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ON A MODEL EQUATION THAT REFLECTS SOME OF THE SHEAR FLOW HYDRODYNAMIC STABILITY PROPERTIES[1]

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A b s t r a c t. A model equation is proposed in the paper that mimics some of the shear flow hydrodynamic stability properties. It contains the basic velocity profile, which can be arbitrarily chosen, and a nonlinear term, whose form can be appropriately adjusted to any particular problem. Full linear and weakly nonlinear theories for the Bickley jet velocity profile are elaborated. The solution of the linear problem is obtained in terms of associated Legendre functions. Within the weakly nonlinear theory a Landau equation is derived that describes the evolution of the perturbations near the critical wave number. The conditions for supercritical stability and subcritical instability are revealed.

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1. *Introduction*

It is well known that two distinct regimes of flow exist in fluid dynamics. They are laminar and turbulent flows. In laminar flow regime trajectories of individual fluid particles are smoothly adjusted to the flow boundaries, and are almost parallel to them. Also, laminar flow can become steady under some conditions. In contrast, turbulent flow is always unsteady, and the trajectories of fluid particles, in addition to some mean motion, which is accommodated with the form of fluid boundaries, are accompanied with intensive irregular fluctuations in the direction perpendicular to the mean flow. These fluctuations make the turbulent flow chaotic. The differences between laminar and turbulent regimes of flow have far reaching consequences upon the pressure drop in channels and pipes, the drag of bodies moving through fluid, heat exchange between the fluid and the boundaries, etc. The transition from laminar to turbulent flow usually takes place in the same flow field, and is seldom sudden, in the sense that a clear boundary between the two regimes can be drawn. Rather, the transition occurs by passing through several different phases, so that the mathematical description of each of them is very important for the understanding of turbulence.

Hydrodynamic stability theory deals with the laminar-turbulent transition in various fluid flows. The first step in this theory is usually a linear theory. Then, it is followed by the so-called weakly nonlinear theory, and this theory is then followed by the secondary stability theory, etc. It turns out within this theory that there is no unique route to turbulence. Namely, a system may pass to turbulence in many different ways depending on the value of some governing parameters (the feature of inhomogeneity!). Also, very different systems may experience the transition in exactly the same way (the feature of universality!). In any case the problems of the hydrodynamic stability theory are delicate and perplex, and they are algebraically tedious and involved. Very often the important physics of the problem is hidden behind the complex mathematical operations. That is why there exist in the literature several so-called model equations that are not physically related directly to any problem of fluid mechanics, but mimic very accurately hydrodynamic stability properties of various flows. These properties can be revealed by much simpler mathematical methods, so that these equations serve as a very good means for the demonstration of the capabilities of the theory. Usually they bear the name of their authors, like Eckhaus equation [1], Segel equation [2], Swift-Hohenberg equation [3], Proudman-Johnson equation [4], Matkowski equation [5], and others. For example Matkowski equation reads

$$u_t - \sin u = \frac{1}{R}\, u_{xx}, \quad t \geq 0,\ 0 \leq x \leq \pi$$

As a rule, in all of these equations the stability of some trivial solution is investigated ($u = 0$, in the case of Matkowski equation), and there is no possibility to chose another, nontrivial solution. In order to overcome this shortcoming of all the existing model equations, we propose in this paper an equation, which contains the basic velocity profile whose stability properties are investigated, and which can be arbitrarily chosen. It reads:

$$\begin{aligned} &u_t - u_0\, f(u - u_0) = \frac{1}{R}\left(u_{xx} + u_{yy} - u_0''\right), \\ &t \geq 0,\ |x| < \infty,\ a \leq y \leq b,\ u_0 = u_0(y),\ f(0) = 0,\ R > 0. \end{aligned} \tag{1}$$

As usual, t is time, x and y are Cartesian coordinates, and u is the velocity in the direction of x. The basic velocity profile in Eq. (1) is denoted by $u_0(y)$, and it is obviously a nontrivial solution of the equation, provided $f(0) = 0$. It represents a parallel shear flow. At that the constants a and b can be finite or infinite, i.e., the flow can be unbounded from both sides, bounded from one side, or bounded from both sides. R is the positive parameter that plays the role of a Reynolds number, and f is an arbitrary function, supposedly odd, which in general makes the Eq. (1) nonlinear, and which allows the following Taylor series representation:

$$f = f'(0)\,(u - u_0) + \frac{f'''(0)}{3!}\,(u - u_0)^3 + \ldots \tag{2}$$

Thus, not only the basic velocity profile can be arbitrarily chosen, but also the form of the nonlinear term in Eq. (1) can be conveniently adjusted to the considered problem by the choice of the coefficients $f'(0)$, $f'''(0)$,...

It will be shown in the paper that the Eq. (1) is particularly suited for studying linear and nonlinear stability properties of free and bounded shear flows. Some of them, which play a very important role in fluid mechanics, are shown in Fig. 1. They are: mixing layer, jet, wake, boundary layer, and channel flow. Both, linear and weakly nonlinear stability theories for these flows are elaborated and presented. It is shown within the linear theory that the unbounded jet type velocity profile experiences long wave instability, and that the eigenfunctions are expressed in terms of associated Legendre functions. Within the weakly nonlinear theory neutral eigen mode is perturbed by introduction of some slowly varying independent variables, and a

Landau type equation is derived, which describes the long time evolution of this mode. The conditions for the appearance of supercritical stability and subcritical instability are illuminated.

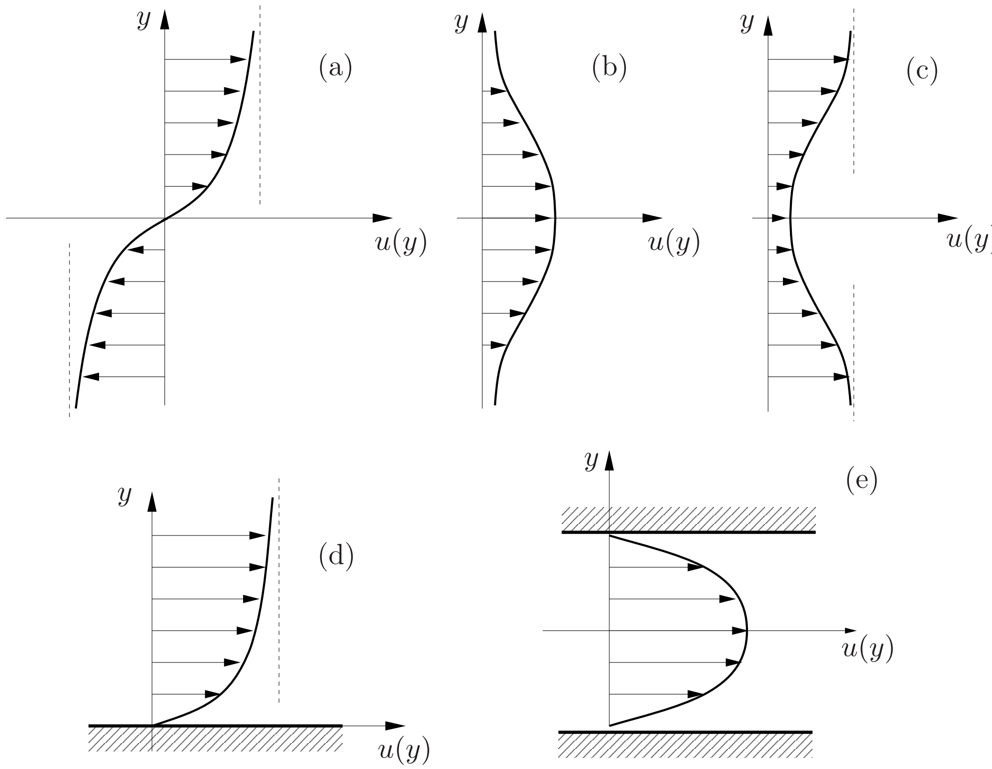

Figure 1. Various characteristic shear flow velocity profiles: (a) mixing layer, (b) jet, (c) wake, (d) boundary layer, and (e) channel or pipe flow

2. *Linear theory*

Within the linear theory we linearize Eq. (1) by taking the first term in the expansion (2) only, and present the solution of (1) in the form

$$u = u_0(y) + \hat{u}(t,\, x\, y),$$

where $\hat{u}(t,\, x\, y)$ is the small perturbation of the basic velocity profile $u_0(y)$. The equation to be satisfied by this perturbation reads

$$\hat{u}_t - u_0\, f'(0)\hat{u} = \frac{1}{R}\,\left(\hat{u}_{xx} + \hat{u}_{yy}\right). \tag{3}$$

The solution of Eq. (3) that suffices for our purposes is sought in the form of a single Fourier component (normal mode approach [6])

$$\hat{u} = \mathrm{Re}\, A\, U(y) \exp\left[i\alpha(x - ct)\right], \tag{4}$$

in which A is an arbitrary complex constant, $U(y)$ is the complex amplitude of the wave, α is the real and positive wave number ($\alpha = 2\pi/\lambda$ where λ is the wave length), and $c = c_r + ic_i$ is the complex speed of the wave. Obviously, c_r is the speed of the wave, and $\alpha\, c_i$ is its growth rate. $c_i > 0$ implies instability, $c_i < 0$ implies stability, while $c_i = 0$ implies neutral stability. Inserting (4) into Eq. (3) we obtain

$$U'' + \left[i\,\alpha\, cR - \alpha^2 + R\, f'(0) u_0\right] U = 0. \tag{5}$$

Together with vanishing boundary conditions on the boundaries: $U(a) = U(b) = 0$, Eq. (5) represents the classical Sturm-Liouville eigenvalue problem, which in addition to the eigenfunction offers also an eigenvalue relation of the form: $c = c(\alpha; R)$. Before we present a concrete solution of this equation for some of the profiles shown in Fig. 1, we will derive a general statement concerned with its eigenvalues. Multiplying Eq. (5) with the complex conjugate U^*, integrating between a and b, and applying the boundary conditions, we get the following relation:

$$\int_a^b \left[i\,\alpha\, cR - \alpha^2 + R\, f'(0) u_0\right] |U|^2 \mathrm{d}y - \int_a^b |U'|^2\, \mathrm{d}y = 0,$$

from which there immediately follows that $c_r = 0$. Thus, the perturbations are stationary. Also, the growth rate of the perturbations satisfies the relation

$$\alpha c_i R \int_a^b |U|^2\, \mathrm{d}y = \int_a^b \left[R f'(0) u_0 - \alpha^2\right] |U|^2\, \mathrm{d}y - \int_a^b |U'|^2\, \mathrm{d}y,$$

and in general it can be both positive and negative. For $c_i = 0$ we may derive an expression for the critical wave number α_c that describes the neutral eigenmode

$$\alpha_c^2 \int_a^b |U|^2\, \mathrm{d}y = R\, f'(0) \int_a^b u_0 |U|^2\, \mathrm{d}y - \int_a^b |U'|^2\, \mathrm{d}y.$$

We will now state a concrete solution for the jet-type flow defined as $u_0 = U_o \mathrm{sech}^2\, y$, $|y| < \infty$, (Bickley jet), where U_0 is an arbitrary positive constant. For that purpose we will transform Eq. (5) by introducing a new independent variable: $T = \tanh y$, thus reducing the range of its variations to $|T| \leq 1$. By using some of the elementary properties of hyperbolic functions, for example that: $\operatorname{sech}^2 y \equiv S^2 = 1 - T^2$, Eq. (5) becomes:

$$\left(1 - T^2\right) \frac{\mathrm{d}^2 U}{\mathrm{d}T^2} - 2T \frac{\mathrm{d}U}{\mathrm{d}T} + \left[R U_0 f'(0) + \frac{i\,\alpha\, cR - \alpha^2}{1 - T^2} \right] U = 0, \tag{6}$$

with the boundary conditions: $U(1) = U(-1) = 0$. This is the associated Legendre equation, whose theory is well established (s. [7], [8]). The only solutions of this equation that can satisfy the prescribed boundary conditions are those for which $R\,U_0\, f'(0) = N(N+1)$, where N is a positive integer, and for which $i\,\alpha\, cR - \alpha^2 = -\mu^2$, where $\mu = n = 1, 2, \ldots N$. By this choice of the governing parameters we confine ourselves to the discrete spectrum of eigenvalues only, and to positive values of the parameter $f'(0)$. The eigenfunctions of the continuous spectrum develop singularities at the boundaries, and cannot be used. The eigenfunctions corresponding to the chosen discrete spectrum are [7]:

$$P_N^n(T) = (-1)^n (1 - T^2)^{n/2} \frac{\mathrm{d}^n P_N(T)}{\mathrm{d}T^n} = U\left[T(y)\right],$$

where $P_N = \dfrac{1}{2^N\, N!} \dfrac{\mathrm{d}^N}{\mathrm{d}T^N} (T^2 - 1)^N$ are Legendre polynomials. It is obvious that for all N, $c_r = 0$, in accordance with the previously derived general statement. For $N = 1\,(\mu = 1)$ the growth rate of the perturbations is determined by $\alpha\, c_i R = 1 - \alpha^2$, so that the critical wave number is $\alpha_c = 1$, and the eigenfunction is $U = -S$. For $N = 2$ there are two modes. For $\mu = 1$ the mode is an odd one. The eigenvalue relation and the critical wave number have the same values as in the previous case, while the eigenfunction is $U = -3ST$. For $\mu = 2$ the mode is even. The eigenvalue relation reads $\alpha\, c_i R = 4 - \alpha^2$. The critical wave number is $\alpha_c = 2$, and the corresponding eigenfunction is $U = 3S^2$. For $N = 3$ there are three modes, etc. It depends on the initial conditions which of the modes will be excited, but we will not treat that problem here.

A general dependence of the growth rate of the perturbations on the imposed wave number is sketched in Fig. 2. For $\alpha < \alpha_c$ (the dotted part of the abscissa) the flow is unstable, while for $\alpha > \alpha_c$ the flow is stable. Thus, the jet type profile in the form of a Bickley jet experiences a long wave instability $-$ exactly as in the classical hydrodynamic stability theory [6].

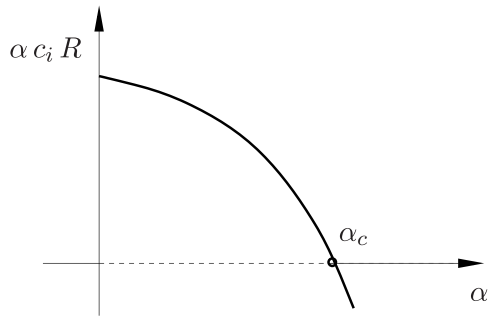

Figure 2. Stability diagram for a Bickley jet, as revealed by linear theory

3. *Weakly nonlinear theory*

Within the weakly nonlinear theory we suppose that the wave number α differs slightly from its critical value α_c, i.e., $|\alpha_c - \alpha| \ll 1$. At that, α can be both less than α_c (linearly unstable case) and greater than α_c (linearly stable case). If α is less than α_c but close to it, perturbations will grow slowly with time. During a long period of time they can become large due to accumulated effect of nonlinearity, so that the linear theory breaks. If $\alpha > \alpha_c$ perturbations decay with time very slowly, so that again nonlinearity may come into play and disrupt the basic result of the linear theory. In both cases a new theory is necessary for the description of long time evolution of such disturbances, and this is the problem the weakly nonlinear theory is dealt with.

In order to develop this theory for the jet type velocity profile elaborated earlier, we will now introduce a small parameter ε defined as the ratio between the maximum amplitude of the perturbations and the thickness of the velocity profile, and suppose for convenience that $|\alpha_c - \alpha| = O(\varepsilon^2)$. In order to make this assumption explicit we will introduce a slow coordinate $\xi = \varepsilon^2 x$ and a slow time $\tau = \varepsilon^2 t$. This form of the slow variables will be justified later. Now we will formally seek the solution of Eq. (1) with the nonlinear term (2) in the form of the following asymptotic series

$$u = u_0(y) + \varepsilon\, u_1(t,\, x,\, \tau,\, \xi,\, y) + \varepsilon^2\, u_2(t,\, x,\, \tau,\, \xi,\, y) + \ldots$$

By inserting this series into Eq. (1) and equating the terms with the same power of ε we obtain a system of recursive partial differential equation for the functions u_1, u_2, ... The equation for u_1 reads:

$$u_{1yy} + u_{1xx} + R\, f'(0)\, u_0(y)\, u_1 = 0. \tag{7}$$

For the Bickley jet $u_0 = U_0 S^2$, and for the mode $N = 1$: $R\, f'(0)\, U_0 = 2$, and $\alpha_c = 1$. Having in mind the result of the linear theory that for $\alpha = \alpha_c$, $c_r = c_i = 0$, the solution of Eq. (7) will now have the form:

$$u_1 = \mathrm{Re}\, A(\tau,\, \xi)\, U_1(y)\, \exp(ix), \tag{8}$$

with: $U_1 = S$. The difference between (8) and (4) is that A is not a constant any more, but a function of slow variables τ and ξ. In what follows we will derive an equation for the amplitude $A(\tau, \xi)$.

At the second order we obtain the following equation for u_2:

$$u_{2yy} + u_{2xx} + R\, f'(0)\, u_0\, u_2 = 0$$

The equation is homogeneous, and there is no interaction between its solution and the first harmonic- fundamental (8). Thus, its solution is of no interest for our purposes. At the next order we get

$$u_{3yy} + u_{3xx} + R\, f'(0)\, u_0\, u_3 = R\left(u_{1\tau} - \frac{f'''(0)}{3!}\, u_0\, u_1^3\right) - 2u_{1x\xi} \tag{9}$$

If evaluated by using (8), the right hand side (r.h.s.) of this equation reads

$$\begin{aligned} \text{r.h.s.} = \quad & \mathrm{Re}\left[(R\, A_\tau - 2\, i\, A_\xi)\; U_1 - R\tfrac{f'''(0)}{8}\, |A|^2\, A\, u_0\, U_1^3\right] \exp(ix) \\ & -R\tfrac{f'''(0)}{24}\, u_0\, \mathrm{Re}\, A^3\, U_1^3\, \exp(3ix), \end{aligned}$$

so that the structure of its solution is to be

$$\begin{aligned} u_3 = \quad & \mathrm{Re}\left[K_1(\tau,\, \xi)\, U_3^{(1)}(y) + K_2(\tau,\, \xi)\, U_3^{(2)}(y) + U_3^{(p)}(\tau,\, \xi,\, y)\right] \exp(ix) \\ & +3^{\mathrm{rd}} \text{ harmonic term.} \end{aligned}$$

Here, $U_3^{(1)}(y)$ and $U_3^{(2)}(y)$ represent the two linearly independent solutions of the homogeneous part of Eq. (9), $K_1(\tau,\, \xi)$ and $K_2(\tau,\, \xi)$ are the "constants" of integration, and $U_3^{(p)}$ is the particular integral. As indicated earlier, we are primarily interested in the derivation of an equation for $A(\tau,\, \xi)$, rather than in the evaluation of the small corrections u_2, u_3, ...

to u_1. This equation is obtained as a solvability condition for the equation for $U_3^{(p)}$:

$$L\left[U_3^{(p)}\right] = (R\,A_\tau - 2\,i\,A_\xi)\;S - \frac{f'''(0)}{4\,f'(0)}\,|A|^2\,A\,S^5,$$

where L is a self-adjoint operator

$$L = \frac{\mathrm{d}^2}{\mathrm{d}y^2} + (2S^2 - 1).$$

The solvability condition for this equation is obtained as [9]:

$$(R\,A_\tau - 2\,i\,A_\xi)\int\limits_{-\infty}^{\infty} S^2\,\mathrm{d}y - \frac{f'''(0)}{4\,f'(0)}\,|A|^2\,A\int\limits_{-\infty}^{\infty} S^6\,\mathrm{d}y = \left(S\,\frac{\mathrm{d}U_3^{(p)}}{\mathrm{d}y}\right)_{-\infty}^{\infty} + \left(S\,T\,U_3^{(p)}\right)_{-\infty}^{\infty},$$

or, after applying the boundary conditions: $U_3^{(p)}(\infty) = U_3^{(p)}(-\infty) = 0$, and evaluating the necessary integrals:

$$R\,A_\tau - 2\,i\,A_\xi - \frac{2f'''(0)}{15f'(0)}\,|A|^2\,A = 0. \tag{10}$$

This is the desired evolution equation. In view of the statement of this problem, as given at the beginning of this Section, we will use the following form of its solution

$$A = |A|\,\exp(-i\,\Delta\alpha\,\xi),$$

where $\Delta\alpha = \alpha_c - \alpha = 1 - \alpha$, and $|A| = f(\tau)$. For convenience we will introduce a simple transformation of variables: $\tilde{\tau} = \dfrac{2\tau}{R}$, $|A| = \sqrt{15}\,B(\tilde{\tau})$ and get

$$\dot{B} = \Delta\alpha\,B + \frac{f'''(0)}{f'(0)}\,B^3, \tag{11}$$

with an arbitrary initial condition: $B(0) = B_0 > 0$.

The Eq. (11) is called the Landau equation in the literature, and its theory is well established [4]. For $\Delta\alpha \neq 0$ the solution is

$$B^2 = \frac{\Delta\alpha\,B_0^2}{\left[\Delta\alpha + \dfrac{f'''(0)}{f'(0)}\,B_0^2\right]\,\exp(-2\,\Delta\alpha\,\tilde{\tau}) - \dfrac{f'''(0)}{f'(0)}\,B_0^2}, \tag{12}$$

while for $\Delta\alpha = 0$ it reads

$$B^2 = \frac{B_0^2}{1 - 2\dfrac{f'''(0)}{f'(0)}\,B_0^2\tilde{\tau}} \tag{13}$$

Qualitative behavior of the solution is shown in Fig. 3, in which arrows symbolically designate the direction in which the amplitude advances with time. It is seen that it crucially depends on the sign of $f'''(0)$.

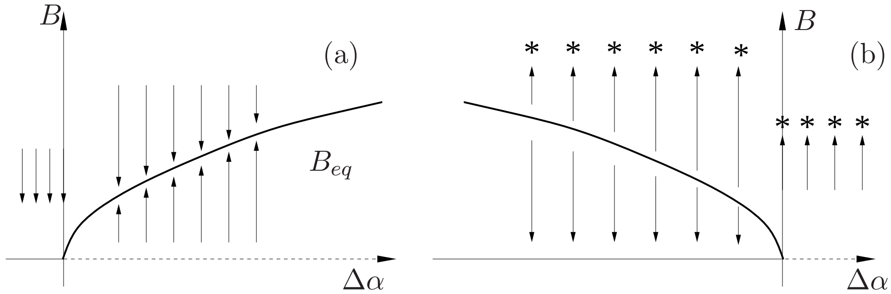

Figure 3. Pitchfork bifurcation for a Bickley jet, as revealed by the weakly nonlinear theory

For $f'''(0) < 0$ (s. Fig. 3a) and $\Delta\alpha > 0$ (linearly unstable case) the amplitude of the perturbations tends to an equilibrium value:

$$B_{eq}^2 = -\frac{\Delta\alpha\, f'(0)}{f'''(0)}.$$

Thus, if initially $B < B_{eq}$ perturbations increase with time, but this increase is not infinite. If initially $B > B_{eq}$, then B decreases with time, again tending to reach the equilibrium (saturated!) value of B_{eq}. Such a flow is called *supercritically stable*, and the effect of nonlinearity is obviously stabilizing in this case. For $\Delta\alpha \leq 0$ (linearly stable case) the perturbations tend to disappear independently of the initial value.

For $f''' > 0$ (s. Fig. 3b) and $\Delta\alpha > 0$ there is no stabilizing effect of nonlinearity - perturbations increase with time for any initial B_0. For $\Delta\alpha < 0$, however, they diminish in accordance with the linear theory if

$$B_0^2 < -\frac{\Delta\alpha\, f'(0)}{f'''(0)}$$

only, and increase if

$$B_0^2 > -\frac{\Delta\alpha\, f'(0)}{f'''(0)}.$$

In such a case the flow is said to be subcritically unstable. The "stars" above the arrows in Fig. 3b indicate that B becomes infinite in finite time, which can be readily revealed from the solutions (12) and (13). Such a singularity of the solution is called the *blow-up*. Of course, when B attains large values, weakly nonlinear theory ceases to be applicable, and another, fully nonlinear theory should be employed. Anyhow, the existence of such a singularity indicates a relatively fast transition to turbulence. As well known [4] the qualitative change in the behavior of the solution of a differential equation when a parameter passes through a characteristic value is called bifurcation. The bifurcation of the solution of Eq. (1) shown in Fig. 3 is known as the pitchfork bifurcation.

4. *Conclusion*

The model equation proposed in this paper is not physically based on the hydrodynamic stability theory. However, it contains an arbitrary basic velocity profile, whose stability properties are investigated. Also, some parameters in the nonlinear term of the equation can be conveniently chosen so as to demonstrate some of the features of the stability properties of shear flows, for which the equation is particularly suitable. It must be admitted, however, that not all of these properties are reproducible. For example, the role of the inflection point on the profile, the existence of the critical layers, the mean flow correction, etc. [6] are missing. Still, we think that the equation can serve as an effective mean for a relatively simple demonstration of the mathematical treatment of perplex linear and weakly nonlinear hydrodynamic stability theories.

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