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WALKS ON COUNTABLE ORDINALS AND SELECTIVE ULTRAFILTERS

S. TODORČEVIĆ

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A b s t r a c t. *In our previous work we have introduced filters on the set of countable ordinals as invariants to standard characteristics of walks in this domain. In this note we examine their projections to the set of natural numbers.*

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0. *Introduction*

Recall that an *ordinal* β in the Von Neumann sense is identified with the set $\{\alpha : \alpha < \beta\}$ of smaller ordinals. Thus,

$$0 = \emptyset,\ 1 = \{0\},\ 2 = \{0, 1\}, ...,\ \omega = \{0, 1, 2, ...\}.$$

This abstraction of the notion of a natural number (which is nothing else than a finite ordinal) has found many uses in mathematics. We shall address here a particularly useful line of research that tries to relate an arbitrary countable ordinal α with the set ω of natural numbers in some uniform and coherent way. Recall that the *Cantor normal form* for an ordinal α is the expression

$$\alpha = n_1\omega^{\alpha_1} + n_2\omega^{\alpha_2} + \cdots + n_k\omega^{\alpha_k},$$

where $\alpha_1 > \alpha_2 > \cdots > \alpha_k \geq 0$ are ordinals and where $n_1, n_2, ..., n_k$ are natural numbers. The form is an abstraction of the standard notion of decimal representation of rational numbers and is made for the purpose to reduce questions about ordinals to questions about natural numbers and vice versa. This normal form is particularly useful for ordinals in the class

$$\varepsilon_0 = \{\alpha : \alpha < \omega^\alpha\}$$

since in this case the ordinals $\alpha_1 > \alpha_2 > \cdots > \alpha_k$ appearing in the Cantor normal form of α are all smaller than α which greatly facilitate recursive definitions and constructions. For example, this is directly responsible for the fact that the normal form is giving us, for each limit ordinal $\alpha < \varepsilon_0$, a canonical choice for a *fundamental sequence* $C_\alpha = \{c_\alpha(n) : n < \omega\}$ for α, an increasing sequence of smaller ordinals converging to α. Here we need to extend this choice to all other countable ordinals α. Thus, for the rest of this note, we fix for each countable ordinal α, a sequence

$$C_\alpha = \{c_\alpha(n) : n < \omega\} \subseteq \alpha$$

such that $c_{\alpha+1}(n) = \alpha$ for all n and such that $c_\alpha(n) < c_\alpha(n+1)$ for all n and $\alpha = \lim_{n\to\infty} c_\alpha(n)$ when α is limit. To any such choice of fundamental sequences C_α we associate the notion of a *walk* from a given countable ordinal β to a lower ordinal α, a finite decreasing sequence

$$\beta = \beta_0 \curvearrowright \beta_1 \curvearrowright \cdots \curvearrowright \beta_k = \alpha$$

of countable ordinals such that for all $i < k$,

$$\beta_{i+1} = c_{\beta_i}(n_i) \text{ where } n_i = \min\{n : c_{\beta_i}(n) \geq \alpha\}.$$

The integer k, denoted usually by $\rho_2(\alpha, \beta)$, is the *length* of the walk and can be formally defined by the recursive formula

$$\rho_2(\alpha, \beta) = \rho_2(\alpha, c_\beta(n(\alpha, \beta))) + 1$$

with the boundary value $\rho_2(\alpha, \alpha) = 0$, where

$$n(\alpha, \beta) = \min\{n : c_\beta(n) \geq \alpha\},$$

i.e where $\beta \curvearrowright c_\beta(n(\alpha,\beta))$ is the first step of the walk from β to α. The finite sequence $\rho_0(\alpha,\beta) = \langle n_0, n_1, ..., n_{k-1}\rangle$ of integers is the *full code* of the walk and the corresponding function

$$\rho_0 : [\omega_1]^2 \to \omega^{<\omega}$$

is an object from which one can draw many canonical structures (see [1], [2], [4], [6]). Its formal recursive definition is

$$\rho_0(\alpha,\beta) = \langle n(\alpha,\beta)\rangle^\frown \rho_0(\alpha, c_\beta(n(\alpha,\beta)))$$

with the boundary value $\rho_0(\alpha,\alpha) = \emptyset$. The number

$$\rho_1(\alpha,\beta) = \max\{n_0, n_1, ..., n_{k-1}\}$$

is the *maximal weight* of the walk and the corresponding function

$$\rho_1 : [\omega_1]^2 \to \omega$$

is another such canonical object. It is recursively given by the formula

$$\rho_1(\alpha,\beta) = \max\{\, n\ (\alpha,\beta), \rho_1(\alpha, c_\beta(n(\alpha,\beta)))$$

with the boundary value $\rho_1(\alpha,\alpha) = 0$. As indicated above there are other characteristics of the walk but we mention only two more because of their relevance to the discussion below. The first of these is the characteristic

$$\rho_3 : [\omega_1]^2 \to \{0,1\}$$

determined by setting

$$\rho_3(\alpha,\beta) = 1 \text{ if and only if } n_{k-1} = \max\{n_0, n_1, ..., n_{k-1}\},$$

i. e. whenever the last step $\beta_{k-1} \curvearrowright \beta_k = \alpha$ realizes the maximal weight. Its recursive definition is given by the formula

$$\rho_3(\alpha,\beta) = \begin{cases} \rho_3(\alpha, c_\beta(n(\alpha,\beta))) & \text{if } n(\alpha,\beta) \le \rho_1(\alpha, c_\beta(n(\alpha,\beta))), \\ 0 & \text{if } n(\alpha,\beta) > \rho_1(\alpha, c_\beta(n(\alpha,\beta))), \end{cases}$$

with the boundary values $\rho_3(\alpha,\alpha) = 0$ and $\rho_3(\alpha,\beta) = 1$ whenever $\alpha \in C_\beta$. The last one that we include in our discussion here is the characteristic $\rho : [\omega_1]^2 \to \omega$ defined by the recursive formula

$$\rho(\alpha,\beta) = \max \begin{cases} n(\alpha,\beta), & \\ \rho(\alpha, c_\beta(n(\alpha,\beta))), & \\ \rho(c_\beta(n),\alpha) & n < n(\alpha,\beta), \end{cases}$$

with the initial value $\rho(\alpha,\alpha)=0$. Each of these characteristics $a=\rho,\rho_0,\rho_1,\rho_2,\rho_3$ is giving us the corresponding distance function, a function $\Delta_a : [\omega_1]^2 \to \omega_1 \cup \{\infty\}$ defined by

$$\Delta_a(\alpha,\beta)=\min\{\xi<\alpha : a(\xi,\alpha)\neq a(\xi,\beta)\}^1$$

which has a strong Lipschitz properties quite useful in applications. For example, in [5] we showed that the strong Lipschitz properties are responsible for the fact that the family

$$\{\Delta_a[X] : X\subseteq\omega_1 \text{ and } X \text{ is uncountable}\}$$

generates a uniform filter $\mathcal{U}_a$ on ω_1, where for $X\subseteq\omega_1$, we set

$$\Delta_a[X]=\{\Delta_a(\alpha,\beta):\alpha,\beta\in X,\alpha<\beta \text{ and } \Delta_a(\alpha,\beta)\neq\infty\}.$$

This turns out to be a quite important invariant that captures many properties of a. Using the Baire category assumption $\mathfrak{m}>\omega_1$, we have shown in [5] that in each of the five cases $a=\rho,\rho_0,\rho_1,\rho_2,\rho_3$ (and more), the filter $\mathcal{U}_a$ is in fact maximal, i.e. an ultrafilter on the set ω_1 of countable ordinals. The purpose of this note is to examine the corresponding Rudin-Keisler projections of these ultrafilters on the set ω of natural numbers, i.e. ultrafilters of the form

$$f[\mathcal{U}_a]=\{X\subseteq\omega : f^{-1}(X)\in\mathcal{U}_a\},$$

for some $f:\omega_1\to\omega$. Of course we shall be interested in such functions $f_a:\omega_1\to\omega$ for which the corresponding ultrafilter

$$\mathcal{V}_a=f_a[\mathcal{U}_a]$$

is not principal and we will show that, modulo a permutation of the index-set ω, the non-principal ultrafilter $\mathcal{V}_a=f_a[\mathcal{U}_a]$ does not depend on the choice of the mapping $f_a:\omega_1\to\omega$. Since ω_1 cannot carry a countably complete non-principal ultrafilter such an $f_a:\omega_1\to\omega$ exists for every of the five characteristics $a=\rho,\rho_0,\rho_1,\rho_2,\rho_3$ of the walk on ω_1 discussed above. In some cases however there is a natural choice for such a map f_a. For example, in case of the characteristic $a=\rho_3$ the following map $d_\Lambda:\omega_1\to\omega$ transfers the ultrafilter $\mathcal{U}_a$ to a non principal ultrafilter $\mathcal{V}_a=d_\Lambda[\mathcal{U}_a]$ on ω,

$$d_\Lambda(\alpha)=|\alpha-\lambda(\alpha)| \text{ where } \lambda(\alpha)=\max\{\lambda:\lambda\leq\alpha,\lambda \text{ limit}\}.^2$$

[1]Here we let $\min\emptyset=\infty$. It will also be conventient to put that $\Delta_a(\alpha,\alpha)=\infty$.

[2]Thus, $d_\Lambda(\alpha)$ is simply the distance from the ordinal α to the set Λ of countable limit ordinals.

This however requires the following assumptions on the choice of fundamental sequences C_α :

1. $c_\alpha(n) = \lambda + n + 1$ whenever $\alpha = \lambda + \omega$ for some $\lambda \in \Lambda \cup \{0\}$,

2. $c_\alpha(n) = \lambda_n + n + 1$ for some increasing sequence λ_n of limit ordinals converging to α whenever α is a limit of limit ordinals.

One of the results that we prove in this note is that under the same Baire category assumption $\mathfrak{m} > \omega_1$, the projections $\mathcal{V}_a = d_\Lambda[\mathcal{U}_a]$ is a *selective* ultrafilter on ω. Recall that an ultrafilter $\mathcal{V}$ on ω is *selective* if for every map $h : \omega \to \omega$, either h is constant on some set X in $\mathcal{V}$ or h is one-to-one on some set X in $\mathcal{V}$. This is an important class of ultrafilters with uses in other areas of mathematics such as, for example, Ramsey theory (see [7]). The Baire category assumption $\mathfrak{m} > \omega_1$ seems first of its kind that guarantees their existence.

1. *Coherent and Lipschitz Mappings*

By a *coherent mapping* in this note we mean a mapping $a : [\omega_1]^2 \to \omega$ such that:

1. $\{\xi < \alpha : a(\xi, \alpha) \neq a(\xi, \beta)\}$ is a finite set for all $\alpha < \beta < \omega_1$,

2. for every uncountable $X \subseteq \omega_1$ there is uncountable $X_0 \subseteq X$ such that $\Delta_a(\alpha, \beta) \neq \infty$ for all $\alpha, \beta \in X_0, \alpha < \beta$.

It follows in particular that the corresponding (partial) distance function $\Delta_a : [\omega_1]^2 \to \omega_1$ has a highly Lipschitz behavior in the following precise sense.

Lemma 1.1. *[5] For every coherent mapping $a : [\omega_1]^2 \to \omega$, every positive integer n, and every uncountable family $\mathcal{F}$ of pairwise disjoint n-element subsets of ω_1 there is an uncountable $\mathcal{F}_0 \subseteq \mathcal{F}$ such that for all $s, t \in \mathcal{F}_0, s \neq t$ and all $i, j < n$, $\Delta_a(s(i), t(i)) = \Delta_a(s(j), t(j)) \neq \infty$.*[3]

Corollary 1.2. *[5] If $a : [\omega_1]^2 \to \omega$ is a coherent mapping then for every pair X and Y of uncountable subsets of ω_1 there is an uncountable subset Z*

[3] For an n-element set $t \subseteq \omega_1$, we let $t(i)$ $(i < n)$ be its increasing enumeration according to the natural order of ω_1.

of X such that $\Delta_a[Z] \subseteq \Delta_a[X] \cap \Delta_a[Y]$.

It turns out that these Lipschitz properties characterize coherent mappings in the following precise sense.

Lemma 1.3. *[5]The following are equivalent assuming $\mathfrak{m} > \omega_1$:*

1. *$a : [\omega_1]^2 \to \omega$ is a coherent mapping.*
2. *For every uncountable family $\mathcal{F}$ of pairwise disjoint 2-element subsets of ω_1 there is an uncountable $\mathcal{F}_0 \subseteq \mathcal{F}$ such that for all $s, t \in \mathcal{F}_0$, $s \neq t$, $\Delta_a(s(0), t(0)) = \Delta_a(s(1), t(1)) \neq \infty$.*

We call maps $a : [\omega_1]^2 \to \omega$ satisfying the condition (2) of Lemma 1.3 *Lipschitz maps.*

Lemma 1.4. *[6] The maps ρ, ρ_1 and ρ_3 are coherent.*

Lemma 1.5. *The maps ρ_0 and ρ_2 are Lipschitz.*

P r o o f. Let $\mathcal{F}$ be a given uncountable family of pairwise disjoint unordered pairs of countable ordinals. We shall use another characteristic of walks on ω_1, the *full lower trace* $F : [\omega_1]^2 \to [\omega_1]^{<\omega}$ given by the recursive formula

$$F(\alpha, \beta) = F(\alpha, c_\beta(n(\alpha, \beta))) \cup \bigcup_{n<n(\alpha,\beta)} F(c_\beta(n), \alpha)$$

with the boundary value $F(\alpha, \alpha) = \{\alpha\}$. We shall need only the following property of the full lower trace, where $\alpha \leq \beta \leq \gamma$ are three countable ordinals (see [6], Lemma 2.1.9):

(a) $\rho_0(\alpha, \beta) = \rho_0(\min(F(\beta, \gamma) \setminus \alpha), \beta)^\frown \rho_0(\alpha, \min(F(\beta, \gamma) \setminus \alpha))$,

(b) $\rho_0(\alpha, \gamma) = \rho_0(\min(F(\beta, \gamma) \setminus \alpha), \gamma)^\frown \rho_0(\alpha, \min(F(\beta, \gamma) \setminus \alpha))$.

In other words, the walk from β to α and the walk from γ to α both pass through the point $\xi = \min(F(\beta, \gamma) \setminus \alpha)$ and after that they coincide with the walk from ξ to α.

Back to our uncountable family $\mathcal{F}$ of pairwise disjoint unordered pairs. For $t \in \mathcal{F}$, let $F(t) = F(t(0), t(1))$. Applying the Δ-system lemma and going to an uncountable subfamily, we may assume that there is a finite set $F \subseteq \omega_1$ such that

$$F(s) \cap F(t) = F \text{ for all } s, t \in \mathcal{F}, s \neq t.$$

Let $\bar{\xi} = \max(F) + 1$, and for $t \in \mathcal{F}$, let $\xi(t) = \min(F(t) \setminus \bar{\xi})$. Refining $\mathcal{F}$ even further we may assume that for some fixed integers $k(0)$ and $k(1)$ and all $t \in \mathcal{F}$,

$$k(i) = \rho_2(\xi(t), t(i)) \text{ for all } i < 2.$$

Relying on the fact that the trees $T(\rho_0)$ and $T(\rho_2)$ associated to the mappings ρ_0 and ρ_2, respectively, can't contain an uncountable subset with no uncountable antichain (see [6], Chapter 2), we can find uncountable $\mathcal{F}_0 \subseteq \mathcal{F}$ such that the following two conditions hold for both $a = \rho_0$ and $a = \rho_2$:

(c) $a(\xi, s(i)) = a(\xi, t(i))$ for all $s, t \in \mathcal{F}_0$, $\xi < \bar{\xi}$ and $i < 2$,

(d) for all $s, t \in \mathcal{F}_0$ and $i < 2$ there is $\xi < \min\{\xi(s), \xi(t)\}$ such that $a(\xi, s(i)) \neq a(\xi, t(i))$.

From this and the two properties of the full lower trace listed above, it follows that for all $s, t \in \mathcal{F}_0, s \neq t$ and $a = \rho_0, \rho_2$,

$$\Delta_a(s(0), t(0)) = \min\{\xi : a(\xi, \xi(s)) \neq a(\xi, \xi(t))\} = \Delta_a(s(1), t(1)).$$

This finishes the proof. □

It follows that the characteristics ρ, ρ_0, ρ_1, ρ_2 and ρ_3 of the walks on ω_1 discussed above are all either coherent or Lipschitz or both. Since we have Lemma 1.3, we shall not make the distinction between the notions of coherent and Lipschitz below and just use the term *coherent* all the time. Following [5] to any coherent mapping $a : [\omega_1]^2 \to \omega$ we attach the following family of subsets of ω_1,

$$\mathcal{U}_a = \{Y \subseteq \omega_1 : (\exists X \subseteq \omega_1)\ [X \text{ is uncountable and } \Delta_a[X] \subseteq Y]\}.$$

Lemma 1.6. *[5] The family $\mathcal{U}_a$ is a uniform filter on ω_1, for every coherent mapping $a : [\omega_1]^2 \to \omega$.*

P r o o f. This is an immediate consequence of Corollary 1.2. □

Recall that $\mathfrak{m}$ is the minimal cardinality of a family of nowhere dense sets that could cover a compact space satisfying the countable chain condition.

Lemma 1.7. *[5] If $\mathfrak{m} > \omega_1$ then for every coherent $a : [\omega_1]^2 \to \omega$ the family $\mathcal{U}_a$ is a uniform ultrafilter on ω_1.*

Lemma 1.8. *Let $a : [\omega_1]^2 \to \omega$ be a coherent mapping, let $f : \omega_1 \to \omega$, and assume $\mathfrak{m} > \omega_1$. Then $f[\mathcal{U}_a]$ is a selective ultrafilter on ω.*

P r o o f. We may assume that $\mathcal{V} = f[\mathcal{U}_a]$ is non-principal since clearly principal ultrafilters are selective. To check the selectivity of $\mathcal{V}$, let $h : \omega \to \omega$ be a given mapping which is not constant on any set belonging to $\mathcal{V}$. We need to find $M \in \mathcal{V}$ such that $h \restriction M$ is one-to-one. This will be done by constructing an uncountable $X \subseteq \omega_1$ such that h is one-to-one on the set

$$M_f[X] = \{f(\Delta_a(\alpha, \beta)) : \alpha, \beta \in X, \alpha < \beta, \text{ and } \Delta_a(\alpha, \beta) \neq \infty\}.$$

Clearly, any such a set $M_f[X]$ belongs to the ultrafilter $\mathcal{V}$. To this end, let $\mathcal{P}$ be the collection of all finite subsets p of ω_1 such that h is one to one on the set

$$M_f[p] = \{f(\Delta_a(\alpha, \beta)) : \alpha, \beta \in p, \alpha < \beta, \text{ and } \Delta_a(\alpha, \beta) \neq \infty\}.$$

We consider $\mathcal{P}$ as a partially ordered set ordered by inclusion. Note that if $\mathcal{P}$ satisfies the countable chain condition our Baire category assumption $\mathfrak{m} > \omega_1$ would give us an uncountable $\mathcal{F} \subseteq \mathcal{P}$ such that $p \cup q \in \mathcal{P}$ for all $p, q \in \mathcal{F}$. Taking $X = \bigcup \mathcal{F}$ gives us an uncountable subset of ω_1 such that h is one-to-one on the corresponding set $M_f[X]$ which, as we know, belongs to the ultrafilter $\mathcal{V}$. To check the countable chain condition of $\mathcal{P}$ let $\mathcal{X}$ be an uncountable subset of $\mathcal{P}$. Refining $\mathcal{X}$ we may assume that $\mathcal{X}$ consists of n-element sets for some fixed positive integer n. By Lemma 1.1 and by the Δ-system lemma, we find uncountable $\mathcal{X}_0 \subseteq \mathcal{X}$ and an integer $n_0 < n$ such that for all $p, q \in \mathcal{X}_0$ such that $p \neq q$, we have that,

(3) $p(i) = q(i)$ for $i < n_0$ and $p(i) \neq q(i)$ for $n_0 \leq i < n$,

(4) $\Delta_a(p(i), q(i)) = \Delta_a(p(j), q(j))$ for $n_0 \leq i, j < n$.

For $p, q \in \mathcal{X}_0, p \neq q$, let $\Delta_a(p, q)$ denote the constant value of the sequence $\Delta_a(p(i), q(i))$ $(n_0 \leq i < n)$. Using the second property of the coherent mapping a and another Δ-system argument we arrive at an uncountable set $\mathcal{X}_1 \subseteq \mathcal{X}_0$ such that for all $p, q \in \mathcal{X}_1, p \neq q$,

$$\Delta_a[p \cup q] = \Delta_a[p] \cup \Delta_a[q] \cup \{\Delta_a(p, q)\}.$$

Find an integer k and uncountable $\mathcal{X}_2 \subseteq \mathcal{X}_1$ such that for all $p, q \in \mathcal{X}_2$,

$$f[\Delta_a(p)] = f[\Delta_a(p)] \subseteq \{0, 1, ..., k\}.$$

Now, find an integer ℓ such that $h(i) \leq \ell$ for all $i \leq k$. Let $Z = f^{-1}(\{0, 1, ..., \ell\})$. Then by our assumption about f the set Z does not belong to $\mathcal{U}_a$. By Corollary 1.2, there is an uncountable $\mathcal{X}_3 \subseteq \mathcal{X}_2$ such that

$$Z \cap \{\Delta_a(p, q) : p, q \in \mathcal{X}_3, p \neq q\} = \emptyset.$$

It follows that for arbitrary $p, q \in \mathcal{X}_3, p \neq q$ the function h is one-to-one on the set,

$$f[\Delta_a(p \cup q)] = f[\Delta_a[p]] \cup f[\Delta_a[q]] \cup \{f(\Delta_a(p, q))\}.$$

So, in particular $p \cup q \in \mathcal{P}$ for all $p, q \in \mathcal{X}_3$. This finishes the proof. □

Lemma 1.9. $\Lambda + k \notin \mathcal{U}_{\rho_3}$ *for all* $k < \omega$.

P r o o f. For a countable ordinal α, set

$$\mathrm{supp}(\rho_3)_\alpha = \{\xi < \alpha : \rho_3(\xi, \alpha) \neq 0\}.$$

Then by [6]; Lemma 2.4.9, for all $k < \omega$ and $\alpha < \omega_1$, the set

$$S_k(\alpha) = (\Lambda + k) \cap \mathrm{supp}(\rho_3)_\alpha$$

is finite. Fix an integer k. We need to find an uncountable set $X \subseteq \omega_1$ such that $\Delta_{\rho_3}[X] \cap (\Lambda + k) = \emptyset$. Toward this end, we apply the Δ-system lemma to the family $S_k(\alpha)$ $(\alpha < \omega_1)$ of finite sets and obtain an uncountable $X \subseteq \omega_1$ and a finite set S such that

$$S_k(\alpha) \cap S_k(\beta) = S \text{ for all } \alpha, \beta \in X, \alpha \neq \beta.$$

We also assume that if $\bar{\xi} = \max(S) + 1$, then

$$\rho_3(\xi, \alpha) = \rho_3(\xi, \beta) \text{ for all } \alpha, \beta \in X \text{ and } \xi < \bar{\xi}.$$

It follows that if for some uncountable $Y \subseteq X$, we have that $S_k(\alpha) = S$ for all $\alpha \in Y$, then $\Delta_{\rho_3}[Y]$ is a member of the filter $\mathcal{U}_{\rho_3}$ disjoint from $\Lambda + k$, as required. So we may assume that $S_k(\alpha) \setminus S \neq \emptyset$ for all $\alpha \in X$. So for $\alpha \in X$, we can define

$$\xi(\alpha) = \min(S_k(\alpha) \setminus S).$$

Then $\xi(\alpha) \neq \xi(\beta)$ for $\alpha, \beta \in X, \alpha \neq \beta$. So as before, using the fact that ρ_3 is a coherent map, we can go to an uncountable subset X_0 of X such that for all $\alpha, \beta \in X_0,\ \alpha \neq \beta$,

$$\bar{\xi} < \Delta_{\rho_3}(\alpha, \beta) < \min\{\xi(\alpha), \xi(\beta)\},$$

So, in particular $\Delta_{\rho_3}(\alpha, \beta) \notin (\Lambda + k)$ for all $\alpha, \beta \in X_0,\ \alpha \neq \beta$. It follows that $\Delta_{\rho_3}[X_0]$ is a member of $\mathcal{U}_{\rho_3}$ disjoint from $\Lambda + k$. This finishes the proof.

□

Recall, that $d_\Lambda : \omega_1 \to \omega$ is the distance function to the set Λ of countable limit ordinals. Then we have the following consequences of Lemma 1.9.

Corollary 1.10. *The filter $\mathcal{V}_{\rho_3} = d_\Lambda[\mathcal{U}_{\rho_3}]$ is non-principal.*

Recall that for any of the other four characteristics $a = \rho, \rho_0, \rho_1, \rho_2$ we can also find a mapping $f_a : \omega_1 \to \omega$ such that the corresponding projection $\mathcal{V}_a = f_a[\mathcal{U}_a]$ is a non-principal filter on ω. Applying Lemmas 1.7, 1.8 and 1.9 we get the following fact.

Corollary 1.11. *If $\mathfrak{m} > \omega_1$, the filters $\mathcal{V}_\rho$, $\mathcal{V}_{\rho_0}$, $\mathcal{V}_{\rho_1}$, $\mathcal{V}_{\rho_2}$, and $\mathcal{V}_{\rho_3}$ are all non-principal selective ultrafilters.*

Remark 1.12. It is interesting to compare this with [3] which contains a description of a *rapid filter*[4] on ω rather than a selective ultrafilter. While the notion of a rapid filter is considerably weaker than the notion of a selective ultrafilter, [3] uses a weaker Baire category assumption than $\mathfrak{m} > \omega_1$ (the assumption that the real line cannot be covered by a family N_α $(\alpha < \omega_1)$ of measure zero sets) to prove the rapidity of the filter. Assuming that that there are uncountably many real numbers that are relatively constructible (in the sense of Gödel) from a single real, both the rapid filter of [3] and our selective ultrafilter $\mathcal{V}_{\rho_3}$ have descriptions that belong to the third level of the projective hierarchy.

2. *Metric Equivalence of Coherent Mappings*

Two coherent mappings $a : [\omega_1]^2 \to \omega$ and $b : [\omega_1]^2 \to \omega$ are *metrically equivalent* if there is an uncountable $X \subseteq \omega_1$ such that

(i) $\Delta_a(\alpha, \beta) \neq \infty$ and $\Delta_b(\alpha, \beta) \neq \infty$ for all $\alpha, \beta \in X$ with $\alpha < \beta$,

(ii) for every quadruple $\alpha, \beta, \gamma, \delta \in X$ such that $\alpha < \beta$ and $\gamma < \delta$,

$$\Delta_a(\alpha, \beta) > \Delta_a(\gamma, \delta) \text{ if and only if } \Delta_b(\alpha, \beta) > \Delta_b(\gamma, \delta).$$

[4]Recall that a *rapid filter* on ω is a filter $\mathcal{F}$ with the property that for every strictly increasing sequence (n_k) of natural numbers there is X in $\mathcal{F}$ such that $|X \cap \{0, 1, ..., n_k\}| \leq k$ for all k. Clearly every selective ultrafilter is rapid.

Note that by Lemma 1.1, if two coherent mappings a and b are metrically equivalent then for every uncountable $Y \subseteq \omega_1$ there is an uncountable $X \subseteq Y$ witnessing the equivalence.

Remark 2.1. It turns out that under a slightly stronger Baire category assumption than $\mathfrak{m} > \omega_1$ (denoted below as $\mathfrak{mm} > \omega_1$), every pair of coherent mappings $a : [\omega_1]^2 \to \omega$ and $b : [\omega_1]^2 \to \omega$ are metrically equivalent (see [6]; Lemma 4.3.4). Our interest here in this notion is based on the following fact which relates the metric equivalence with the Rudin-Keisler equivalence of the corresponding filters $\mathcal{U}_a$ (see [6]; pp.117-118).

Lemma 2.2. *If two coherent mappings $a : [\omega_1]^2 \to \omega$ and $b : [\omega_1]^2 \to \omega$ are metrically equivalent then the corresponding uniform filters $\mathcal{U}_a$ and $\mathcal{U}_b$ are Rudin-Keisler equivalent, i.e. there is a bijection $f : \omega_1 \to \omega_1$ such that $f[\mathcal{U}_a] = \mathcal{U}_b$.*

P r o o f. Fix an uncountable set $X \subseteq \omega_1$ witnessing the metric equivalence of a and b. It suffices to find a bijection $f : \Delta_a[X] \to \Delta_b[X]$ which maps a set in $\mathcal{U}_a$ to a set in $\mathcal{U}_b$. Consider $\xi \in \Delta_a[X]$. Choose $\alpha, \beta \in X, \alpha \neq \beta$ such that $\xi = \Delta_a(\alpha, \beta)$. Let $f(\xi) = \Delta_b(\alpha, \beta)$. Note that this is a well-defined map, i.e that $f(\xi)$ does not depend on the choice of $\alpha, \beta \in X, \alpha \neq \beta$ such that $\xi = \Delta_a(\alpha, \beta)$. Since in this definition a and b play symmetric roles, we also have a well-defined inverse mapping $g : \Delta_b[X] \to \Delta_a[X]$. So in particular f is one-to-one (and, in fact, strictly increasing). Choose a generator $\Delta_a[Y]$ of $\mathcal{U}_a$. We need to show that the image $f[\Delta_a[Y]]$ belongs to $\mathcal{U}_b$. By Corollary 1.2, shrinking Y, we may assume that it is a subset of X. However, if this is true, then according to the definition of f, we have that

$$f[\Delta_a[Y]] = \Delta_b[Y]$$

which is clearly a member of $\mathcal{U}_b$. This finishes the proof. □

The purpose of this section is to examine the Rudin-Keisler projections to ω of the uniform filters associated to metrically equivalent coherent mappings.

Lemma 2.3. *Suppose $a : [\omega_1]^2 \to \omega$ and $b : [\omega_1]^2 \to \omega$ are two metrically equivalent mappings and that mappings $f : \omega_1 \to \omega$ and $g : \omega_1 \to \omega$ map $\mathcal{U}_a$ and $\mathcal{U}_b$ to two non-principal filters $f[\mathcal{U}_a]$ and $g[\mathcal{U}_b]$ on ω. Then if $\mathfrak{m} > \omega_1$, the selective ultrafilters $f[\mathcal{U}_a]$ and $g[\mathcal{U}_b]$ are equivalent.*

P r o o f. Fix uncountable $X \subseteq \omega_1$ witnessing the metric equivalence of

a and b. Let $\mathcal{P}$ be the collection of all pairs $p = (X_p, h_p)$, where

(a) X_p is a finite subset of X,

(b) h_p is a finite one-to-one map,

(c) $\mathrm{Dom}(h_p) = \{f(\Delta_a(\alpha, \beta)) : \alpha, \beta \in X_p\}$,

(d) $\mathrm{Rang}(h_p) = \{g(\Delta_b(\alpha, \beta)) : \alpha, \beta \in X_p\}$,

(e) $h_p(f(\Delta_a(\alpha, \beta))) = g(\Delta_b(\alpha, \beta))$ for $\alpha, \beta \in X_p, \alpha < \beta$.

We order $\mathcal{P}$ by coordinatewise inclusions. Note that if we show that $\mathcal{P}$ satisfies the countable chain condition then the Baire category assumption $\mathfrak{m} > \omega_1$ applied to $\mathcal{P}$ gives us an uncountable set $X_0 \subseteq X$ and a one-to-one map

$$h : \{f(\Delta_a(\alpha, \beta)) : \alpha, \beta \in X_0\} \to \{g(\Delta_b(\alpha, \beta)) : \alpha, \beta \in X_0\}$$

such that $h(f(\Delta_a(\alpha, \beta))) = g(\Delta_b(\alpha, \beta))$ for all $\alpha, \beta \in X_0, \alpha < \beta$. To see that h is a witness for the Rudin-Keisler equivalence between the selective ultrafilters $f[\mathcal{U}_a]$ and $g[\mathcal{U}_b]$, note that a typical generator of the ultrafilter $f[\mathcal{U}_a]$ is a set of the form $f[\Delta_a[Y]]$, where Y is an uncountable subset of X_0. Our mapping h maps this set to the set $g[\Delta_b[Y]]$ which is a typical generator of the selective ultrafilter $f[\mathcal{U}_b]$. So, it remains to show that $\mathcal{P}$ satisfies the countable chain condition. Let $\mathcal{X}$ be an uncountable subset of $\mathcal{P}$. Refining $\mathcal{X}$ we may assume that for some fixed set R,

$$X_p \cap X_q = R \text{ for all } p, q \in \mathcal{X}, p \neq q.$$

Refining $\mathcal{X}$ if necessary we may assume that for some fixed positive integers n and k and some fixed mapping h we have that

(f) $|Y_p| = n$ and $h_p = h$ for all $p \in \mathcal{X}$,

(g) $\mathrm{Dom}(h), \mathrm{Rang}(h) \subseteq \{0, 1, ..., k\}$.

Let $U = f^{-1}\{0, 1, ..., k\}$ and $V = g^{-1}\{0, 1, ..., k\}$ and for $p \in \mathcal{X}$, let $Y_p = X_p \setminus R$. Note that by our assumption, $U \notin \mathcal{U}_a$ and $V \notin \mathcal{U}_b$. Applying Lemma 1.1, we find uncountable $\mathcal{X}_0 \subseteq \mathcal{X}$ such that for all $p, q \in \mathcal{X}_0, p \neq q$ and all $i, j < n$,

$$\Delta_a(Y_p(i), Y_q(i)) = \Delta_a(Y_p(j), Y_q(j)),$$
$$\Delta_b(Y_p(i), Y_q(i)) = \Delta_b(Y_p(j), Y_q(j)).$$

For $p, q \in \mathcal{X}_0, p \neq q$, let $\Delta_a(Y_p, Y_q)$ denote the constant value of the sequence $\Delta_a(Y_p(i), Y_q(i))$ $(i < n)$ and let $\Delta_b(Y_p, Y_q)$ denote the constant value of $\Delta_b(Y_p(i), Y_q(i))$ $(i < n)$. By Corollary 1.2, we can find uncountable $\mathcal{X}_1 \subseteq \mathcal{X}_0$ such that

$$Y \cap \{\Delta_a(Y_p, Y_q) : p, q \in \mathcal{X}_1, p \neq q\} = \emptyset,$$

$$Z \cap \{\Delta_b(Y_p, Y_q) : p, q \in \mathcal{X}_1, p \neq q\} = \emptyset.$$

Now we find an uncountable $\mathcal{X}_2 \subseteq \mathcal{X}_1$ such that for all $p, q \in \mathcal{X}_2, p \neq q$,

$$\Delta_a[X_p \cup X_q] = \Delta_a[X_p] \cup \Delta_a[X_q] \cup \{\Delta_a(Y_p, Y_q)\},$$

$$\Delta_b[X_p \cup X_q] = \Delta_b[X_p] \cup \Delta_b[X_q] \cup \{\Delta_b(Y_p, Y_q)\}.$$

Fix $p, q \in \mathcal{X}_2, p \neq q$. Let $r = (X_r, h_r)$, where $X_r = X_p \cup X_r$ and where the mapping h_r is determined as follows. Let

$$\mathrm{Dom}(h_r) = f[\Delta_a[X_p \cup X_q]] = f[\Delta_a[X_p]] \cup \{f(\Delta_a(Y_p, Y_q))\},$$

$$\mathrm{Rang}(h_r) = g[\Delta_b[X_p \cup X_q]] = g[\Delta_b[X_p]] \cup \{g(\Delta_b(Y_p, Y_q))\}.$$

Let $h_r \restriction f[\Delta_a[X_p]] = h_p (= h_q)$, and let

$$h_r(f(\Delta_a(Y_p, Y_q))) = g(\Delta_b(Y_p, Y_q)).$$

Note that such a defined map h_r is one-to one and that it extends h_p and h_q. So, this gives us the desired element r of $\mathcal{P}$ which extends both p and q finishing the proof that $\mathcal{P}$ satisfies the countable chain condition and the proof of the Lemma. $\square$

Corollary 2.4. *If* $\mathfrak{mm} > \omega_1$, *there is only one, up to a permutation of* ω, *non-principal ultrafilter of the form* $f[\mathcal{U}_a]$, *where* $a : [\omega_1]^2 \to \omega$ *is a coherent mapping and where* $f : \omega_1 \to \omega$.

Remark 2.5. It follows in particular that, modulo the Baire category assumption, the five selective ultrafilters $\mathcal{V}_a$ for $a = \rho, \rho_0, \rho_1, \rho_2, \rho_3$ are pairwise equivalent. In other words, there is a canonical non-principal selective ultrafilter on ω that one can attach as invariant to the notion of walk along a given sequence C_α $(\alpha < \omega_1)$ of fundamental converging sequences which, up to a permutation of ω, does not depend on the choice of the sequence C_α $(\alpha < \omega_1)$ at all.

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Matematički institut
SANU
Kneza Mihaila 35
11000 Beograd
Srbija
stevo@mi.sanu.ac.rs