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REGULAR NONLINEAR GENERALIZED FUNCTIONS AND APPLICATIONS[1]

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A b s t r a c t. We present new types of regularity for Colombeau nonlinear generalized functions, based on the notion of regular growth with respect to the regularizing parameter of the simplified model. This generalizes the notion of $\mathcal{G}^\infty$-regularity introduced by M. Oberguggenberger. As a first application, we show that these new spaces are useful in a problem of representation of linear maps by integral operators, giving an analogon to Schwartz kernel theorem in the framework of nonlinear generalized functions. Secondly, we remark that these new regularities can be characterized, for compactly supported generalized functions, by a property of their Fourier transform. This opens the door to microlocal analysis of singularities of generalized functions, with respect to these regularities.

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1. *Introduction*

The various theories of nonlinear generalized functions are suitable frameworks to set and solve mathematical problems with irregular operators or data. We are going to follow the theory introduced by J.-F. Colombeau [2, 3, 8, 14]. Throughout the paper, Ω will denote an open subset of $\mathbb{R}^d$, $d \in \mathbb{N}$. The Colombeau simplified algebra $\mathcal{G}(\Omega)$ is the space $\mathcal{X}_M(\Omega)/\mathcal{N}(\Omega)$ with

$$\mathcal{X}_M(\Omega) = \{(f_\varepsilon) \in \mathcal{E}(\Omega)\ \forall K \Subset \Omega,\ \forall l \in \mathbb{N},\ \exists N = \mathbf{N}(K,l) \in \mathbb{N}$$
$$p_{K,l}(f_\varepsilon) = \mathrm{O}\left(\varepsilon^{-\mathbf{N}(\mathrm{K},\mathrm{l})}\right)\Big\},$$
$$\mathcal{N}(\Omega) = \left\{(f_\varepsilon) \in \mathcal{E}(\Omega)\ \middle|\ \forall K \Subset \Omega,\ \forall (l,m) \in \mathbb{N}^2 \quad p_{K,l}(f_\varepsilon) = \mathrm{O}(\varepsilon^{\mathrm{m}})\right\}, \quad (1)$$

where $\mathcal{E}(\Omega) = \mathrm{C}^\infty(\Omega)^{(0,1]}$, $p_{K,l}(f) = \sup_{x\in K, |\alpha| \le l} |\partial^\alpha f(x)|$ and relations $\mathrm{O}(\cdot)$ are considered for $\varepsilon \to 0$, this precision being omitted in the sequel.

Some subspaces of $\mathcal{G}(\Omega)$ have been considered in which some conditions of growth with respect to l are added for $\mathbf{N}(K,l)$. The most important one is the space $\mathcal{G}^\infty(\Omega)$, for which $\mathbf{N}(K,l)$ only depends on K [15]. This space plays a great part in the local and microlocal analysis of nonlinear generalized functions. (See [9, 10, 11, 14] among other references.)

We include $\mathcal{G}^\infty(\Omega)$ and $\mathcal{G}(\Omega)$ in a new framework of $\mathcal{R}$*-regular spaces of nonlinear generalized functions*, in which the growth bounds $\mathbf{N}(K,l)$ belong to regular spaces of sequences, that is spaces satisfying natural conditions of stability.

We illustrate the usefulness of these new spaces by two examples. First application, developed in [4], is a problem of Schwartz kernel type theorem in the framework of Colombeau generalized functions. We show that some nets of linear maps (parametrized by $\varepsilon \in (0,1]$) satisfying some growth conditions similar to those introduced for $\mathcal{R}$-regular spaces, give rise to linear maps between spaces of generalized functions. Moreover, they can be represented by integral kernel on regular subspaces of $\mathcal{G}(\Omega)$ in which the growth of $\mathbf{N}(K,l)$ is at most sublinear with respect to l. The second application, developed in [5], is the introduction of the $\mathcal{R}$-local and microlocal analysis. This is possible since the $\mathcal{R}$-regular elements with compact supports can be characterized by a "$\mathcal{R}$-property" of their Fourier transform. (This Fourier Transform belongs to some regular subspaces of spaces of *rapidly decreasing generalized functions* [6, 16].) Thus, the parallel is complete with the C^∞-regularity of compactly supported distributions and the $\mathcal{G}^\infty$-regularity [7, 11] of nonlinear generalized functions appears to be one a remarkable particular case of $\mathcal{R}$-regularity.

2. *The sheaf of Colombeau simplified algebras and $\mathcal{R}$-regular subsheaves*

Definition 1. *A non empty subspace $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$ is* regular *if*
(i) *$\mathcal{R}$ is "overstable" by translations and by maximum*

$$\forall N \in \mathcal{R},\ \forall (k,k') \in \mathbb{N}^2,\ \exists N' \in \mathcal{R},\ \forall n \in \mathbb{N} \quad N(n+k)+k' \leq N'(n), \tag{2}$$

$$\forall N_1 \in \mathcal{R},\ \ \forall N_2 \in \mathcal{R},\ \ \exists N \in \mathcal{R},\ \ \forall n \in \mathbb{N} \qquad \max(N_1(n), N_2(n)) \leq N(n), \tag{3}$$

(ii) *For all N_1 and N_2 in $\mathcal{R}$, there exists $N \in \mathcal{R}$ such that*

$$\forall (l_1, l_2) \in \mathbb{N}^2 \qquad N_1(l_1) + N_2(l_2) \leq N(l_1 + l_2). \tag{4}$$

Example 1. The set $\mathbb{R}_+^{\mathbb{N}}$ of all positive valued sequences and the set $\mathcal{B}$ of bounded sequences are regular.

Let $\mathcal{R}$ be a regular subset of $\mathbb{R}_+^{\mathbb{N}}$ and set

$$\mathcal{X}^{\mathcal{R}}(\Omega) = \left\{ (f_\varepsilon) \in \mathcal{E}(\Omega) \,\middle|\, \forall K \Subset \Omega,\ \exists N \in \mathcal{R},\ \forall l \in \mathbb{N}\ \ p_{K,l}(f_\varepsilon) = \mathrm{O}\left(\varepsilon^{-\mathrm{N(l)}}\right) \right\},$$
$$\mathcal{N}^{\mathcal{R}}(\Omega) = \left\{ (f_\varepsilon) \in \mathcal{E}(\Omega) \,\middle|\, \forall K \Subset \Omega,\ \forall m \in \mathcal{R},\ \forall l \in \mathbb{N}\ \ p_{K,l}(f_\varepsilon) = \mathrm{O}\left(\varepsilon^{\mathrm{m(l)}}\right) \right\}.$$

Proposition 1.
(i) *For all regular subspace $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$, $\mathcal{X}^{\mathcal{R}}(\cdot)$ is a sheaf of differential algebras on the ring $\mathcal{X}_M(\mathbb{C})$ with*

$$\mathcal{X}_M(\mathbb{K}) = \left\{ (r_\varepsilon) \in \mathbb{K}^{(0,1]} \mid \exists q \in \mathbb{N} \quad |r_\varepsilon| = \mathrm{O}\left(\varepsilon^{-\mathrm{q}}\right) \right\}, \quad \mathbb{K} = \mathbb{R} \text{ or } \mathbb{K} = \mathbb{C}.$$

(ii) *$\mathcal{N}^{\mathcal{R}}(\cdot)$ is equal to Colombeau's sheaf of ideal $\mathcal{N}(\cdot)$, defined by relation (1).*
(iii) *For all regular subspaces $\mathcal{R}_1$ and $\mathcal{R}_2$ of $\mathbb{R}_+^{\mathbb{N}}$, with $\mathcal{R}_1 \subset \mathcal{R}_2$, $\mathcal{X}^{\mathcal{R}_1}(\cdot)$ is a subsheaf of $\mathcal{X}^{\mathcal{R}_2}(\cdot)$.*

P r o o f. We split the proof in two parts.
(a) *Algebraical properties.* Let us first show that for any open set $\Omega \subset \mathbb{R}^d$, $\mathcal{X}^{\mathcal{R}}(\Omega)$ is a subalgebra of $\mathrm{C}^\infty(\Omega)^{(0,1]}$. Take $(f_\varepsilon)_\varepsilon$ and $(g_\varepsilon)_\varepsilon$ in $\mathcal{X}^{\mathcal{R}}(\Omega)$ and $K \Subset \Omega$. There exist $N_f \in \mathcal{R}$ and $N_g \in \mathcal{R}$ such that

$$\forall l \in \mathbb{N} \quad p_{K,l}(h_\varepsilon) = \mathrm{O}\left(\varepsilon^{-\mathrm{N_h(l)}}\right), \ \text{ for } \mathrm{h}_\varepsilon = \mathrm{f}_\varepsilon,\, \mathrm{g}_\varepsilon.$$

We get immediately that $p_{K,l}(f_\varepsilon + g_\varepsilon) = \mathrm{O}\left(\varepsilon^{-\max(\mathrm{N_f(l)},\mathrm{N_g(l)})}\right)$, with $\max(N_f, N_g) \le N$ for some $N \in \mathcal{R}$ according to (3). Then, $(f_\varepsilon + g_\varepsilon)_\varepsilon$ belongs to $\mathcal{X}^{\mathcal{R}}(\Omega)$.

For $(c_\varepsilon)_\varepsilon \in \mathcal{X}(\mathbb{C})$, there exists q_c such that $|c_\varepsilon| = \mathrm{O}(\varepsilon^{-\mathrm{q_c}})$. Then $p_{K,l}(c_\varepsilon f_\varepsilon) = \mathrm{O}\left(\varepsilon^{-\mathrm{N_f(l)}-\mathrm{q_c}}\right)$. From (2), there exists $N \in \mathcal{R}$ such that $N_f + q_c \le N$. Thus, $(c_\varepsilon f_\varepsilon)_\varepsilon \in \mathcal{X}^{\mathcal{R}}(\Omega)$. It follows that $\mathcal{X}^{\mathcal{R}}(\Omega)$ is a submodule of $\mathrm{C}^\infty(\Omega)^{(0,1]}$ over $\mathcal{X}(\mathbb{C})$.

Consider now $l \in \mathbb{N}$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| = l$. By the Leibniz rule, we have, for all $\varepsilon \in (0,1]$ and $x \in K$,

$$\begin{aligned} |\partial^\alpha (f_\varepsilon g_\varepsilon)(x)| &= \sum\nolimits_{\gamma \le \alpha} C_\alpha^\gamma \left|\partial^\gamma f_\varepsilon\ (x)\, \partial^{\alpha-\gamma} g_\varepsilon(x)\right| \\ &\le \sum\nolimits_{\gamma \le \alpha} C_\alpha^\gamma p_{K,|\gamma|}(f_\varepsilon)\, p_{K,|\alpha-\gamma|}(g_\varepsilon), \end{aligned}$$

where C_α^γ is the generalized binomial coefficient. We have, for all $\gamma \le \alpha$,

$$p_{K,|\gamma|}(f_\varepsilon)\, p_{K,|\alpha-\gamma|}(g_\varepsilon) = \mathrm{O}\left(\varepsilon^{-\mathrm{N_f}(|\gamma|)-\mathrm{Ng}(|\alpha-\gamma|)}\right).$$

As $\gamma \le \alpha$, we get $|\gamma| + |\alpha - \gamma| = |\alpha| = l$. According to (4), there exists $N \in \mathcal{R}$ such that, for all k and $k' \le k$ in $\mathbb{N}$, $N_f(k') + N_g(k-k') \le N(k)$. Then $\sup_{x\in K} |\partial^\alpha (f_\varepsilon g_\varepsilon)(x)| = \mathrm{O}\left(\varepsilon^{-\mathrm{N(l)}}\right)$.Thus $p_{K,l}(f_\varepsilon\, g_\varepsilon) = \mathrm{O}\left(\varepsilon^{-\mathrm{N(l)}}\right)$, and $(f_\varepsilon\, g_\varepsilon)_\varepsilon \in \mathcal{X}^{\mathcal{R}}(\Omega)$.

For the properties related to $\mathcal{N}^{\mathcal{R}}(\Omega)$, take $(f_\varepsilon)_\varepsilon \in \mathcal{N}^{\mathcal{R}}(\Omega)$. For any $K \Subset \Omega$, $l \in \mathbb{N}$ and $m \in \mathbb{N}$, choose $N \in \mathcal{R}$. According to (2) there exists $N' \in \mathcal{R}$ such that $N + m \le N'$. Thus, $p_{K,l}(f_\varepsilon) = \mathrm{O}\left(\varepsilon^{\mathrm{N'(l)}}\right) = \mathrm{O}(\varepsilon^{\mathrm{m}})$ and $(f_\varepsilon)_\varepsilon \in \mathcal{N}(\Omega)$. Conversely, given $(f_\varepsilon)_\varepsilon \in \mathcal{N}(\Omega)$ and $N \in \mathcal{R}$, we have $p_{K,l}(f_\varepsilon) = \mathrm{O}\left(\varepsilon^{\mathrm{N(l)}}\right)$, since this estimates holds for all $m \in \mathbb{N}$.

(b) *Sheaf properties.* The proof follows the same lines as the one of Colombeau simplified algebras. (See for example [8], theorem 1.2.4.) First, the definition of restriction (by the mean of the restriction of representatives) is straightforward as in Colombeau's case. For the sheaf properties, we have to replace Colombeau's usual estimates by $\mathcal{X}^{\mathcal{R}}$-estimates. But, at each place this happens, we have only to consider a finite number of terms, by compactness properties. Thus, the stability by maximum of $\mathcal{R}$ (property (3)) induces the result. Finally, point (iii) of the proposition follows directly from the obvious inclusion $\mathcal{X}^{\mathcal{R}_1}(\Omega) \subset \mathcal{X}^{\mathcal{R}_2}(\Omega)$.

The sheaf $\mathcal{G}^{\mathcal{R}}(\cdot) = \mathcal{X}^{\mathcal{R}}(\cdot)/\mathcal{N}^{\mathcal{R}}(\cdot) = \mathcal{X}^{\mathcal{R}}(\cdot)/\mathcal{N}(\cdot)$ turns to be a sheaf of differentiable algebras on the ring $\mathcal{X}_M(\mathbb{C})/\mathcal{N}(\mathbb{C})$ with

$$\mathcal{N}(\mathbb{K}) = \left\{ (r_\varepsilon) \in \mathbb{K}^{(0,1]} \mid \forall q \in \mathbb{N} \quad |r_\varepsilon| = \mathrm{O}(\varepsilon^q) \right\}, \quad \mathbb{K} = \mathbb{R} \text{ or } \mathbb{K} = \mathbb{C}.$$

Definition 2. *For all regular subset $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$, the sheaf of factor algebras $\mathcal{G}^{\mathcal{R}}(\cdot)$ is called the sheaf of* $\mathcal{R}$-regular algebras of nonlinear generalized functions*.*

Example 2. Taking $\mathcal{R} = \mathbb{R}_+^{\mathbb{N}}$, we recover the sheaf of *Colombeau simplified algebras*. Taking $\mathcal{R} = \mathcal{B}$, we obtain the sheaf of $\mathcal{G}^\infty$-generalized functions [15].

Notation 1. We shall write $\mathcal{G}(\Omega)$ (resp. $\mathcal{X}_M(\Omega)$) instead of $\mathcal{G}^{\mathbb{R}_+^{\mathbb{N}}}(\Omega)$ (resp. $\mathcal{X}^{\mathbb{R}_+^{\mathbb{N}}}(\Omega)$). For (f_ε) in $\mathcal{X}_M(\Omega)$ or in $\mathcal{X}^{\mathcal{R}}(\Omega)$, $[(f_\varepsilon)]$ will be its class in $\mathcal{G}(\Omega)$ or in $\mathcal{G}^{\mathcal{R}}(\Omega)$, since these classes are obtained modulo the same ideal. (We consider $\mathcal{G}^{\mathcal{R}}(\Omega)$ as a subspace of $\mathcal{G}(\Omega)$.)

As $\mathcal{G}$ is a sheaf, the notion of support of a section $f \in \mathcal{G}(\Omega)$ makes sense. Then, the *support* of a generalized function $f \in \mathcal{G}(\Omega)$ is the complement in Ω of the largest open subset of Ω where f is null. For a regular subset $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{N}}$, we denote by $\mathcal{G}_C(\Omega)$ (*resp.* $\mathcal{G}_C^{\mathcal{R}}(\Omega)$) the subset of $\mathcal{G}(\Omega)$ (*resp.* $\mathcal{G}^{\mathcal{R}}(\Omega)$) of elements with compact support. Let us remark that every $f \in \mathcal{G}_C^{\mathcal{R}}(\Omega)$ has a representative (f_ε) in $\mathcal{X}^{\mathcal{R}}(\Omega)$ such that each f_ε has the same compact support.

3. *Application 1: Schwartz Kernel type theorem*

Definition 3. *Let H be in $\mathcal{G}(\mathbb{R}^m \times \mathbb{R}^n)$. The* integral operator of kernel *H is the map $\widetilde{H}$ defined by*

$$\widetilde{H} : \mathcal{G}_C(\mathbb{R}^n) \to \mathcal{G}(\mathbb{R}^m) : f \mapsto \widetilde{H}(f) = \left[\left(x \mapsto \textstyle\int_{supp\, f} H_\varepsilon(x,y) f_\varepsilon(y)\,\mathrm{dy} \right) \right],$$

where (H_ε) (resp. (f_ε)) is any representative of H (resp. f).

In the references [1] and [6], the generalized function H satisfies some additive conditions such as being properly supported. This assumption is not needed here, since we consider operators acting on compactly supported generalized functions. For H in $\mathcal{G}(\mathbb{R}^m \times \mathbb{R}^n)$, the operator $\widetilde{H}$ defines a linear map from $\mathcal{G}_C(\mathbb{R}^n)$ to $\mathcal{G}(\mathbb{R}^m)$ continuous for the respective sharp topologies of $\mathcal{G}_C(\mathbb{R}^n)$ and $\mathcal{G}(\mathbb{R}^m)$. Moreover the map$\,\widetilde{}\,$: $H \mapsto \widetilde{H}$ from $\mathcal{G}(\mathbb{R}^m \times \mathbb{R}^n)$ to $\mathcal{L}(\mathcal{G}_C(\mathbb{R}^n), \mathcal{G}(\mathbb{R}^m))$ is injective.

Take a in $[0, +\infty]$ and set $\mathcal{L}_a = \left\{N \in \mathbb{R}_+^{\mathbb{N}} \,\middle|\, \mathrm{limsup}_{n\to+\infty}(N(l)/l) < a\right\}$. (For $a = 0$ the limsup is replaced by a simple limit which should be equal to 0.) For all a in $[0, +\infty]$, $\mathcal{L}_a$ is a regular subset of $\mathbb{R}_+^{\mathbb{N}}$. The corresponding sheaves $\mathcal{G}^{\mathcal{L}_a}(\cdot) = \mathcal{X}^{\mathcal{L}_a}(\cdot)/\mathcal{N}(\cdot)$ are the sheaves of algebras of generalized functions with slow growth introduced in [4]. The following lemma is crucial for the proof of the main result (Theorem 4) and was the initial motivation for the introduction of $\mathcal{G}^{\mathcal{L}_a}(\cdot)$ classes of spaces:

Lemma 2. *Let a be a real in $[0,1]$, d be an integer and $(\theta_\varepsilon) \in \mathcal{D}\left(\mathbb{R}^d\right)^{(0,1]}$ a net of mollifiers satisfying, for all $k \in \mathbb{N}$,*

$$\int \theta_\varepsilon(x)\,\mathrm{dx} = 1 + \mathrm{O}\left(\varepsilon^{\mathrm{k}}\right), \quad \forall \mathrm{m} \in \mathbb{N}^{\mathrm{d}} \backslash \{0\} \quad \int \mathrm{x}^{\mathrm{m}} \theta_\varepsilon(\mathrm{x})\,\mathrm{dx} = \mathrm{O}\left(\varepsilon^{\mathrm{k}}\right). \tag{5}$$

For any $(g_\varepsilon) \in \mathcal{X}^{\mathcal{L}_a}\left(\mathbb{R}^d\right)$, we have $(g_\varepsilon * \theta_\varepsilon - g_\varepsilon)_\varepsilon \in \mathcal{N}\left(\mathbb{R}^d\right)$.

We turn to the definition and the properties of the nets of linear maps used in the main theorem. Fix $(L_\varepsilon) \in (\mathcal{L}(\mathcal{D}(\mathbb{R}^n), \mathrm{C}^\infty(\mathbb{R}^{\mathrm{m}})))^{(0,1]}$.

Definition 4.
(i) We say that (L_ε) is continuously moderate *(resp.* negligible*) if*

$$\begin{gathered} \forall j \in \mathbb{N}, \quad \forall K \Subset \mathbb{R}^m, \quad \forall l \in \mathbb{N}, \quad \exists (C_\varepsilon) \in \mathcal{X}_M(\mathbb{R}_+) \ \ \textit{(resp. } \mathcal{N}(\mathbb{R}_+)\textit{)} \\ \exists l' \in \mathbb{N}, \quad \forall f \in \mathcal{D}_j(\mathbb{R}^n) \quad p_{K,l}(L_\varepsilon(f)) \le C_\varepsilon\, p_{j,l'}(f). \end{gathered} \tag{6}$$

(ii) Let (b, c) be in $(\mathbb{R}^+ \cup \{+\infty\}) \times \mathbb{R}^+$. We say that (L_ε) is $\mathcal{L}_{b,c}$-strongly continuously moderate *if: $\forall j \in \mathbb{N}$, $\forall K \Subset \mathbb{R}^m$,*

$$\begin{gathered} \exists \lambda \in \mathbb{N}^{\mathbb{N}} \textit{ with } \limsup_{l\to+\infty}(\lambda(l)/l) < b, \ \exists r \in \mathbb{R}_+^{\mathbb{N}} \textit{ with } \limsup_{l\to+\infty}(r(l)/l) < c, \\ \forall l \in \mathbb{N}, \ \exists C \in \mathbb{R}_+, \quad \forall f \in \mathcal{D}_j(\mathbb{R}^n) \quad p_{K,l}(L_\varepsilon(f)) \le C\varepsilon^{-r(l)} p_{j,\lambda(l)}(f). \end{gathered}$$

Proposition 3. *[4]*
(i) Any continuously moderate net $(L_\varepsilon) \in (\mathcal{L}(\mathcal{D}(\mathbb{R}^n), \mathrm{C}^\infty(\mathbb{R}^{\mathrm{m}})))^{(0,1]}$ can be extended in a map $L \in \mathcal{L}(\mathcal{G}_C(\mathbb{R}^n), \mathcal{G}(\mathbb{R}^m))$ defined by $L(f) = [(L_\varepsilon(f_\varepsilon))]$ where (f_ε) is any representative of f.
(ii) The extension L depends only on the family (L_ε) in the following sense: If (N_ε) is a net of negligible maps, then the extensions of (L_ε) and $(L_\varepsilon + N_\varepsilon)$

are equal.
(iii) Let (a,b,c) *be in* $(\mathbb{R}^+)^3$. *If the net* (L_ε) *is* $\mathcal{L}_{b,c}$*-strongly continuously moderate, then* $L\left(\mathcal{G}_C^{\mathcal{L}_a}(\mathbb{R}^n)\right)$ *is included in* $\mathcal{G}^{\mathcal{L}_{ab+c}}(\mathbb{R}^m)$.
Moreover, $L\left(\mathcal{G}_C^{\mathcal{L}_0}(\mathbb{R}^n)\right)$ *is included in* $\mathcal{G}^{\mathcal{L}_c}(\mathbb{R}^m)$ *even if* $b=+\infty$.

Theorem 4. *[4]* ***Schwartz kernel type theorem****. Fix* $(a,b,c)\in\mathbb{R}_+^3$ *such that* $a\leq 1$ *and* $ab+c\leq 1$. *Let* $(L_\varepsilon)\in(\mathcal{L}(\mathcal{D}(\mathbb{R}^n),\mathrm{C}^\infty(\mathbb{R}^{\mathrm{m}})))^{(0,1]}$ *be a net of* $\mathcal{L}_{b,c}$*-strongly continuously moderate linear maps and* $L\in\mathcal{L}(\mathcal{G}_C(\mathbb{R}^n),\mathcal{G}(\mathbb{R}^m))$ *its canonical extension. There exists* $H_L\in\mathcal{G}(\mathbb{R}^m\times\mathbb{R}^n)$ *such that*

$$\forall f\in\mathcal{G}_C^{\mathcal{L}_a}(\mathbb{R}^n)\quad L(f)=\widetilde{H}_L(f)=\left[\left(x\longmapsto\int_{\mathrm{supp}f}H_{L,\varepsilon}(x,y)f_\varepsilon(y)\,\mathrm{dy}\right)\right],\quad(7)$$

where $(H_{L,\varepsilon})$ *(resp.* (f_ε)*) is any representative of* H_L *(resp.* f*).*

The principal significance of the parameters a, b and c is the following: More irregular the net (L_ε) is (that is: b big and c close to 1), smaller the space on which (7) holds is. The limiting conditions $a\leq 1$ and $ab+c\leq 1$ are induced by Lemma 2 and Proposition 3. We can give a version of Theorem 4 valid for more irregular nets of maps. If the family (L_ε) is moderate, with the assumption that the net of constants (C_ε) in (6) satisfies $C_\varepsilon=\mathrm{O}\left(\varepsilon^{-\mathrm{r(l)}}\right)$ with $\limsup_{l\to+\infty}(r(l)/l)<c$, then the extension L satisfies $L(\mathcal{G}_C^\infty(\mathbb{R}^n))\subset\mathcal{G}^{\mathcal{L}_c}(\mathbb{R}^m)$ and the conclusion of Theorem 4 holds on $\mathcal{G}_C^\infty(\mathbb{R}^n)$. We refer the reader to [4] for further comments on these results and discussions about the question of uniqueness.

These results are strongly related to Schwartz Kernel theorem in the following sense. We can associate to each linear operator $\Lambda:\mathcal{D}(\mathbb{R}^n)\to\mathcal{D}'(\mathbb{R}^m)$, continuous for the strong topology of $\mathcal{D}'(\mathbb{R}^m)$, a strongly moderate map L_Λ and consequently a kernel $H_{L_\Lambda}\in\mathcal{G}(\mathbb{R}^m\times\mathbb{R}^n)$ with the following equality property: For all f in $\mathcal{D}(\mathbb{R}^n)$, $\Lambda(f)$ and $\widetilde{H}_{L_\Lambda}(f)$ are equal in the generalized distribution sense [14] that is, for all $k\in\mathbb{N}$ and $\left(H_{L_{\Lambda,\varepsilon}}\right)$ representative of H_{L_Λ},

$$\forall\Psi\in\mathcal{D}(\mathbb{R}^m),\ \ \langle\Lambda(f),\Psi\rangle-\int\left(\int H_{L_{\Lambda,\varepsilon}}(x,y)\,f(y)\,\mathrm{dy}\right)\Psi(x)\,\mathrm{dx}=\mathrm{O}\left(\varepsilon^{\mathrm{k}}\right).$$

4. *Application 2: Microlocal analysis*

A well known result asserts that a compactly supported distribution is C^{∞} iff its Fourier transform, which is *a priori* a slowly increasing function, israpidly decreasing. This result can be refined by the following equivalences, which hold for $u \in \mathcal{E}'\left(\mathbb{R}^d\right)$,

$$u \in \mathrm{C}^{\infty}\left(\mathbb{R}^{\mathrm{d}}\right) \Leftrightarrow \mathcal{F}(\mathrm{u}) \in \mathcal{S}_*\left(\mathbb{R}^{\mathrm{d}}\right) \Leftrightarrow \mathcal{F}(\mathrm{u}) \in \mathcal{O}'_{\mathrm{M}}\left(\mathbb{R}^{\mathrm{d}}\right) \Leftrightarrow \mathcal{F}(\mathrm{u}) \in \mathcal{O}'_{\mathrm{C}}\left(\mathbb{R}^{\mathrm{d}}\right),$$

with $\mathcal{S}_*(\Omega) = \{f \in \mathrm{C}^{\infty}(\Omega) \mid \forall \mathrm{q} \in \mathbb{N} \;\; \mu_{\mathrm{q},0}(\mathrm{f}) < +\infty\}$ and

$$\mu_{q,l}(f) = \sup_{x\in\Omega, |\alpha|\le l} (1+|x|)^q \left|\partial^{\alpha} f(x)\right| \text{ for all } (q,l) \in \mathbb{N}^2.$$

In the framework of generalized functions, an analogue results hold. More precisely, for the characterization of regular elements, there is no need to consider a space of generalized functions based on $\mathcal{S}$, that is on a space of functions with all the derivatives rapidly decreasing. It suffices to consider a space of generalized functions based on $\mathcal{S}_*$, the above introduced space of rapidly decreasing functions, with no further hypothesis on the derivatives. Set, for any regular set $\mathcal{R} \subset \mathbb{R}_+^{\mathbb{N}}$,

$$\mathcal{X}_{\mathcal{S}_*}^{\mathcal{R}}(\Omega) = \left\{(f_\varepsilon) \in \mathcal{E}(\Omega) \,\middle|\, \exists N \in \mathcal{R},\ \forall q \in \mathbb{N} \;\; \mu_{q,0}(f_\varepsilon) = \mathrm{O}\left(\varepsilon^{-\mathrm{N(q)}}\right)\right\},$$
$$\mathcal{N}_{\mathcal{S}_*}(\Omega) = \left\{(f_\varepsilon) \in \mathcal{E}(\Omega) \,\middle|\, \forall (q,m) \in \mathbb{N}^2 \;\; \mu_{q,0}(f_\varepsilon) = \mathrm{O}(\varepsilon^{\mathrm{m}})\right\},$$
$$\mathcal{X}_{\mathcal{B}}^{\mathcal{R}}(\Omega) = \left\{(f_\varepsilon) \in \mathcal{E}(\Omega) \,\middle|\, \exists N \in \mathcal{R},\ \forall l \in \mathbb{N} \;\; \mu_{0,l}(f_\varepsilon) = \mathrm{O}\left(\varepsilon^{-\mathrm{N(l)}}\right)\right\},$$
$$\mathcal{N}_{\mathcal{B}}(\Omega) = \left\{(f_\varepsilon) \in \mathcal{E}(\Omega) \,\middle|\, \forall (l,m) \in \mathbb{N}^2 \;\; \mu_{0,l}(f_\varepsilon) = \mathrm{O}\left(\varepsilon^{\mathrm{m})}\right)\right\}.$$

The space $\mathcal{X}_{\mathcal{S}_*}^{\mathcal{R}}(\Omega)$ (*resp.* $\mathcal{X}_{\mathcal{B}}^{\mathcal{R}}(\Omega)$) is a subalgebra of $\mathcal{S}_*(\Omega)^{(0,1]}$ (*resp.* $\mathcal{E}(\Omega)$) and $\mathcal{N}_{\mathcal{S}_*}(\Omega)$ (*resp.* $\mathcal{N}_{\mathcal{B}}(\Omega)$) an ideal of $\mathcal{X}_{\mathcal{S}_*}^{\mathcal{R}}(\Omega)$ (*resp.* $\mathcal{X}_{\mathcal{B}}^{\mathcal{R}}(\Omega)$). (The proof of these results follows the same line as that of Proposition 1).

Definition 5. *The factor space* $\mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}}(\Omega) = \mathcal{X}_{\mathcal{S}_*}^{\mathcal{R}}(\Omega)/\mathcal{N}_{\mathcal{S}_*}(\Omega)$ *(resp.* $\mathcal{G}_{\mathcal{B}}^{\mathcal{R}}(\Omega) = \mathcal{X}_{\mathcal{B}}^{\mathcal{R}}(\Omega)/\mathcal{N}_{\mathcal{B}}(\Omega)$*) is called the algebra of* $\mathcal{R}$-regular rough rapidly decreasing *(resp.* $\mathcal{R}$-regular bounded*)* generalized functions.

In particular, we set $\mathcal{X}_{\mathcal{S}_*}(\Omega) = \mathcal{X}_{\mathcal{S}_*}^{\mathbb{R}_+^{\mathbb{N}}}(\Omega)$ (*resp.* $\mathcal{X}_{\mathcal{B}}(\Omega) = \mathcal{X}_{\mathcal{B}}^{\mathbb{R}_+^{\mathbb{N}}}(\Omega)$) and $\mathcal{G}_{\mathcal{S}_*}(\Omega) = \mathcal{G}_{\mathcal{S}_*}^{\mathbb{R}_+^{\mathbb{N}}}(\Omega)$ (*resp.* $\mathcal{G}_{\mathcal{B}}(\Omega) = \mathcal{G}_{\mathcal{B}}^{\mathbb{R}_+^{\mathbb{N}}}(\Omega)$) which is called the space of *rough rapidly decreasing* (*resp. bounded*) *generalized functions.*

Lemma 5. *For all $u \in \mathcal{G}_{\mathcal{S}_*}\left(\mathbb{R}^d\right)$ and $(u_\varepsilon) \in \mathcal{X}_{\mathcal{S}_*}\left(\mathbb{R}^d\right)$ any representative of u, the formula*

$$\mathcal{F}_*(u) = \widehat{u} : \left[\widehat{u}_\varepsilon = \left(\xi \mapsto \int \mathrm{e}^{-\mathrm{i}\mathrm{x}\xi} \mathrm{u}_\varepsilon(\mathrm{x})\, \mathrm{dx}\right)\right]_{\mathcal{G}_\mathcal{B}} \tag{8}$$

defines an element $\widehat{u}$ of $\mathcal{G}_\mathcal{B}(\Omega)$ depending only on u.
Moreover, for any regular subspace $\mathcal{R}$ of $\mathbb{R}_+^\mathbb{N}$ and $(u_\varepsilon) \in \mathcal{X}_{\mathcal{S}_*}^\mathcal{R}(\Omega)$, we have $(\widehat{u}_\varepsilon) \in \mathcal{X}_\mathcal{B}^\mathcal{R}(\Omega)$.

Lemma 5 is mainly a consequence of a classical estimate which links $\mu_{0,l}(\widehat{u})$ and $\mu_{l+d+1,0}(u)$, for all $u \in \mathcal{S}_*\left(\mathbb{R}^d\right)$ and $l \in \mathbb{N}$. We define the *Fourier transform* $\mathcal{F}_* : \mathcal{G}_{\mathcal{S}_*}\left(\mathbb{R}^d\right) \to \mathcal{G}_\mathcal{B}\left(\mathbb{R}^d\right)$ by the equality (8). (The inverse Fourier on $\mathcal{G}_{\mathcal{S}_*}\left(\mathbb{R}^d\right)$ is defined analogously.) Lemma 5 implies

Proposition 6. *[5]* ***Regularity theorem****. We have $\mathcal{F}_*\left(\mathcal{G}_{\mathcal{S}_*}^\mathcal{R}\right) \subset \mathcal{G}_\mathcal{B}^\mathcal{R}\left(\mathbb{R}^d\right)$.*

This proposition is the main tool for the proof of the:

Theorem 7. *[5]* ***Characterization of regular compactly supported elements****. An element u of $\mathcal{G}_C\left(\mathbb{R}^d\right)$ belongs to $\mathcal{G}^\mathcal{R}\left(\mathbb{R}^d\right)$ iff $\mathcal{F}_*(u)$ belongs to $\mathcal{G}_{\mathcal{S}_*}^\mathcal{R}\left(\mathbb{R}^d\right)$.*

With this material, we can turn to local and microlocal analysis. We follow the presentation of [12] showing that the $\mathcal{G}^\mathcal{R}$-wavefront of a generalized function can be defined exactly like the C^∞-wavefront of a distribution.

Notation 2. For $(x,\xi) \in \Omega \times \left(\mathbb{R}^d \backslash \{0\}\right)$, we shall note
(i) $\mathcal{V}_x$ (resp. $\mathcal{V}_\xi^\Gamma$) the set of all open neighborhoods (resp. open convex conic neighborhoods) of x (resp. ξ),
(ii) $\mathcal{D}_x(\Omega)$ the set of elements $\mathcal{D}(\Omega)$ non vanishing at x.

As $\mathcal{G}^\mathcal{R}(\cdot)$ is a subsheaf of $\mathcal{G}(\cdot)$, we can define the *singular $\mathcal{G}^\mathcal{R}$-support* of $u \in (\Omega)$ by

$$\operatorname{singsupp}_\mathcal{R} u = \Omega \backslash \left\{x \in \Omega \mid \exists V \in \mathcal{V}_x \quad u \in \mathcal{G}^\mathcal{R}(V)\right\}.$$

For a fixed regular subset $\mathcal{R}$ of $\mathbb{R}_+^\mathbb{N}$, we set

$$O^\mathcal{R}(u) = \left\{\xi \in \mathbb{R}^d \backslash \{0\} \,\middle|\, \exists \Gamma \in \mathcal{V}_\xi^\Gamma \quad (\widehat{u})_{|\Gamma} \in \mathcal{G}_{\mathcal{S}_*}^\mathcal{R}(\Gamma)\right\}.$$

This definition makes sense since the functor $\mathcal{G}_{\mathcal{S}_*} : \Omega \to \mathcal{G}_{\mathcal{S}_*}(\Omega)$ defines a presheaf (It allows restrictions.). From this property, the following definition also makes sense: An element $u \in \mathcal{G}(\Omega)$ is said to be *$\mathcal{R}$-microregular on* $(x,\xi) \in \mathbb{R}^d \times \left(\mathbb{R}^d \setminus \{0\}\right)$ if there exists $\varphi \in \mathcal{D}_x(\Omega)$, $\Gamma \in \mathcal{V}_\xi^\Gamma$, such that $(\widehat{\varphi u})_{|\Gamma} \in \mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}}(\Gamma)$.

Set, for $u \in \mathcal{G}(\Omega)$,

$$O_x^{\mathcal{R}}(u) = \cup_{\varphi \in \mathcal{D}_x} O^{\mathcal{R}}(\varphi u) = \left\{ \xi \in \mathbb{R}^d \setminus \{0\} \mid u \text{ is } \mathcal{R}\text{-microregular on } (x,\xi) \right\},$$
$$WF_{\mathcal{R}}(u) = \left\{ (x,\xi) \in \mathbb{R}^d \times \left(\mathbb{R}^d \setminus \{0\}\right) \,\middle|\, \xi \notin O_x^{\mathcal{R}}(u) \right\}.$$

This last set is called the *$\mathcal{R}$-wavefront* set of u. The interest of this definition comes from the following

Proposition 8. *For $u \in \mathcal{G}(\Omega)$, the projection on the first component of $WF_{\mathcal{R}}u$ is equal to* $\operatorname{singsupp}_{\mathcal{R}} u$.

The **proof** of this proposition lies on the fundamental following property, equivalent in the setting of $\mathcal{R}$-microregularity to Lemma 8.1.1. in [12] for the $\mathcal{D}'$-microregularity: For $u \in \mathcal{G}_C\left(\mathbb{R}^d\right)$ and $\varphi \in \mathcal{D}\left(\mathbb{R}^d\right)$, $O^{\mathcal{R}}(u) \subset O^{\mathcal{R}}(\varphi u)$. In fact, from this property, the proof of Proposition 8 - or of the similar proposition for the case of the $\mathcal{D}'$-wavefront - follows from topological considerations, which are independent of the particular context of regularity.

Example 3. Taking $\mathcal{R} = \mathcal{B}$, the set of bounded sequence, we recover the $\mathcal{G}^\infty$-wavefront, which has here a definition independent of representatives. Taking

$$\mathcal{R} = \left\{ N \in \mathbb{R}_+^{\mathbb{N}} \mid \exists b \in \mathbb{R}_+, \forall l \in \mathbb{R} \quad N(l) \leq l + b \right\},$$

we get a wavefront which "contains" the distributional microlocal singularities of a generalized function, since $\mathcal{D}'(\cdot)$ is embedded in $\mathcal{G}^{(1)}(\cdot)$ [5].

In [13], it is shown that an analogon of this theory holds for the analytic singularities of a generalized function giving rise to the corresponding wavefront set. We also refer the reader to [7, 9, 10, 11, 14] and the literature therein for other presentations of the $\mathcal{G}^\infty$-wavefront and for some applications.

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REFERENCES

[1] S. Bernard, J. F. Colombeau, A. Delcroix, *Generalized Integral Operators and Applications*, To appear in Math. Proc. Cambridge Philos. Soc. **141**(1), 2006.

[2] J. F. Colombeau, *New Generalized Functions and Multiplication of Distributions.* North-Holland, Amsterdam, Oxford, New-York, 1984.

[3] J. F. Colombeau, *Elementary introduction to New generalized Functions.* North-Holland, Amsterdam, Oxford, New-York, 1985.

[4] A. Delcroix, *Generalized Integral Operators and Schwartz Kernel Theorem*, J. Math. Anal. Appl. **306**(2):481–501, 2005.

[5] A. Delcroix, *Fourier transform of rapidly decreasing generalized functions. Application to microlocal regularity*, Preprint AOC, 2005.

[6] C. Garetto, *Pseudo-differential Operators in Algebras of Generalized Functions and Global Hypoellipticity*, Acta Appl. Math. **80**(2):123–174, 2004.

[7] C. Garetto, G. Hörmann, *Microlocal analysis of generalized functions: pseudodifferential techniques and propagation of singularities*, Proc. Edinb. Math. Soc. **48**: 603–629, 2005.

[8] M. Grosser, M. Kunzinger, M. Oberguggenberger, R. Steinbauer *Geometric Theory of Generalized Functions with Applications to General Relativity.* Kluwer Academic Press, 2001.

[9] G. Hörmann, M.-V. DeHoop, *Microlocal analysis and global solutions for some hyperbolic equations with discontinuous coefficients*, Acta Appl. Math. **67**:173-224, 2001.

[10] G. Hörmann, M. Kunzinger, *Microlocal properties of basic operations in Colombeau algebras*, J. Math. Anal. Appl. **261**:254–270, 2001.

[11] G. Hörmann, M. Oberguggenberger, S. Pilipović, *Microlocal hypoellipticity of linear differential operators with generalized functions as coefficients*, To appear in Trans. Amer. Math. Soc.

[12] L. Hörmander, *The analysis of Linear Partial Differential Operators I, distribution theory and Fourier Analysis.* Grundlehren der mathematischen Wissenchaften 256. Springer Verlag, Berlin, Heidelberg, New York, 2nd edition 1990.

[13] J.-A. Marti, *$\mathcal{G}^L$-microanalysis of generalized functions*, To appear in Integral transf. Spec. Funct. 17/2-3 (2006), 119-125.

[14] M. Nedeljkov, S. Pilipović, D. Scarpalezos, *The linear theory of Colombeau generalized functions.* Pitman Research Notes in Mathematics Series, 385. Longman Scientific & Technical, 1998.

[15] M. Oberguggenberger, *Multiplication of Distributions and Applications to Partial Differential Equations.* Pitman Research Notes in Mathematics Series 285. Longman Scientific & Technical, 1992.

[16] Ya. V. Radyno, NgoFuTkhan, S. Ramadan, *The Fourier transform in an algebra of new generalized functions*, Russian acad. Sci. Dokl. Math. **46**(3):414–417, 1992.

[17] D. Scarpalezos, *Colombeau's generalized functions: Topological structures; Microlocal properties. A simplified point of view*, Bull. Cl. Sci. Math. Nat. Sci. Math. **25**:89–114, 2000.

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