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SOME PROPERTIES OF THE LOCALLY CONFORMAL KÄHLER MANIFOLD

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A b s t r a c t. We determine the holomorphic curvature tensor of locally conformal Kähler manifold and find the expression of the Riemannian curvature tensor of such manifold if it is of pointwise constant holomorphic sectional curvature. We examine the geodesic mapping using the holomorphic curvature tensor and apply the obtained results to the locally conformal Kähler manifolds.

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1. *The locally conformal Kähler manifolds*

Let (M, g, J) be a real $2n$-dimensional almost Hermitian manifold, where J is almost complex structure and g is Hermitian metric. Then

$$J^2 = -Id. \,, \qquad g(JX, JY) = g(X, Y) \,,$$

for any vector fields X, Y tangent to M. The fundamental 2-form $F(X, Y)$ is

$$F(X, Y) = g(JX, Y) = -F(Y, X) \,.$$

(M, g, J) is a Kähler manifold if $\nabla J = 0$, where ∇ denotes the covariant differentiation with respect to Levi-Civita connection.

The manifold (M, g, J) is called locally conformal Kähler (bf. l.c.Kähler) manifold if each point $p \in M$ has an open neighborhood $\mathcal{U}$ with a differentiable function

$$\sigma : \mathcal{U} \to R^{2n}$$

such that

$$\dot{g} = e^{-2^{\sigma}} g|_{\mathcal{U}} \tag{1.1}$$

is a Kähler metric on $\mathcal{U}$.

The necessary and the sufficient condition for (M, g, J) to be l.c. Kähler manifold is that it admits a global 1-form α satisfying the condition [2]:

$$(\nabla_Z F)(X, Y) = \alpha(JX)g(Z, Y) - \alpha(JY)g(Z, X) - \alpha(X)F(Z, Y) + \alpha(Y)F(Z, X). \tag{1.2}$$

With respect to the local coordinates, (1.2) is

$$\begin{gathered} \nabla_s F_{ij} = -\alpha_i F_{sj} + \alpha_j F_{si} - \alpha^a F_{ai} g_{sj} + \alpha^a F_{aj} g_{si}, \\ \nabla_i \alpha_j = \nabla_j \alpha_i \ , \qquad \alpha^i = \alpha_p g^{pi}, \end{gathered} \tag{1.3}$$

and the indices run over the range $\{1, 2, \ldots, 2n\}$.

From (1.3) we obtain

$$\begin{gathered} \nabla_r \nabla_s F_{ij} - \nabla_s \nabla_r F_{ij} = \\ = P_{rs} J^a_j g_{si} - P_{ra} J^a_i g_{sj} - P_{sa} J^a_j g_{ri} + P_{sa} J^a_i g_{rj} \\ -P_{rj} F_{si} + P_{ri} F_{sj} + P_{sj} F_{ri} - P_{si} F_{rj} \quad , \end{gathered}$$

where

$$P_{ri} = -\nabla_r \alpha_i - \alpha_r \alpha_i + \frac{1}{2} \alpha_p \alpha^p g_{ri}.$$

We note that $P_{ri} = P_{ir}$.

Using the Ricci identity, we get

$$\begin{gathered} -R_{sria} J^a_j + R_{srja} J^a_i = \\ = P_{ri} F_{sj} + P_{sj} F_{ri} - P_{rj} F_{si} - P_{si} F_{rj} \\ -P_{ra} J^a_i g_{sj} - P_{sa} J^a_j g_{ri} + P_{ra} J^a_j g_{si} + P_{sa} J^a_i g_{rj}, \end{gathered} \tag{1.4}$$

or

$$\begin{aligned} R_{srab}J_j^aJ_h^b &= R_{srjh} \\ &+P_{rh}g_{sj} - P_{rj}g_{sh} + P_{sj}g_{rh} - P_{sh}g_{rj} \\ &+P_{ra}J_h^aF_{sj} - P_{ra}J_j^aF_{sh} + P_{sa}J_j^aF_{rh} - P_{sa}J_h^aF_{rj}. \end{aligned} \tag{1.5}$$

Transecting (1.4) with g^{ir} and denoting by ρ the Ricci tensor, we get

$$\begin{aligned} &-\rho_{sa}J_j^a + R_{srja}J_i^ag^{ir} = \\ &= -(2n-3)P_{sa}J_j^a - P_{ja}J_s^a + (P_{ab}g^{ab})F_{sj}, \end{aligned}$$

from which, taking the symmetric part, we obtain [3]:

$$\rho_{ia}J_j^a + \rho_{ja}J_i^a = 2(n-1)(P_{ja}J_i^a + P_{ia}J_j^a). \tag{1.6}$$

2. *Holomorphic curvature tensor*

A 2-plane π in $T_p(M)$, $p \in M$, is said to be holomorphic if $J\pi = \pi$. The manifold M has pointwise constant holomorphic sectional curvature if the sectional curvature relative to π does not depend on the holomorphic 2-plane π in $T_p(M)$.

The curvature tensor of Kähler manifold satisfies the condition

$$R(X,Y,JZ,JW) = R(X,Y,Z,W). \tag{2.1}$$

Using it, in [4] (Proposition 7.3, p.167) is determined the curvature tensor of Kähler manifold of constant holomorphic sectional curvature. But, if $\nabla J \neq 0$, (2.1) does not hold. Nevertheless, there exist for any almost Hermitian manifold, the algebraic curvature tensor, satisfying the condition of type (2.1). It is ([4],[7]):

$$\begin{aligned} &(HR)(X,Y,Z,W) = \\ &= \frac{1}{16}\{3\,[R(X,Y,Z,W) + R(JX,JY,Z,W) + R(X,Y,JZ,JW) \\ &+R(JX,JY,JZ,JW)] - R(X,Z,JW,JY) - R(JX,JZ,W,Y) \\ &-R(X,W,JY,JZ) - R(JX,JW,Y,Z) + R(JX,Z,JW,Y) \\ &+R(X,JZ,W,JY) + R(JX,W,Y,JZ) + R(X,JW,JY,Z)\}\ . \end{aligned} \tag{2.2}$$

Namely, it is easy to see that

$$
\begin{aligned}
&(HR)(X,Y,Z,W) = -(HR)(Y,X,Z,W) = \\
&= -(HR)(X,Y,W,Z) = (HR)(Z,W,X,Y)\ , \\
(HR)(X,Y,Z,W) &+ (HR)(X,Z,W,Y) + (HR)(X,W,Y,Z) = 0\ ,
\end{aligned}
\tag{2.3}
$$

as well as

$$(HR)(X,Y,JZ,JW) = (HR)(X,Y,Z,W), \tag{2.4}$$

$$(HR)(X,JX,JX,X) = R(X,JX,JX,X). \tag{2.5}$$

The tensor (2.2) is said to be *the holomorphic curvature tensor.*

The relation (2.5) shows that the holomorphic sectional curvatures relative to R and HR are the same. Thus, applying the way of [4] to the tensor HR, we can state

Proposition 2.1 *If an almost Hermitian manifold* (M, g, J) *is of pointwise constant holomorphic sctional curvature* $H(p)$, $p \in M$, *then the holomorphic curvature tensor is*

$$
\begin{aligned}
(HR)(X,Y,Z,W) = \frac{H(p)}{4}\,&[g(X,W)g(Y,Z) - g(X,Z)g(Y,W) \\
+F(X,W)F(Y,Z) - F(X,Z)F(Y,W) &- 2F(X,Y)F(Z,W)]\,.
\end{aligned}
\tag{2.6}
$$

Now, we shall determine the holomorphic curvature tensor for a l.c. Kähler manifold.

With respect to the local coordinates, (2.2) reads

$$
\begin{aligned}
&(HR)_{ijhk} = \\
&= \frac{1}{16}\Big\{3\Big[R_{ijhk} + R_{abhk}J_i^aJ_j^b + R_{ijab}J_h^aJ_k^b + R_{abcd}J_i^aJ_j^bJ_h^cJ_k^d\Big] \\
&\quad -R_{ihab}J_k^aJ_j^b - R_{abkj}J_i^aJ_h^b - R_{ikab}J_j^aJ_h^b - R_{abjh}J_i^aJ_k^b \\
&\quad +R_{ahbj}J_i^aJ_k^b + R_{iakb}J_h^aJ_j^b + R_{akjb}J_i^aJ_h^b + R_{iabh}J_k^aJ_j^b\Big\}.
\end{aligned}
\tag{2.7}
$$

In view of (1.5), we find

$$\begin{aligned}
&3\left[R_{ijhk}+R_{abhk}J_i^aJ_j^b+R_{ijab}J_h^aJ_k^b+R_{abcd}J_i^aJ_j^bJ_h^cJ_k^d\right]\\
&=3\,[4R_{ijhk}\\
&\quad+g_{ih}(3P_{jk}-P_{ab}J_j^aJ_k^b)-g_{ik}(3P_{jh}-P_{ab}J_j^aJ_h^b)\\
&\quad+g_{jk}(3P_{ih}-P_{ab}J_i^aJ_h^b)-g_{jh}(3P_{ik}-P_{ab}J_i^aJ_k^b)\\
&\quad+F_{ih}(P_{ja}J_k^a-P_{ka}J_i^a)-F_{ik}(P_{ja}J_h^a-P_{ha}J_j^a)\\
&\quad+F_{jk}(P_{ia}J_h^a-P_{ha}J_i^a)-F_{jh}(P_{ia}J_k^a-P_{ka}J_i^a)].
\end{aligned}\tag{2.8}$$

Using (1.5), we also obtain

$$\begin{aligned}
&-(R_{ihab}J_k^aJ_j^b+R_{abkj}J_i^aJ_h^b+R_{ikab}J_j^aJ_h^b+R_{abjh}J_i^aJ_k^b)\\
&=2R_{ijhk}\\
&\quad-2(g_{ik}P_{jh}+g_{jh}P_{ik}-g_{ih}P_{jk}-g_{jk}P_{ih})\\
&\quad-F_{ik}(P_{ha}J_j^a-P_{ja}J_h^a)-F_{jh}(P_{ka}J_i^a-P_{ia}J_k^a)\\
&\quad-F_{ih}(P_{ja}J_k^a-P_{ka}J_j^a)-F_{jk}(P_{ia}J_h^a-P_{ha}J_i^a)\\
&\quad-2F_{ij}(P_{ka}J_h^a-P_{ha}J_k^a)-2F_{hk}(P_{ja}J_i^a-P_{ia}J_j^a)\;.
\end{aligned}\tag{2.9}$$

Using (1.4), as well as the relation

$$R_{akht}-R_{ahkt}=-R_{khat}\;,$$

we find

$$\begin{aligned}
&R_{ahbj}J_i^aJ_k^b+R_{akjb}J_i^aJ_h^b=\\
&=-R_{khab}J_i^aJ_j^b\\
&\quad-P_{hj}g_{ik}+P_{ka}J_i^aF_{hj}+P_{hk}g_{ij}-P_{ja}J_i^aF_{hk}\\
&\quad-P_{ha}J_j^aF_{ik}-P_{ab}J_i^aJ_k^bg_{jh}+P_{ha}J_k^aF_{ij}+P_{ab}J_i^aJ_j^bg_{hk}\\
&\quad+P_{kj}g_{ih}-P_{ha}J_i^aF_{kj}-P_{hk}g_{ij}+P_{ja}J_i^aF_{kh}\\
&\quad+P_{ka}J_j^aF_{ih}+P_{ab}J_i^aJ_h^bg_{jk}-P_{ka}J_h^aF_{ij}-P_{ab}J_i^aJ_j^bg_{hk},
\end{aligned}$$

such that, applying (1.5), we have

$$\begin{aligned} & R_{ahbj}J_i^aJ_k^b + R_{akjb}J_i^aJ_h^b = \\ & = R_{ijhk} + 2g_{ih}P_{jk} - 2g_{ik}P_{jh} \\ & \quad +g_{jk}(P_{ih} + P_{ab}J_i^aJ_h^b) - g_{jh}(P_{ik} + P_{ab}J_i^aJ_k^b) \\ & \quad -2P_{ja}J_i^aF_{hk} + (P_{ha}J_k^a - P_{ka}J_h^a)F_{ij}. \end{aligned} \tag{2.10}$$

In a similar way, we obtain

$$\begin{aligned} & R_{iakb}J_h^aJ_j^b + R_{iabh}J_k^aJ_j^b = \\ & = R_{ijhk} - 2P_{ik}g_{jh} + g_{ih}(P_{jk} + P_{ab}J_j^aJ_k^b) \\ & \quad -g_{ik}(P_{jh} + P_{ab}J_j^aJ_h^b) + 2P_{ih}g_{jk} \\ & \quad +2P_{ia}J_j^aF_{hk} + (P_{ha}J_k^a - P_{ka}J_h^a)F_{ij}. \end{aligned} \tag{2.11}$$

Substituting (2.8)-(2.11) into (2.7), we get

$$\begin{aligned} & (HR)_{ijhk} = R_{ijhk} \\ & +\frac{1}{8}\Big\{g_{ih}(7P_{jk} - P_{ab}J_j^aJ_k^b) - g_{ik}(7P_{jh} - P_{ab}J_j^aJ_h^b) \\ & \quad +g_{jk}(7P_{ih} - P_{ab}J_i^aJ_h^b) - g_{jh}(7P_{ik} - P_{ab}J_i^aJ_k^b) \\ & \quad +F_{ih}(P_{ja}J_k^a - P_{ka}J_j^a) - F_{ik}(P_{ja}J_h^a - P_{ha}J_j^a) \\ & \quad +F_{jk}(P_{ia}J_h^a - P_{ha}J_i^a) - F_{jh}(P_{ia}J_k^a - P_{ka}J_i^a) \\ & \quad +2F_{ij}(P_{ha}J_k^a - P_{ka}J_h^a) + 2F_{hk}(P_{ia}J_j^a - P_{ja}J_i^a)\Big\}. \end{aligned} \tag{2.12}$$

Thus, we can state

Theorem 2.1 *The holomorphics curvature tensor of l.c.Kähler manifold has the form (2.12).*

In view of Proposition (2.1), l.c.Kähler manifold has pointwise constant holomorphic sectional curvature if its curvature tensor has the form

$$\begin{aligned}R_{ijhk} = {} & \frac{H(p)}{4}\left[g_{ik}g_{jh} - g_{ih}g_{jk} + F_{ik}F_{jh} - F_{ih}F_{jk} - 2F_{ij}F_{hk}\right] \\ & + \frac{1}{8}\Big\{g_{ik}(7P_{jh} - P_{ab}J_j^a J_h^b) - g_{ih}(7P_{jk} - P_{ab}J_j^a J_k^b) \\ & + g_{jh}(7P_{ik} - P_{ab}J_i^a J_k^b) - g_{jk}(7P_{ih} - P_{ab}J_i^a J_h^b) \\ & + F_{ik}(P_{ja}J_h^a - P_{ha}J_j^a) - F_{ih}(P_{ja}J_k^a - P_{ka}J_j^a) \\ & + F_{jh}(P_{ia}J_k^a - P_{ka}J_i^a) - F_{jk}(P_{ia}J_h^a - P_{ha}J_i^a) \\ & - 2F_{ij}(P_{ha}J_k^a - P_{ka}J_h^a) - 2F_{hk}(P_{ia}J_j^a - P_{ja}J_i^a)\Big\}\,.\end{aligned} \tag{2.13}$$

Conversely, if (2.13) holds, then the holomorphic curvature tensor has the form (2.6). Thus, we can state

Theorem 2.2 *L.c. Kähler manifold has pointwise constant holomorphic sectional curvature if and only if its curvature tensor can be expressed in the form (2.13).*

If the tensor P_{ij} is hybrid, i.e. if

$$P_{ab}J_i^a J_j^b = P_{ij}, \quad P_{ia}J_j^a = -P_{ja}J_i^a,$$

the relations (2.12) and (2.13) reduce, respectively to

$$\begin{aligned}(HR)_{ijhk} = {} & R_{ijhk} \\ & + \frac{3}{4}(P_{jk}g_{ih} + P_{ih}g_{jk} - P_{jh}g_{ik} - P_{ik}g_{jh}) \\ & + \frac{1}{4}(F_{ih}P_{ja}J_k^a + F_{jk}P_{ia}J_h^a - F_{ik}P_{ja}J_h^a - F_{jh}P_{ia}J_k^a \\ & + 2F_{hk}P_{ia}J_j^a + 2F_{ij}P_{ha}J_k^a)\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}R_{ijhk} = {} & \frac{H(p)}{4}\left[g_{ik}g_{jh} - g_{ih}g_{jk} + F_{ik}F_{jh} - F_{ih}F_{jk} - 2F_{ij}F_{hk}\right] \\ & + \frac{3}{4}(g_{ik}P_{jh} + g_{jh}P_{ik} - g_{ih}P_{jk} - g_{jk}P_{ih}) \\ & + \frac{1}{4}(F_{ik}P_{ja}J_h^a + F_{jh}P_{ia}J_k^a - F_{ih}P_{ja}J_k^a - F_{jk}P_{ia}J_h^a \\ & - 2F_{ij}P_{ha}J_k^a - 2F_{hk}P_{ia}J_j^a).\end{aligned} \tag{2.15}$$

The relation (2.15) is just the form of the curvature tensor of l.c.Kähler manifold obtained in [3] and [5].

According (1.6), the tensor P_{ij} is hybrid if and only if the Ricci tensor ρ_{ij} is hybrid. Thus, we can state ([3], [5]):

Corollary 2.1. *L.c.Kähler manifold whose Ricci tensor is hybrid is of pointwise constant holomorphic sectional curvature if and only if its curvature tensor has the form (2.15).*

3. *The expression of the holomorphic curvature tensor which does not include the tensor P_{ij}*

We shall show that, if $n > 2$, the tensors HR and R in the relations (2.12)-(2.15) can be written in the form which include the second Ricci tensor instead of the tensor P_{ij}.

The second Ricci tensor is defined as follows

$$\overset{*}{\rho}_{jh} = R_{ijab} J_h^a J_k^b g^{ik}.$$

It has, for any almost Hermitian manifold, the property

$$\overset{*}{\rho}_{ab} J_i^a J_j^b = \overset{*}{\rho}_{ji}.$$

In general, it is not symmetric, but for l.c.Kähler manifold, it is. Namely, the relation (1.5) yields

$$(2n-3)P_{jh} - P_{ab} J_j^a J_h^b = \rho_{jh} - \overset{*}{\rho}_{jh} - P g_{jh}, \tag{3.1}$$

where

$$P = P_{ab} g^{ab} = \frac{\kappa - \overset{*}{\kappa}}{4(n-1)},$$

and

$$\kappa = \rho_{ab} g^{ab} \quad \text{and} \quad \overset{*}{\kappa} = \overset{*}{\rho}_{ab} g^{ab}$$

are the first and the second scalar curvatures.

From (3.1) immediately follows $\overset{*}{\rho}_{jh} = \overset{*}{\rho}_{hj}$. Thus, for any l.c.Kähler manifold, we have

$$\overset{*}{\rho}_{jh} = \overset{*}{\rho}_{hj}, \quad \overset{*}{\rho}_{ab} J_j^a J_h^b = \overset{*}{\rho}_{jh}.$$

The relation (3.1), together with

$$-P_{jh} + (2n-3)P_{ab}J_j^a J_h^b = \rho_{ab}J_j^a J_h^b - \overset{*}{\rho}_{jh} - Pg_{jh},$$

implies

$$4(n-1)(n-2)P_{jh} = (2n-3)\rho_{jh} + \rho_{ab}J_j^a J_h^b - 2(n-1)\overset{*}{\rho}_{jh} - 2(n-1)Pg_{jh}, \quad (3.2)$$

such that, if $n > 2$, we have

$$P_{jh} = \frac{1}{4(n-1)(n-2)}\Big\{(2n-3)\rho_{jh} + \rho_{ab}J_j^a J_h^b - 2(n-1)\overset{*}{\rho}_{jh} - 2(n-1)Pg_{jh}\Big\}. \quad (3.3)$$

Substituting (3.3) into (2.12) and (2.13), we get, respectively,

$$\begin{aligned}
(HR)_{ijhk} &= R_{ijhk} \\
&+ \frac{P}{4(n-2)}\{3(g_{ik}g_{jh} - g_{ih}g_{jk}) - (F_{ik}F_{jh} - F_{ih}F_{jk} - 2F_{ij}F_{hk})\} \\
&+ \frac{7n-11}{16(n-1)(n-2)}(g_{ih}\rho_{jk} + g_{jk}\rho_{ih} - g_{ik}\rho_{jh} - g_{jh}\rho_{ik}) \\
&- \frac{n-5}{16(n-1)(n-2)}(g_{ih}\rho_{ab}J_j^a J_k^b + g_{jk}\rho_{ab}J_i^a J_h^b - g_{ik}\rho_{ab}J_j^a J_h^b - g_{jh}\rho_{ab}J_i^a J_k^b) \\
&- \frac{3}{8(n-2)}(g_{ik}\overset{*}{\rho}_{jk} + g_{jk}\overset{*}{\rho}_{ih} - g_{ik}\overset{*}{\rho}_{jh} - g_{jh}\overset{*}{\rho}_{ik}) \\
&+ \frac{1}{16(n-2)}\Big\{F_{ih}(\rho_{ja}J_k^a - \rho_{ka}J_j^a - 2\overset{*}{\rho}_{ja}J_k^a) \\
&\quad + F_{jk}(\rho_{ia}J_h^a - \rho_{ha}J_i^a - 2\overset{*}{\rho}_{ia}J_h^a) \\
&\quad - F_{ik}(\rho_{ja}J_h^a - \rho_{ha}J_j^a - 2\overset{*}{\rho}_{ja}J_h^a) \\
&\quad - F_{jh}(\rho_{ia}J_k^a - \rho_{ka}J_i^a - 2\overset{*}{\rho}_{ia}J_k^a) \\
&\quad + 2F_{ij}(\rho_{ha}J_k^a - \rho_{ka}J_h^a - 2\overset{*}{\rho}_{ha}J_k^a) \\
&\quad + 2F_{hk}(\rho_{ia}J_j^a - \rho_{ja}J_i^a - 2\overset{*}{\rho}_{ia}J_j^a)\Big\}
\end{aligned} \quad (3.4)$$

and

$$
\begin{aligned}
R_{ijhk} = {} & \frac{1}{4}(H - \frac{3P}{n-2})(g_{ik}g_{jh} - g_{ih}g_{jk}) \\
& + \frac{1}{4}(H + \frac{P}{n-2})(F_{ik}F_{jh} - F_{ih}F_{jk} - 2F_{ij}F_{hk}) \\
& + \frac{7n-11}{16(n-1)(n-2)}(g_{ik}\rho_{jh} + g_{jh}\rho_{ik} - g_{ih}\rho_{jk} - g_{jk}\rho_{ih}) \\
& - \frac{n-5}{16(n-1)(n-2)}(g_{ik}\rho_{ab}J_j^aJ_h^b + g_{jh}\rho_{ab}J_i^aJ_k^b - g_{ih}\rho_{ab}J_j^aJ_k^b - g_{jk}\rho_{ab}J_i^aJ_h^b) \\
& - \frac{3}{8(n-2)}(g_{ik}\overset{*}{\rho}_{jh} + g_{jh}\overset{*}{\rho}_{ik} - g_{ih}\overset{*}{\rho}_{jk} - g_{jk}\overset{*}{\rho}_{ih}) \\
& + \frac{1}{16(n-2)}\Big\{F_{ik}(\rho_{ja}J_h^a - \rho_{ha}J_j^a - 2\overset{*}{\rho}_{ja}J_h^a) \\
& \quad + F_{jh}(\rho_{ia}J_k^a - \rho_{ka}J_i^a - 2\overset{*}{\rho}_{ia}J_k^a) \\
& \quad - F_{ih}(\rho_{ja}J_k^a - \rho_{ka}J_j^a - 2\overset{*}{\rho}_{ja}J_k^a) \\
& \quad - F_{jk}(\rho_{ia}J_h^a - \rho_{ha}J_i^a - 2\overset{*}{\rho}_{ia}J_h^a) \\
& \quad - 2F_{ij}(\rho_{ha}J_k^a - \rho_{ka}J_h^a - 2\overset{*}{\rho}_{ha}J_k^a) \\
& \quad - 2F_{hk}(\rho_{ia}J_j^a - \rho_{ja}J_i^a - 2\overset{*}{\rho}_{ia}J_j^a)\Big\} .
\end{aligned}
\tag{3.5}
$$

Thus, we can state

Theorem 3.1. *If $n > 2$, the holomorphic curvature tensor of a l.c.Kähler manifold can be expressed in the form (3.4). If such a manifold has pointwise constant holomorphic sectional curvature, its Riemannian curvature tensor can be expressed in the form (3.5).*

Remark. If $n = 2$, we can not eliminate the tensor P_{ij} from the equations (2.12)-(2.15). On the other hand, according (3.2), we have

$$\rho_{jh} + \rho_{ab}J_j^aJ_h^b = 2\overset{*}{\rho}_{jh} + 2Pg_{jh},$$

and, if ρ_{jh} is hybrid,

$$\rho_{jh} - \overset{*}{\rho}_{jh} = Pg_{jh}.$$

4. *Geodesic mapping*

The diffeomorphism transforming all geodesics of the Riemannian manifold $(\bar{M}, \bar{g})$ onto the geodesics of the Riemannian manifold (M, g) is said to be the geodesic mapping. If we consider the manifolds $\bar{M}$ and M with respect to the coordinate system which is common for $\bar{M}$ and M, then the necessary and the sufficient condition for geodesic mapping in [6]

$$\bar{\Gamma}^t_{ij} = \Gamma^t_{ij} + \delta^t_i \psi_j + \delta^t_j \psi_i. \tag{4.1}$$

where $\bar{\Gamma}$ and Γ are the Levi-Civita connections of the metrics $\bar{g}$ and g respectively, and ψ_i is the gradient vector field. As for the curvature tensors $\bar{R}$ and R, they are related as follows

$$\bar{R}^t{}_{jhk} = R^t{}_{jhk} + \delta^t_k \psi_{jh} - \delta^t_h \psi_{jk} \tag{4.2}$$

where

$$\psi_{jh} = \nabla_h \psi_j - \psi_j \psi_h.$$

Because ψ_j is a gradient, we have $\psi_{jh} = \psi_{hj}$.

Now, let us consider the geodesic mapping

$$f : (\bar{M}, \bar{g}, J) \to (M, g, J)$$

of an almost Hermitian manifold $(\bar{M}, \bar{g}, J)$ onto an almost Hermitian manifold (M, g, J).

For $(\bar{M}, \bar{g}, J)$, the holomorphic curvature tensor is

$$\begin{aligned}
&(H\bar{R})^t{}_{jhk} = \\
&\frac{1}{16}\Big\{3\Big[\bar{R}^t{}_{jhk} + \bar{R}^t{}_{jab} J^a_h J^b_k - \bar{R}^a{}_{bhk} J^t_a J^b_j - \bar{R}^a{}_{bcd} J^t_a J^b_j J^c_h J^d_k\Big] \\
&-\bar{R}^t{}_{hab} J^a_k J^b_j + \bar{R}^a{}_{bkj} J^t_a J^b_h - \bar{R}^t{}_{kab} J^a_j J^b_h + \bar{R}^a{}_{bjh} J^t_a J^b_k \\
&-\bar{R}^a{}_{hbj} J^t_a J^b_k + \bar{R}^t{}_{akb} J^a_h J^b_j - \bar{R}^a{}_{kjb} J^t_a J^b_h + \bar{R}^t{}_{abh} J^a_k J^b_j\Big\} \ .
\end{aligned}$$

Substituting (4.2), we find

$$\begin{aligned}
8(H\bar{R})^t{}_{jhk} = 8(HR)^t{}_{jhk} \\
\delta^t_k Q_{jh} - \delta^t_h Q_{jk} + J^t_k Q_{ja} J^a_h - J^t_h Q_{ja} J^a_k - 2J^t_j Q_{ha} J^a_k,
\end{aligned} \tag{4.3}$$

where

$$Q_{jh} = \psi_{jh} + \psi_{ab} J^a_j J^b_h.$$

Contracting (4.3) with respect to t and k, we get

$$Q_{jh} = \frac{4}{n+1}\left[\rho(H\bar{R})_{jh} - \rho(HR)_{jh}\right], \tag{4.4}$$

where $\rho(H\bar{R})$ and $\rho(HR)$ are the Ricci tensors of $H\bar{R}$ and HR respectively. Substituting (4.4) into (4.3), we find

$$(H\bar{W})^t{}_{jhk} = (HW)^t{}_{jhk},$$

where

$$\begin{aligned}(HW)^t{}_{jhk} &= (HR)^t{}_{jhk}\\ &-\frac{1}{2(n+1)}\Big\{\delta^t_k\rho(HR)_{jh} - \delta^t_h\rho(HR)_{jk}\\ &+J^t_k\rho(HR)_{ja}J^a_h - J^t_h\rho(HR)_{ja}J^a_k - 2J^t_j\rho(HR)_{ha}J^a_k\Big\},\end{aligned} \tag{4.5}$$

and $(H\bar{W})^t{}_{jhk}$ is constructed in the same way, but with respect to the tensor $H\bar{R}$.

Thus we can state

Theorem 4.1. *For an almost Hermitian manifold, the tensor (4.5) is invariant with respect to the geodesic mapping (4.1), preserving the complex structure.*

Now, let us suppose that $HW = 0$. Then

$$\begin{aligned}(HR)_{ijhk} &= \frac{1}{2(n+1)}\left[g_{ik}\rho(HR)_{jh} - g_{ih}\rho(HR)_{jk}\right.\\ &\left.-F_{ik}\rho(HR)_{ja}J^a_h + F_{ih}\rho(HR)_{ja}J^a_k + 2F_{ij}\rho(HR)_{ha}J^a_k\right],\end{aligned} \tag{4.6}$$

from which, transvecting with g^{jh} and taking into account that, as a consequence of (2.4), the Ricci tensor $\rho(HR)$ is hybrid, we get

$$\rho(HR)_{ik} = \frac{\kappa(HR)}{2n}g_{ik},$$

where $\kappa(HR) = \rho(HR)_{jh}g^{jh}$.

Substituting this into (4.6), we obtain

$$(HR)_{ijhk} = \frac{\kappa(HR)}{4n(n+1)}(g_{ik}g_{jh} - g_{ih}g_{jk} + F_{ik}F_{jh} - F_{ih}F_{jk} - 2F_{ij}F_{hk}).$$

But, this is just the equation (2.6). Conversely, if (2.6) holds, the tensor (4.5) vanishes. Thus, we can state

Theorem 4.2. *For an almost Hermitian manifold (M, g, J) the tensor (4.5) vanishes if and only if (M, g, J) is of pointwise constant holomorphic sectional curvature.*

The l.c.Kähler manifold can serve as an example. Indeed, for such a manifold the holomorphic curvature tensor has the form (2.12). Therefore

$$\rho(HR)_{ik} = \rho_{ik} - \frac{3}{4}Pg_{ik} - \frac{7n-10}{4}P_{ik} + \frac{n+2}{4}P_{ab}y_i^a y_k^b.$$

Substituting this and (2.12) into (4.5), we find that for l.c.Kähler manifold, the tensor $(HW)_{ijhk}$ is

$$\begin{aligned}
&(HW)_{ijhk} = R_{ijhk} \\
&+\frac{1}{8}\Big\{g_{ih}(7P_{jk} - P_{ab}J_j^a J_k^b) - g_{ik}(7P_{jh} - P_{ab}J_j^a J_h^b) \\
&\quad g_{jk}(7P_{ih} - P_{ab}J_i^a J_h^b) - g_{jh}(7P_{ik} - P_{ab}J_i^a J_k^b) \\
&\quad +F_{ih}(P_{ja}J_k^a - P_{ka}J_j^a) - F_{ik}(P_{ja}J_h^a - P_{ha}J_j^a) \\
&\quad +F_{jk}(P_{ia}J_h^a - P_{ha}J_i^a) - F_{jh}(P_{ia}J_k^a - P_{ka}J_i^a) \\
&\quad +2F_{ij}(P_{ha}J_k^a - P_{ka}J_h^a) + 2F_{hk}(P_{ia}J_j^a - P_{ja}J_i^a)\Big\} \\
&-\frac{1}{2(n+1)}\Big\{g_{ik}(\rho_{jh} - \frac{3}{4}Pg_{jh} - \frac{7n-10}{4}P_{jh} + \frac{n+2}{4}P_{ab}J_j^a J_h^b) \\
&\quad -g_{ih}(\rho_{jk} - \frac{3}{4}Pg_{jk} - \frac{7n-10}{4}P_{jk} + \frac{n+2}{4}P_{ab}J_j^a J_k^b) \\
&\quad -F_{ik}(\rho_{ja} - \frac{3}{4}Pg_{ja} - \frac{7n-10}{4}P_{ja} + \frac{n+2}{4}P_{st}J_j^s J_a^t)J_h^a \\
&\quad +F_{ih}(\rho_{ja} - \frac{3}{4}Pg_{ja} - \frac{7n-10}{4}P_{ja} + \frac{n+2}{4}P_{st}J_j^s J_a^t)J_k^a \\
&\quad +2F_{ij}(\rho_{ha} - \frac{3}{4}Pg_{ha} - \frac{7n-10}{4}P_{ha} + \frac{n+2}{4}P_{st}y_h^s y_a^t)y_k^a\Big\}.
\end{aligned} \tag{4.7}$$

If this tensor vanishes, then

$$\begin{aligned}
(2n+3)\rho_{ik} - 3\rho_{ab}J_i^a J_k^b &= \frac{7n^2+2n-12}{2}P_{ik} - \frac{n^2+14n-12}{2}P_{ab}J_i^a J_k^b \\
&\left[\kappa - \frac{3(n-2)}{2}P\right] g_{ik}.
\end{aligned}$$

This relation, together with

$$-3\rho_{ik} + (2n+3)\rho_{ab}J_i^aJ_k^b = -\frac{n^2+14n-12}{2}P_{ik}$$
$$+\frac{7n^2+2n-12}{2}P_{ab}J_i^aJ_k^b + \left[\kappa - \frac{3(n-2)}{2}P\right]g_{ik},$$

yields

$$\rho_{ik} - \frac{3}{4}Pg_{ik} - \frac{7n-10}{4}P_{ik} + \frac{n+2}{4}P_{ab}J_i^aJ_k^b$$
$$= \frac{1}{2n}\left[\kappa - 3(n-1)P\right]g_{ik},$$

and the relation (4.7), if $HW = 0$, gives

$$R_{ijhk} = \frac{\kappa - 3(n-1)P}{4n(n+1)}(g_{ik}g_{jh} - g_{ih}g_{jk} + F_{ik}F_{jh} - F_{ih}F_{jk} - 2F_{ij}F_{hk})$$
$$\frac{1}{8}\Big\{g_{ik}(7P_{jh} - P_{ab}J_j^aJ_h^b) + g_{jh}(7P_{ik} - P_{ab}J_i^aJ_k^b)$$
$$-g_{ih}(7P_{jk} - P_{ab}J_j^aJ_k^b) - g_{jk}(7P_{ih} - P_{ab}J_i^aJ_h^b)$$
$$+F_{ik}(P_{ja}J_h^a - P_{ha}J_j^a) + F_{jh}(P_{ia}J_k^a - P_{ka}J_i^a)$$
$$-F_{ih}(P_{ja}J_k^a - P_{ka}J_j^a) - F_{jk}(P_{ia}J_h^a - P_{ha}J_i^a)$$
$$-2F_{ij}(P_{ha}J_k^a - P_{ka}J_h^a) - 2F_{hk}(P_{ia}J_j^a - P_{ja}J_i^a)\Big\}.$$

But, this is just the relation (2.13), where

$$H(p) = \frac{\kappa - 3(n-1)P}{n(n+1)} = \frac{\kappa + 3\overset{*}{\kappa}}{4n(n+1)}.$$

Thus, and also as a consequence of theorems 4.1 and 4.2, we have

Theorem 4.3 *Let (M, g, J) be l.c.Kähler manifold. Then the tensor (4.7) is invariant with respect to the geodesic mapping. The tensor (4.7) vanishes if and only if (2.13) holds.*

5. *The remark on Kähler manifolds*

A curve $x^i = x^i(t)$ of an almost Hermitian manifold (M, g, J) having the property that the holomorphic planes determined by its tangent vectors are parallel along the curve itself is called a holomorphically planer curve. The manifold $(M, \bar{g}, J)$ and (M, g, J) are said to be holomorphically projectively related if they have all holomorphically planer curves in common, and the corresponding invariant tensor is holomorphically projective curvature tensor. In the case of Kähler manifolds, this tensor is [8]:

$$R^i{}_{jhk} - \frac{1}{2(n+1)} \left(\delta^t_k \rho_{jh} - \delta^t_h \rho_{jk} + J^t_k \rho_{ja} J^a_h - J^t_h \rho_{ja} J^a_k - 2J^t_j \rho_{ha} J^a_k \right) \tag{5.1}$$

On the other hand, if (M, g, J) is a Kähler manifold, then

$$(HR)^t{}_{jhk} = R^t{}_{jhk}, \quad \rho(HR)_{jh} = \rho_{jh},$$

such that (4.5) reduces just to the tensor (5.1).

Thus, in the case of Kähler manifolds, the tensor (5.1) can be obtained considering holomorphically planer curves or applying geodesic mapping to the holomorphic curvature tensor (2.2).

This does not hold for l.c.Kähler manifolds.

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