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A METHOD TO SOLVE AN EQUATION WITH LEFT AND RIGHT FRACTIONAL DERIVATIVES

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A b s t r a c t. We give a procedure to find a solution to the equation $D^{\alpha}_{b-}D^{\alpha}_{0+}y(x) + mD^{\alpha}_{0+}D^{\alpha}_{b-}y = C_0(x),\ 0 < x < b,\ 0 < \alpha < 1,$ *solving the corresponding Fredholm integral equation of the second kind.*

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1. *Introduction*

Differential equations with the left and right fractional derivatives (denoted by D^{α}_{0+} and D^{α}_{b-}, respectively) arise in mathematical models in many branches of sciences as: mechanics, rheology, fluid flows, electrical networks, viscoelasticity, chemical physics, polymer science, biophyscis,.. See for example the books [4] and [5]. The theory of such equations with the left fractional derivatives has been elaborated in many papers and books (see for example [5]). But equations in which arises the left fractional derivative combined with the right fractional derivative have been left unexamined. One of the reason lies in the fact that the connection between these two aspects of fractional derivatives is very complex.

We treat of the equation

$$D_{b-}^{\alpha} D_{0+}^{\alpha} y(x) + m D_{0+}^{\alpha} D_{b-}^{\alpha} y(x) = C_0(x), \ 0 < x < b, \ 0 < \alpha < 1, \qquad (1)$$

where m is a constant, bringing down to a Fredholm integral equation of the second kind. This equation is interesting not only as a mathematical model in mechanics, but also it gives the connection between two operators $D_{b-}^{\alpha} D_{0+}^{\alpha}$ and $D_{0+}^{\alpha} D_{b-}^{\alpha}$.

To find a solution to (1) we first construct a singular integral equation, then a quasi-Fredholm integral equation which can be brought by integration to a Fredholm integral equation with the continuous kernel.

In this paper we use the notation and defnitions as they are given in [9]. Let us repeat definitions of three classes of functions we use (see [9]):

The function f defined on the interval $[a, b]$ belongs to the class $\mathcal{H}(\lambda)$ if and only if $|f(t_2) - f(t_1)| \le A|t_2 - t_1|^{\lambda}, \ 0 < \lambda \le 1$, for every $t_1, t_2 \in [a, b]$.

The function $f \in \mathcal{H}$ if $f \in \mathcal{H}(\lambda)$, for a fixed $\lambda \in (0, 1]$.

The function f defined on $[0, b]$ belongs to $h^{\lambda}[0, b], \ 0 < \lambda \le 1$, if

$$\frac{f(x_2) - f(x_1)}{|x_2 - x_1|^{\lambda}} \to 0, \ \ x_2 \to x_1 \, .$$

Let $\rho(x) = x^{\alpha}$. We denote by $\mathcal{L}^p(\rho, (0, 1))$ the space of functions f, such that $\rho(x) f(x) \in \mathcal{L}^p(0, 1)$.

2. *A method to solve equation (1)*

2.1. *Construction of a singular integral equation related to (1)*

Let $C_0(x) = C^{(1)}(x)$. It is well-known that $D_{0+}^{\alpha} = \dfrac{d}{dx} I_{0+}^{1-\alpha}$ and $D_{b-}^{\alpha} = -\dfrac{d}{dx} I_{b-}^{1-\alpha}$, where $I_{0+}^{1-\alpha}$ an $I_{b-}^{1-\alpha}$ are the left and right fractional integral operators, respectively. Then (1) is equivalent to

$$-I_{b-}^{1-\alpha} D_{0+}^{\alpha} y(x) + m I_{0+}^{1-\alpha} D_{b-}^{\alpha} y(x) = C(x) + k, \ 0 < x < b \, . \qquad (2)$$

We use the connection between I_{0+}^{α} and I_{b-}^{α} (see [9], p. 206):

$$I_{0+}^{1-\alpha} \varphi(x) = \cos(1-\alpha)\pi I_{b-}^{1-\alpha} \varphi(x) - \sin(1-\alpha)\pi \Big(I_{b-}^{1-\alpha} (b-t)^{\alpha-1} S (b-\tau)^{1-\alpha} \varphi \Big)(x),$$

which holds for $\varphi \in \mathcal{L}^q(0,b),\ q(1-\alpha) > 1$ and $x \in (0,b)$, where

$$S(\varphi) = \frac{1}{\pi} VP \int_0^b (t-x)^{-1}\varphi(t)dt\,.$$

Then (2) is:

$$\begin{aligned}&-I_{b^-}^{1-\alpha} D_{0+}^{\alpha} y(x) + m I_{b^-}^{1-\alpha}\Big[\cos(1-\alpha)\pi D_{b-}^{\alpha} y(t)\\ &-\sin(1-\alpha)\pi(b-t)^{\alpha-1}\Big(S(b-\tau)^{1-\alpha} D_{b-}^{\alpha} y(\tau)\Big)(t)\Big](x) = C(x)+k\,,\end{aligned}\tag{3}$$

where $D_{b-}^{\alpha} y \in \mathcal{L}^q(0,b),\ q(1-\alpha) > 1$.

Appliyng the operator $D_{b^-}^{1-\alpha}$ to (3), we have:

$$\begin{aligned}&-D_{0+}^{\alpha} y(x) + m\Big[\cos(1-\alpha)\pi D_{b-}^{\alpha} y(x) - \sin(1-\alpha)\pi(b-x)^{\alpha-1}\\ &\cdot\Big(S(b-\tau)^{1-\alpha} D_{b-}^{\alpha} y(\tau)\Big)(x)\Big] = D_{b^-}^{1-\alpha}(C(x)+k)\,.\end{aligned}\tag{4}$$

We introduce a new function g by the connection with y as

$$y(x) = I_{b-}^{\alpha} g(x),\ 0 < x < b\,,\tag{5}$$

supposing that $g \in L^p(0,b),\ \alpha p > 1$. Then $y = I_{b-}^{\alpha} g \in h^{\alpha-1/p}[0,b]$ (see [9], p. 69). With the new form of y, given by (5), equation (4) can be written as

$$\begin{aligned}&-D_{0+}^{\alpha} I_{b-}^{\alpha} g(x) + m\Big[\cos(1-\alpha)\pi g(x) - \sin(1-\alpha)\pi(b-x)^{\alpha-1}\ \mathrm{x}\\ &\Big(S(b-\tau)^{1-\alpha} g(\tau)\Big)(x)\Big] = D_{b^-}^{1-\alpha}(C(x)+k)\,.\end{aligned}\tag{6}$$

Now we again use the connection of I_{b-}^{α} and I_{0+}^{α} given by (see [9], p. 206)

$$I_{b-}^{\alpha}\varphi(x) = \cos\alpha\pi I_{0+}^{\alpha}\varphi(x) + \sin\alpha\pi(I_{0+}^{\alpha} t^{-\alpha} S\tau^{\alpha}\varphi)(x),$$

which holds for $\varphi \in \mathcal{L}^p(0,b),\ p\alpha > 1, x \in (0,b)$. With such expression of I_{b-}^{α} (6) is as

$$\begin{aligned}&-D_{0+}^{\alpha}\Big(\cos\alpha\pi I_{0+}^{\alpha} g(x) + \sin\alpha\pi(I_{0+}^{\alpha} t^{-\alpha} S\tau^{\alpha} g)(x)\Big)\\ &+m\Big[\cos(1-\alpha)\pi g(x) - \sin(1-\alpha)\pi(b-x)^{\alpha-1}\Big(S(b-\tau)^{1-\alpha} g\Big)(x)\Big]\\ &= D_{b^-}^{1-\alpha}(C(x)+k)\,.\end{aligned}\tag{7}$$

Since $D^{\alpha}_{0+}I^{\alpha}_{0+}\varphi = \varphi$ for any sommable function φ on $(0,b)$, and the operator S is bounded on $\mathcal{L}^q(\tau^{\alpha},(0,1)),\ \alpha < q-1$ (see [9], p. 200), we can write (7) as

$$\begin{aligned} &-[\cos\alpha\pi g(x) + \sin\alpha\pi x^{-\alpha}(S\tau^{\alpha}g)(x)] + m[\cos(1-\alpha)\pi g(x) \\ &-\sin(1-\alpha)\pi(b-x)^{\alpha-1}\Big(S(b-\tau)^{1-\alpha}g(\tau)\Big)(x) \\ &= D^{\alpha}_{b-}(C(x)+k)\,. \end{aligned} \tag{8}$$

Since $0<\alpha<1,\ \cos(1-\alpha)\pi = -\cos\alpha\pi$ and $\sin(1-\alpha)\pi = \sin\alpha\pi$, finally, we obtain the singular integral equation

$$A(x)g(x) + \frac{1}{\pi i}\int_0^b \frac{K(x,t)}{t-x}g(t) = F(x),\ 0<x<b\,, \tag{9}$$

where

$$\begin{aligned} &A(x) = (m+1)\cos\alpha\pi x^{\alpha}(b-x)^{1-\alpha}, \\ &F(x) = -(D^{\alpha}_{b-}C(t)+k)x)x^{\alpha}(b-x)^{1-\alpha}, \\ &K(x,t) = i\sin\alpha\pi\Big(t^{\alpha}(b-x)^{1-\alpha} + m(b-t)^{1-\alpha}x^{\alpha}\Big)\,. \end{aligned} \tag{10}$$

The function $g(\tau)\in\mathcal{L}^p(0,b)$ and $\tau^{\alpha}g(\tau)\in\mathcal{L}^q(0,b),\ \dfrac{1}{p}<\alpha<q-1$.

We show that A and K belong to $\mathcal{H}$. It is well-known that x^{α} and $(b-x)^{1-\alpha}$ belong to $\mathcal{H}$ and are bounded on $[0,b]$. The product and the sum of two functions which belong to $\mathcal{H}$, belong to $\mathcal{H}$, as well (see [6], p. 24). Consequently A and K belong to $\mathcal{H}$ (K in two variables). We can now prove the following

Proposition. *If g is a solution to (9) such that $g\in\mathcal{L}^p(0,1),\ p = \max(\dfrac{1}{\alpha},\ \dfrac{1}{1-\alpha},\alpha+1)$, then $y = I^{\alpha}_{b-}g$ is a solution to (1) and belongs to $h^{\alpha-1/p}[0,b],\ \alpha-1/p>0$.*

P r o o f. The function g can be given in the form

$$g = D^{\alpha}_{0+}I^{\alpha}_{0}g\,.$$

To prove this we introduce the function

$$F(x) = (D^{\alpha}_{0+}I^{\alpha}_{0}g)(x),\ 0<x<1\,.$$

Then (see [9], p. 45)

$$\begin{aligned} I^{\alpha}_{0+}F &= I^{\alpha}_{0+}g - (I^{1-\alpha}_{0+}I^{\alpha}_{0+}g)(0) \\ &= I^{\alpha}_{0+}g - (I^{1}_{0+}g)(0) = I^{\alpha}_{0+}h\,. \end{aligned}$$

Consequently,

$$I^{\alpha}_{0+}(F-g)=0, \quad \text{or } (F-g)(x)*x^{-\alpha}=0\,.$$

By the Tichmarsh theorem $F=\varphi$.

We know (see [9], p. 69) that the operator I^{α}_{0+} has the property

$$I^{\alpha}_{0+}:\mathcal{L}^p(a,b)\to h^{\alpha-1/p}[0,b].$$

Since $I^{\alpha}_{b-}=QI^{\alpha}_{0+}Q$, where the operator Q maps a function f defined on $(0,b)$ as $Q:f\to f(b-x)$, the operator I^{α}_{b-} has the same cited property of I^{α}_{0+}. This gives that $y\in h^{\alpha-1/p}[0,b]$.

Suppose that we find a solution $g\in\mathcal{L}^p(0,1)$ to (9), $p=\max\Big(\dfrac{1}{\alpha},\dfrac{1}{1-\alpha},1+\alpha\Big)$. Now we can return from (9) to (1) taking in consideration that $L^p(0,b)\subset L^q(0,b)$, for $q=1,\ q=\dfrac{1}{\alpha},q=\dfrac{1}{1-\alpha}$ and $q=\alpha+1$.

2.2. *The characteristic equation of (9)*

Our aim is to construct a Fredholm integral equation instead of singular integral equation (9). There exist different methods to do it (see for example $[M]$ and $[G]$). We propose the method which uses the characteristic equation. Then the first step is to solve the characteristic equation.

Equation (9) can be written as

$$A(x)g(x)+\frac{B(x)}{\pi i}\int_0^b\frac{g(t)}{t-x}dt+\frac{1}{\pi i}\int_0^b k(x,t)g(t)dt=F(x)\,, \tag{11}$$

where

$$\begin{aligned}&B(x)=K(x,x)=i(1+m)\sin\alpha\pi x^{\alpha}(b-x)^{1-\alpha}\ \text{and}\\ &k(x,t)=\frac{K(x,t)-B(x)}{t-x}\,.\end{aligned} \tag{12}$$

For the function $k(x,t)$ we know (see [6], p. 90) that

$$k(x,t)=\frac{k^*(x,t)}{|t-x|^{\lambda}}\,,\ 0\le\lambda<1,\ k^*\in\mathcal{H}\,. \tag{13}$$

The equation

$$A(x)g_0(x)=\frac{B(x)}{\pi i}\int\limits_0^b\frac{g_0(t)}{t-x}dt=F(x),\quad 0<x<b, \tag{14}$$

is called the characteristic equation to (9). Because of (10) and (12) is can be given as

$$ag_0(x) + \frac{b}{\pi i}\int_0^b \frac{g_0(t)}{t-x}dt = d(x),\ 0 < x < b, \tag{15}$$

where

$$a = (m+1)\cos\alpha\pi,\ b = i(m+1)\sin\alpha\pi,\ d(x) = \Big(-D^{\alpha}_{b^-}(C(t)+k)\Big)(x),$$

To solve equation (15) we follow the procedure given in [6], §97.

Let $G(t)$ be the function

$$G(t) = \frac{a-b}{a+b} = \frac{(m+1)\cos\alpha\pi - i\sin\alpha\pi}{(m+1)\cos\alpha\pi + i\sin\alpha\pi} = (\cos\alpha\pi - i\sin\alpha\pi)^2 = e^{-2\alpha\pi i}.$$

Then $\ell n G(t) = (k-\alpha)2\pi i, k = 0, \pm 1, \pm 2, ...$ We fix

$$\ell n G(t) = (-\alpha)2\pi i\,. \tag{16}$$

Let us analyze the function

$$\gamma(x) = \frac{1}{2\pi i}\int_0^b \frac{\ell n G(t)}{t-z}\,dt\,.$$

With G given by (16) we have

$$\gamma(z) = \frac{1}{2\pi i}\ell n G(t)\ell n\Big(\frac{b-z}{-z}\Big) = (-\alpha)\ell n\Big(\frac{b-z}{-z}\Big) \tag{17}$$

(see [6], p. 47). Then the function

$$e^{\gamma(z)} = \Big(\frac{b-z}{-z}\Big)^{(-\alpha)}$$

satisfies the equation

$$\phi^+(t) = G(t)\phi^-(t),\ \ 0 < t < b\,. \tag{18}$$

In the neighborhood of $c = 0$ and $c = b$ we have

$$\gamma(z) = A\,m(z-c),\ \ A = \mp(-\alpha)\,, \tag{19}$$

where we take the sign "-" if $c = 0$, and the sigh "+" if $c = b$. We determine integers λ_0 and λ_b such that $-1 < \alpha + \lambda_0 < 0$ and $-1 < -\alpha + \lambda_b < 0$. This gives $\lambda_0 = -1$ and $\lambda_b = 0$. Then the canonical solution $X(z)$ to (18) is

$$X(z) = e^{\gamma(z)} z^{-1} = \Big(\frac{b-z}{-z}\Big)^{-\alpha} z^{-1} = (b-z)^{-\alpha} z^{-(1-\alpha)}\,. \tag{20}$$

This solution is of the class $h(0,b)$ and the index of the class $h(0,b)$ is $\kappa = 1$ (for the definition of the class $h(0,b)$ and the index κ see [6], p. 315). The function $X^+(x)$ and $X^-(x)$ are

$$X^+(x) = e^{-i\alpha\pi}(b-x)^{-\alpha} x^{-(1-\alpha)},\ X^-(x) = e^{i\alpha\pi}(b-x)^{-\alpha} x^{-(1-\alpha)}\,.$$

With the canonical solution $X(z)$ to (18) we construct the general solution $\phi_0(z) = X(z)P(z)$ to (18) of the given class $h(0,b)$; $P(z)$ is any polynomial.

The function

$$\phi(z) = \frac{X(z)}{2\pi i}\int_0^b \frac{d(t)\,dt}{(a+b)X^+(t)(t-z)} + \phi_0(z) \tag{21}$$

is the general solution of the class $h(0,b)$ to equation

$$\phi^+(x) = G(x)\phi^-(x) + \frac{d(x)}{a+b}\,. \tag{22}$$

Since we need that $\phi(\infty) = 0,\ P(z)$ has to be only a constant, denoted by P.

Finally, the solution to (16) is

$$g_0(x) = \phi^+(t_0) - \phi^-(t_0)\,. \tag{23}$$

If $d(x)$ belong to the class $\mathcal{H}$, then $g_0(x) \in \mathcal{H}$ and is the general solution in the class $\mathcal{H}$ to (15).

To give the explicit form of $g_0(x)$ we introduce the function $Z(x) = (a+b)X^+(x)$ and the constants

$$a^* = \frac{a}{a^2-b^2} = \frac{(m+1)\cos\alpha\pi}{(m+1)^2\cos^2\alpha\pi + (m+1)^2\sin^2\alpha\pi} = \frac{a}{(m+1)^2}$$

$$b^* = \frac{b}{a^2-b^2} = \frac{b}{(m+1)^2}\,.$$

By the formula of Sohocki-Plemelj (see [6], p. 66) we have

$$g_0(x) = a^* d(x) - \frac{b^* Z(x)}{\pi i} \int_0^b \frac{d(t)dt}{Z(t)(t-x)} + b^* Z(x) P^*,\ 0 < x < b. \tag{24}$$

2.3. *The quasi Fredholm integral equation*

We use the solution to the characteristic equation of (9) to construct a quasi Fredholm integral equation (cf. [3], §49.1). Equation (11) can be written as

$$\begin{aligned} ag(x) \quad &+ \frac{b}{\pi i} \int_0^b \frac{g(t)}{t-x} dy = d(x) \\ &- x^{-\alpha}(b-x)^{\alpha-1} \frac{1}{\pi i} \int_0^b k(x,t) g(t) dt,\ 0 < x < b\,. \end{aligned} \tag{25}$$

By the result of 2.2 the formal solution to (25) is

$$\begin{aligned} g(x) \quad &= a^* \Big(d(x) - x^{-\alpha}(b-x)^{\alpha-1} \frac{1}{\pi i} \int_0^b k(x,t) g(t) dt \Big) \\ &- \frac{b^* Z(x)}{\pi i} \int_0^b \frac{1}{Z(t)(t-x)} \Big[d(t) \\ &- t^{-\alpha}(b-t)^{\alpha-1} \frac{1}{\pi i} \int_0^b k(t,\tau) g(\tau) d\tau \Big] dt \\ &+ b^* Z(x) P^*,\ \ 0 < x < b\,, \end{aligned}$$

or

$$g(x) + \int_0^b M(x,t) g(t) dt = E(x),\ \ 0 < x < b\,, \tag{26}$$

where

$$
\begin{aligned}
M(x,t) \quad &= a^* x^{-\alpha}(b-x)^{\alpha-1}\frac{1}{\pi i}k(x,t) \\
&\quad -\frac{b^* Z(x)}{\pi i}\int\limits_0^b \frac{e^{\alpha\pi i}}{(a+b)t^{-1+2\alpha}(b-t)^{1-2\alpha}}\frac{1}{\pi}k(t,\tau)dt,
\end{aligned}
\tag{27}
$$

$$
E(x) = a^* d(x) - \frac{b^* Z(x)}{\pi i}\int\limits_0^b \frac{d(t)}{Z(t)(t-x)}dt + b^* Z(x) P^* . \tag{28}
$$

Equation (26) is a quasi Fredholm integral equation. It can be transformed to a Fredholm integral equation (see for example [7]) by using the process of iteration. The quasi Fredholm equations have the same properties as Fredholm's integral equations (see [G], p. 152).

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