as a subspace of the Cantor cube $2^{\mathbb{Q}}$, has universal measure 0 (see [19]). This finishes the proof. □

Theorem 12. *Suppose T is a universally Baire A-subtree of* w$\mathbb{Q}$ *which admits no strictly increasing mapping into the rationals. Then the one-point compactification of T has the Namioka property but the weak topology of $\mathcal{C}_0(T)$ is not σ-fragmented* [8] *by the norm.*

P r o o f. Note that if $(\mathcal{C}_0(T), \tau_{\mathrm{p}})$ is σ-fragmented by the norm, then its subspace $(\mathcal{C}_0(T, 2)$ is σ-scattered and so in particular (T, τ_{c}) is σ-scattered. So the theorem will be proved once we show that (T, τ_{c}) is not σ-scattered. To see this, call a subset X of T *special* if it is a countable union of antichains of T. By our assumption, T is not special. So in order to show that (T, τ_{c}) is not a σ-scattered space, it suffices to show that for every nonspecial subset X of T and every well-ordering $<_{\mathrm{w}}$ on X there is $t \in X$ and a sequence $(x_n) \subseteq X$ converging to t relative to the topology τ_{c} such that $t <_{\mathrm{w}} x_n$ for all n. Otherwise, for every $t \in X$ there will be a finite set $F_t \subseteq \mathbb{Q} \cap [\sup(t), \infty)$ such that $\min(x \setminus t) \in F_t$ for every $x \in X$ such that $t \sqsubseteq x$ and $t <_{\mathrm{w}} x$. Find a nonspecial $Y \subseteq X$ and a finite $F \subseteq \mathbb{Q}$ such that $F_t = F$ for all $t \in Y$. Since Y is nonspecial, it must contain an infinite sequence (t_n) such that $t_n \sqsubset t_m$ whenever $n < m$. Since $<_{\mathrm{w}}$ is a well-ordering, there must exist $n < m$ such that $t_n <_{\mathrm{w}} t_m$. By the choice of $F_{t_n} = F$, we conclude that

[8] Recall that a topological space (X, τ) is *σ-fragmented* by a metric ρ on X if for every $\varepsilon > 0$ there is a decomposition $X = \bigcup_{n=0}^{\infty} X_n^\varepsilon$ such that for every n and $A \subseteq X_n^\varepsilon$ there is $U \in \tau$ such that $U \cap A \neq \emptyset$ and ρ-diam$(U \cap A) < \varepsilon$.

$\min(t_m \setminus t_n) \in F$, contradicting the fact that F_{t_m} is also equal to F. This completes the proof. □

The following result follows from Theorem 12 and the results of Section 1.

Theorem 13. *Under the assumption of Theorem 12, there is a scattered compact subset K of the first Baire class satisfying the Namioka property though the weak topology of its function space $\mathcal{C}(K)$ is not σ-fragmented by the norm, and so in particular, $\mathcal{C}(K)$ admits no locally uniformly convex renorming.* □

Remark 14.

1. Recall that in the context of function spaces $\mathcal{C}_0(T)$ over a tree T, the norm σ-fragmentability of the weak topology of $\mathcal{C}_0(T)$ is equivalent to the existence of an equivalent norm on $\mathcal{C}_0(T)$ on whose unit sphere the weak and the norm topologies coincide (see [3]).
2. Regarding Theorem 13, we should also note that the existence of a scattered compactum K with the Namioka property such that $\mathcal{C}(K)$ is not σ-fragmented by the norm was first established by Namioka and Pol [13] assuming the existence of a co-analytic set of reals of cardinality continuum containing no perfect subset.

4. *The hypothesis of Theorem 12*

We finish the paper with comments about the consistency of assumption of Theorem 12. First of all, we should mention that the hypothesis of Theorem 12 is satisfied in the constructible universe. However, it's consistency is considerably easier to show using the following construction based on the ideas from [18]. We start by fixing a C-sequence C_α $(\alpha < \omega_1)$ such that $C_{\alpha+1} = \{\alpha\}$ while for a limit ordinal α, C_α is a set of ordinals $< \alpha$ such that $\mathrm{otp}(C_\alpha) = \omega$ and $\sup(C_\alpha) = \alpha$. Let $C_\alpha(0) = 0$ and for $0 < n < \omega$, let $C_\alpha(n))$ denote the n'th element of C_α according to its increasing enumeration with the convention that $C_{\alpha+1}(n) = \alpha$ for all $n > 0$. From C_α $(\alpha < \omega_1)$ one easily obtains a sequence $e_\alpha : \alpha \to \omega$ $(\alpha < \omega_1)$ of one-to-one mappings which is *coherent* in the sense that

$$\{\xi < \min\{\alpha, \beta\} : e_\alpha(\xi) \neq e_\beta(\xi)\}$$

is finite for all α and β (see [18]) though the reader can take this simply as an additional parameter in the definition of the functor $r \mapsto T(\rho_1^r)$ that we are now going to give. For each $r \in ([\omega]^{<\omega})^{\omega}$, we associate another sequence C^r_α $(\alpha < \omega_1)$ by letting $C^r_\alpha = \bigcup_{n\in\omega} D^r_\alpha(n)$, where

$$D^r_\alpha(n) = \{\xi \in [C_\alpha(n), C_\alpha(n+1)) : e_\alpha(\xi) \in r(n)\}.$$

(This definition really takes place only when α is a limit ordinal; for successor ordinals we put $C^r_{\alpha+1} = \{\alpha\}$.) Note that in general C^r_α is a subset of α of order type $\leq \omega$ which in general may not be unbounded in α if α is a limit ordinal. However, for every $r \in ([\omega]^{<\omega})^{\omega}$ for which C^r_α is unbounded in α for every limit ordinal α, using C^r_α $(\alpha < \omega_1)$, we can recursively define $\rho^r_1 : [\omega_1]^2 \longrightarrow \omega$ as follows (see [18]):

$$\rho^r_1(\alpha, \beta) = \max\{\rho^r_1(\alpha, \min(C^r_\beta \setminus \alpha)), |C^r_\beta \cap \alpha|\}.$$

Thus, given two countable ordinals $\alpha < \beta$, the integer $\rho^r_1(\alpha, \beta)$ is simply the maximal of the 'weights' $|C^r_{\beta_i} \cap \alpha|$ of the 'minimal walk' $\beta = \beta_0 > \beta_1 > ... > \beta_k = \alpha$ from β down to α along the C-sequence C^r_α $(\alpha < \omega_1)$ determined by the condition that $\beta_{i+1} = \min(C^r_{\beta_i} \setminus \alpha)$ for every $i < k$. It is known (see [18]) that the corresponding fiber mappings

$$(\rho^r_1)_\beta : \beta \longrightarrow \omega \ (\beta < \omega_1)^9$$

are all finite-to-one maps satisfying the coherence property saying that

$$\{\xi < \min\{\alpha, \beta\} : \rho^r_1(\xi, \alpha) \neq \rho^r_1(\xi, \beta)\}$$

is finite for all α and β. It follows that the corresponding tree

$$T(\rho^r_1) = \{(\rho^r_1)_\beta \restriction \alpha : \alpha \leq \beta < \omega_1\}$$

is an Aronszajn-tree that admits a strictly increasing mapping into the real line. Hence, as we have seen it above, the tree $T(\rho^r_1)$ is isomorphic to a downwards closed subtree of w$\mathbb{Q}$. Using the corresponding arguments from Section 6 of [18], we shall show the following property of the functor $r \mapsto T(\rho^r_1)$.

Theorem 15. *If r is a Cohen real, then for every antichain $A \subseteq T(\rho^r_1)$ there is a closed and unbounded set $\Gamma \subseteq \omega_1$ such that* length$(t) \notin \Gamma$ *for all $t \in A$.*

[9]Defined by, $(\rho^r_1)_\beta(\alpha) = \rho^r_1(\alpha, \beta)$.

P r o o f. First of all, let us define what we mean by a 'Cohen real' refereing the reader to some of the sources like [7] for more details if necessary. To express this, we consider the family $[\omega]^{<\omega}$ of finite subsets of ω equipped with the discrete topology and its power $([\omega]^{<\omega})^\omega$ with the corresponding product topology. Thus $([\omega]^{<\omega})^\omega$ is just another topological copy of the Baire space ω^ω. A real $r \in ([\omega]^{<\omega})^\omega$ is Cohen over a given universe of sets (typically the one in which we put ourselves) if r belongs to every dense-open subset of $([\omega]^{<\omega})^\omega$ belonging to the universe. Thus in particular, if r is a Cohen real then C^r_α is unbounded in α for every limit ordinal α, so the corresponding tree $T(\rho^r_1)$ is well-defined. What we need to show is that for every subset A of $T(\rho^r_1)$ for which the set $\{\mathrm{lenght}(t) : t \in A\}$ is stationary[10] contains two comparable nodes. Since nodes of the tree $T(\rho^r_1)$ are pairwise coherent this amounts to showing that for every stationary $\Gamma \subseteq \omega_1$ there exists $\gamma, \delta \in \Gamma$ such that $(\rho^r_1)_\gamma \sqsubset (\rho^r_1)_\delta$. This in turn amounts to showing that for every stationary $\Gamma \subseteq \omega_1$, the set

$$G = \{x \in ([\omega]^{<\omega})^\omega : (\exists \gamma, \delta \in \Gamma)\ (\rho^x_1)_\gamma \sqsubset (\rho^x_1)_\delta\}$$

is an subset of $([\omega]^{<\omega})^\omega$ that is comeager relative to the set G_{ω_1} of all $x \in ([\omega]^{<\omega})^\omega$ for which C^x_α is unbounded in α for every limit ordinal α. So, given a finite partial function p from ω into $[\omega]^{<\omega}$, it is sufficient to find a finite extension q of p such that the basic open subset of $([\omega]^{<\omega})^\omega$ determined by q is included in G modulo the set G_{ω_1}. Let n be the minimal integer that is bigger than all integers appearing in the domain of p or any set of the form $p(j)$ for $j \in \mathrm{dom}(p)$. For $\gamma \in \Gamma$, set

$$F_n(\gamma) = \{\xi < \gamma : e_\gamma(\xi) \leq n\}.$$

Applying the Pressing Down Lemma, we obtain a finite set $F \subseteq \omega_1$ and a stationary set $\Delta \subseteq \Gamma$ such that $F_n(\gamma) = F$ for all $\gamma \in \Delta$ and such that if $\alpha = \max(F) + 1$ then $e_\gamma \restriction \alpha = e_\delta \restriction \alpha$ for all $\gamma, \delta \in \Delta$. A similar application of the Pressing Down Lemma will give us an integer $m > n$ and two ordinals $\gamma < \delta$ in Δ such that $C_\gamma(j) = C_\delta(j)$ for all $j \leq m$, $C_\delta(m+1) > \gamma$, and

$$e_\gamma \restriction (C_\gamma(m) + 1) = e_\delta \restriction (C_\gamma(m) + 1).$$

Consider the graph $\mathcal{H}$ on the vertex-set ω where two different integers i and j are connected by an edge if we can find $\xi < \gamma$ such that $i = e_\gamma(\xi)$ and

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[10] Recall that a subset Γ of ω_1 is *stationary* if it intersects every closed and unbounded subset of ω_1.

$j = e_\delta(\xi)$, or vice versa $i = e_\delta(\xi)$ and $j = e_\gamma(\xi)$. Since the mappings e_γ and e_δ are one-to-one and have only finite disagreement on ordinals $< \gamma$, one easily shows that the maximal proper $\mathcal{H}$-path

$$P = \{i_0, i_2, ..., i_k\}$$

that starts from $i_0 = e_\delta(\gamma)$ is finite (and in fact included in the e_δ-image of the finite set $\{\xi < \gamma : e_\gamma(\xi) \neq e_\delta(\xi)\} \cup \{\gamma\})$. The maximality of P means that i_k is not in the range of e_γ. This and the fact that P is an $\mathcal{H}$-path gives that

$$e_\gamma^{-1}(P) = e_\gamma^{-1}(P \setminus \{i_k\}),$$

and that if we let $D = e_\gamma^{-1}(P)$, then $e_\delta^{-1}(P) = D \cup \{\gamma\}$. Let $l > m$ be an integer such that $D \subseteq C_\gamma(l+1)$. Extend the partial function p to a partial function q with domain $\{0, 1, ..., l\}$ such that:

(1) $q(j) = P$ for all j such that $m \leq j \leq l$, and

(2) $q(j) = \emptyset$ for any $j < m$ not belonging to $\mathrm{dom}(p)$.

Choose any $x \in ([\omega]^{<\omega})^\omega$ extending the partial mapping q and having the property that C^x_β is unbounded in β for every limit ordinal β. Then from the choices we made above about the objects n, Δ, F,m, γ, δ and q, we conclude that

(3) $\gamma \in C^x_\delta$, and

(4) $C^x_\delta \cap \gamma$ is an initial segment of C^x_γ.

It follows that, given a $\xi < \gamma$, the walk from δ to ξ along the C-sequence C^x_β $(\beta < \omega_1)$ either leads to the same finite string of the corresponding weights as the walk from γ to ξ, or else it starts with the first step equal to γ and then follows the walk from γ to ξ. Since $\rho^x_1(\xi, \delta)$ and $\rho^x_1(\xi, \gamma)$ are by definitions maximums of these two strings of weights, we conclude that $\rho^x_1(\xi, \delta) \geq \rho^x_1(\xi, \gamma)$. On the other hand, note that by (4), in the second case, the weight $|C^x_\delta \cap \xi|$ of the first step from δ to ξ is less than or equal to the weight $|C^x_\gamma \cap \xi|$ of the first step from γ to ξ. It follows that we have also the other inequality $\rho^x_1(\xi, \delta) \leq \rho^x_1(\xi, \gamma)$. Hence we have shown that

$$(\forall \beta < \gamma)\ \rho^x_1(\beta, \gamma) = \rho^x_1(\beta, \delta),$$

or in other words, that $(\rho^x_1)_\gamma \sqsubset (\rho^x_1)_\delta$. This finishes the proof. □

It follows that if r is a Cohen real, then $T(\rho_1^r)$ admits no strictly increasing mapping into the rationals. This gives us one part of the hypothesis of Theorem 12. The other part is obtained assuming that $\mathfrak{p} > \omega_1$ and the existence of an uncountable co-analytic set of reals without a perfect subset. (Recall that $\mathfrak{p}$ is the minimal cardinality of a centered family $\mathcal{F}$ of infinite subsets of $\mathbb{N}$ for which one cannot find an infinite set $M \subseteq \mathbb{N}$ such that $M \setminus N$ is finite for all $N \in \mathcal{F}$.) Note that these two assumptions are preserved when a single Cohen real is added [15]. Recall also that these two assumptions imply that every set of reals of size at most $\aleph_1$ is co-analytic and therefore that every A-subtree of w$\mathbb{Q}$ is co-analytic (see [10]). Finally, note that co-analytic sets of reals are universally Baire. Thus, we have established the following

Theorem 16. *If there is an uncountable co-analytic set of reals without a perfect subset , if $\mathfrak{p} > \omega_1$, and if r is a Cohen real, then $T(\rho_1^r)$ is a universally Baire A-subtree of* w$\mathbb{Q}$ *that admits no strictly increasing mapping into the rationals, and therefore its one-point compactification has the Namioka property and is homeomorphic to a compact subset of the first Baire class though the corresponding function space $\mathcal{C}_0(T(\rho_1^r))$ cannot be renormed by an equivalent locally uniformly convex norm.*

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