

Relationship between Laplacian Operator and D'Alembertian Operator

Relación entre los Operadores Laplaciano y D'Alembertiano

Graciela S. Birman * (gbirman@exa.unicen.edu.ar)

Graciela M. Desideri (gmdeside@exa.unicen.edu.ar)

NUCOMPA - Fac. Exact Sc. - UNCPBA
Pinto 399 - B7000GHG, Tandil - Argentina.

Abstract

Laplacian and D'Alembertian operators on functions are very important tools for several branches of Mathematics and Physics. In addition to their relevance, both operators are very used in vector calculus.

In this paper, we show a relationship between the Laplacian and the D'Alembertian operators, not only on functions but also on vector fields defined on hypersurfaces in the m -dimensional Lorentzian spaces.

We also define the $B_m^{k_1, \dots, k_l}$ -product and B_m -congruence.

Key words and phrases: Laplacian, D'Alembertian, Lorentzian space, operator, $B_m^{k_1, \dots, k_l}$ -product.

Resumen

Los operadores Laplaciano y D'Alembertiano aplicados a funciones son herramientas muy importantes en varias ramas de la Matemática y de la Física. Sumada a su relevancia, ambos operadores se destacan por ser muy utilizados en el cálculo vectorial.

En este artículo mostramos la relación entre los operadores Laplaciano y D'Alembertiano tanto sobre funciones como sobre campos vectoriales definidos sobre hipersuperficies del espacio Lorentziano m -dimensional. Además, definimos los $B_m^{k_1, \dots, k_l}$ - productos y la B_m - congruencia entre operadores.

Palabras y frases clave: Laplaciano, D'Alembertiano, espacios Lorentzianos, operador, $B_m^{k_1, \dots, k_l}$ -producto.

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1 Introduction

In the last three decades the interest in Lorentzian geometry has increased, [1]. We will concentrate on two differential operators of particular interest here: the Laplacian and the D'Alembertian.

Laplacian and D'Alembertian operators on functions are very important tools for several branches of Mathematics and Physics, specifically in investigating many geometrical and physical properties. In addition to relevance, both operators are very used in vector calculus.

Moreover, the Laplacian operator on functions is quite different from the Laplacian operator on vector fields and the D'Alembertian on functions is quite different from the D'Alembertian on vector fields.

There are many interesting vector fields in differential geometry, for example the mean curvature vector field. In [5], Bang-yen Chen developed the Laplacian on vector fields, and he studied its application on mean curvature vector field for submanifolds in Riemannian space. In [3], we studied the Laplacian operator of the mean curvature vector fields on surfaces in the 3-dimensional Lorentzian space, \mathbb{R}_1^3 , and we showed the Laplacian operator of the mean curvature vector fields on the non-lightlike surfaces S_1^2 , H_0^2 , $S_1^1 \times R$, $H_0^1 \times R$, $\mathbb{R}_1^1 \times S^1$ and \mathbb{R}_1^2 .

The purpose of this article is to show the relationship between the Laplacian and the D'Alembertian operators, not only on functions but also on vector fields for non null hypersurfaces in the $n + 1$ -dimensional Lorentzian space.

In order to do that we will first give the definitions of these operators on functions in both Euclidean and Lorentzian spaces.

In the third section, we will generalize the Laplacian and the D'Alembertian on vector fields of Riemannian geometry to Lorentzian geometry, specifically of the hypersurfaces in Riemannian space to non null hypersurfaces in the $n + 1$ -dimensional Lorentzian space, \mathbb{R}_1^{n+1} . We will introduce the B_{n+1}^k -product, from which the relationship between Laplacian and D'Alembertian derives.

In the fourth section, we will study the $B_{n+1}^{k_1, \dots, k_l}$ -product. We will show that the $B_{n+1}^{k_1, \dots, k_l}$ -product becomes a B_{n+1} -congruence.

In the fifth section we will show many examples of operators on vector fields and B_{n+1}^k -products.

2 Preliminaries and definitions

Let \mathbb{R}^n be the n -dimensional Euclidean space with natural coordinates u_1, \dots, u_n . In classical notation, the metric tensor is

$$g = g_{ij} du^i \otimes du^j \quad \text{with } g = \text{diag}(+1, \dots, +1)$$

The Laplacian and D'Alembertian operators on functions defined on \mathbb{R}^n are well known operators, defined as follows.

Definition 1. Let u_1, \dots, u_n be the natural coordinates in \mathbb{R}^n . The differential operators

$$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial u_i^2} \tag{1}$$

$$\square = -\frac{\partial^2}{\partial u_1^2} + \sum_{i=2}^n \frac{\partial^2}{\partial u_i^2} \tag{2}$$

are called the Laplacian operator and the D'Alembertian operator in \mathbb{R}^n , respectively. They are defined on smooth real-valued functions on \mathbb{R}^n .

Let $(\mathbb{R}_1^n, \mathbf{g})$ be an n -dimensional Lorentzian space of zero curvature where the signature of \mathbf{g} is $(-, +, \dots, +)$. We will indicate with $\langle \cdot, \cdot \rangle$ the corresponding inner product.

In Lorentzian spaces there are three kinds of vectors: timelike, spacelike and lightlike, according to the inner product of the vector with itself is negative, positive or zero, respectively.

We say that a hypersurface M in \mathbb{R}_1^n is spacelike or timelike if at every point $p \in M$ its tangent space $T_p(M)$ is spacelike or timelike, that is if the normal vector is timelike or spacelike, respectively, (cf. [2] for more details). We will call these hypersurfaces *non null hypersurfaces* from now onwards.

Considering $\mathbb{R}^n = \mathbb{R}_0^n$, we denote the set of all smooth real-valued functions on \mathbb{R}_ν^n with $\mathcal{F}(\mathbb{R}_\nu^n)$, where $\nu : 0, 1$.

It is natural then to define Laplacian and D'Alembertian operators on functions in the Lorentzian space \mathbb{R}_1^n . Some operators on functions in the Lorentzian space \mathbb{R}_1^n are well known, (cf. [1] and [7]).

Definition 2. Let u_1, \dots, u_n be the natural coordinates in \mathbb{R}_1^n . The differential operators Δ and \square are given by:

$$\Delta = \sum_{i=1}^n \varepsilon_i \frac{\partial^2}{\partial u_i^2} \tag{3}$$

and

$$\square = -\varepsilon_1 \frac{\partial^2}{\partial u_1^2} + \sum_{i=2}^n \varepsilon_i \frac{\partial^2}{\partial u_i^2}, \quad (4)$$

respectively, where $\varepsilon_i = \begin{cases} -1 & \text{if } i = 1, \\ +1 & \text{if } 2 \leq i \leq n. \end{cases}$

Both operators are defined on functions $f \in \mathcal{F}(\mathbb{R}_1^n)$.

According to Definition 1 and Definition 2, the Laplacian operator is defined by using the tensor metric of the respective structure. In some contexts, the Laplacian is defined with opposite sign and others name are used to call it (cf. [7]).

3 Relationships between the Laplacian and D'Alembertian operators

We denote the Laplacian and the D'Alembertian operators on functions in \mathbb{R}_1^n with Δ_1^n and \square_1^n , and on functions in \mathbb{R}^n with Δ_0^n and \square_0^n , respectively.

Proposition 3. *According to Definitions 1 and 2,*

$$\Delta_1^n(f) = \square_0^n(f) \text{ and } \Delta_0^n(f) = \square_1^n(f).$$

Proof. By Definition 2, $\square_0^n(f) = -\frac{\partial^2 f}{\partial u_1^2} + \sum_{i=2}^n \frac{\partial^2 f}{\partial u_i^2}$.

$$\text{By Definition 1, } \Delta_1^n(f) = -\frac{\partial^2 f}{\partial u_1^2} + \sum_{i=2}^n \frac{\partial^2 f}{\partial u_i^2}.$$

$$\text{Thus } \Delta_1^n(f) = \square_0^n(f).$$

$$\text{Similarly, } \square_1^n(f) = -\left(-\frac{\partial^2 f}{\partial u_1^2}\right) + \sum_{i=2}^n \frac{\partial^2 f}{\partial u_i^2} = \sum_{i=1}^n \frac{\partial^2 f}{\partial u_i^2} = \Delta_0^n(f). \quad \square$$

The Laplacian operator on vector fields for submanifolds in Riemannian manifolds is known (cf. [5]). Now, we show the Laplacian and D'Alembertian operators on vector fields for hypersurfaces in a $n+1$ -dimensional Lorentzian space of zero curvature, \mathbb{R}_1^{n+1} .

Let M be an n -dimensional non null hypersurface in \mathbb{R}_1^{n+1} with induced connection $\bar{\nabla}$.

Let

$$\bar{\Xi}(M) = \{X : M \rightarrow \mathbb{R}_1^{n+1}; X \text{ is a vector field and } X(p) \in \mathbb{R}_1^{n+1}\}$$

and

$$\Xi(M) = \left\{ X : M \rightarrow \bigcup_{p \in M} T_p(M); X \text{ is vector field and } X(p) \in T_p(M) \right\}.$$

We say E_1, \dots, E_n is a basis of $\Xi(M)$ and E_{n+1} is the unit normal vector field on M if at every point $p \in M$, $\{E_1(p), \dots, E_n(p)\}$ is a basis of $T_p(M)$ and $E_{n+1}(p)$ is the unit normal vector at p , respectively. Thus, E_1, \dots, E_{n+1} is a basis of $\bar{\Xi}(M)$. If $\{E_1(p), \dots, E_n(p)\}$ is an orthonormal basis of $T_p(M)$ and $E_{n+1}(p)$ is the unit normal vector at $p, \forall p \in M$, E_1, \dots, E_{n+1} is an orthonormal basis of $\bar{\Xi}(M)$

We recall the well known fact that if $X \in \bar{\Xi}(M)$ and $E_i \in \Xi(M)$, then $\bar{\nabla}_{E_i} X$ is vector field of $\bar{\Xi}(M)$. Consequently, if $X \in \bar{\Xi}(M)$ and $E_{i_j} \in \Xi(M)$ then $\bar{\nabla}_{E_{i_1}} \dots \bar{\nabla}_{E_{i_m}} X \in \bar{\Xi}(M)$. Thus it is possible to define the Laplacian and the D'Alembertian operators on vector fields of $\bar{\Xi}(M)$.

Definition 4. Let M be an n -dimensional non null hypersurface in \mathbb{R}_1^{n+1} with induced connection $\bar{\nabla}$. Let E_1, \dots, E_n be an orthonormal basis of $\Xi(M)$.

a) The Laplacian Δ on vector fields of $\bar{\Xi}(M)$ is given by:

$$\Delta = \sum_{i=1}^n \varepsilon_i \bar{\nabla}_{E_i} \bar{\nabla}_{E_i}, \tag{5}$$

b) The D'Alembertian \square on vector fields of $\bar{\Xi}(M)$ is given by:

$$\square = -\varepsilon_1 \bar{\nabla}_{E_1} \bar{\nabla}_{E_1} + \sum_{i=2}^n \varepsilon_i \bar{\nabla}_{E_i} \bar{\nabla}_{E_i}, \tag{6}$$

where $\varepsilon_i = \langle E_i, E_i \rangle, i = 1, \dots, n$.

Now we introduce some notation which will be used later. Let $\bar{\nabla}_{i_1}^1 = \bar{\nabla}_{E_{i_1}}, \bar{\nabla}_{i_1, i_2}^2 = \bar{\nabla}_{E_{i_1}} \bar{\nabla}_{E_{i_2}}, \dots, \bar{\nabla}_{i_1, \dots, i_m}^m = \bar{\nabla}_{E_{i_1}} \dots \bar{\nabla}_{E_{i_m}}$, where $1 \leq i_1, \dots, i_m \leq n$ and E_1, \dots, E_{n+1} is basis of $\bar{\Xi}(M)$. Let $\mathcal{F}(M)$ be the set of all smooth real-valued functions on M . Let

$$\mathcal{P}(M) = \left\{ Q \neq 0; Q = \sum_{i_1=1}^n q_{i_1} \bar{\nabla}_{i_1}^1 + \dots + \sum_{i_1, \dots, i_m=1}^n q_{i_1, \dots, i_m} \bar{\nabla}_{i_1, \dots, i_m}^m, \text{ where } m = m(Q) < \infty \text{ and } q_{i_1}, \dots, q_{i_1, \dots, i_m} \in \mathcal{F}(M) \right\}.$$

We define a new application which produces a certain change of sign in some terms of the operators of $\mathcal{P}(M)$. Since this application satisfies properties of inner products, we shall call it "product". We shall make use of this product when we relate the Laplacian and the D'Alembertian operators.

Definition 5. Let M be an n -dimensional non null hypersurface in \mathbb{R}_1^{n+1} with induced connection $\bar{\nabla}$. Let E_1, \dots, E_{n+1} be an orthonormal basis of $\bar{\Xi}(M)$.

For $k : 0, \dots, n$, the B_{n+1}^k -product is an application on $\mathcal{P}(M)$ to $\mathcal{P}(M)$ which is characterized by

$$(b_{i_1, \dots, i_m}^k)_{jt} = \begin{cases} -\varepsilon_{jt} & \text{if } k \in \{i_1, \dots, i_m\} \\ \varepsilon_{jt} & \text{if } k \notin \{i_1, \dots, i_m\} \end{cases}, \quad (7)$$

where $\varepsilon_{jt} = \langle E_j, E_t \rangle$, $j, t = 1, \dots, n+1$, and $\{i_1, \dots, i_m\} \subset \{1, \dots, n\}$.

We denote the B_{n+1}^k -product with $\langle \cdot, \cdot \rangle_{B_{n+1}}^k$.

The equality $Q = \sum_{t=1}^{n+1} \langle Q, E_t \rangle_{B_{n+1}}^k E_t$ means $QX = \sum_{t=1}^{n+1} \langle QX, E_t \rangle_{B_{n+1}}^k E_t$ for all $X \in \overline{\Xi}(M)$. Hence, the B_{n+1}^k -product is well defined.

Remark 6. The B_{n+1}^k -product is $\mathcal{F}(M)$ -bilinear.

Remark 7. From Definition 5, if $\overline{\nabla}_{i_1, \dots, i_m}^m X = \sum_{j=1}^{n+1} X_{i_1, \dots, i_m}^j E_j$ then we have

$$\begin{aligned} \langle \overline{\nabla}_{i_1, \dots, i_m}^m X, E_t \rangle_{B_{n+1}}^k &= \sum_{j=1}^{n+1} X_{i_1, \dots, i_m}^j \langle E_j, E_t \rangle_{B_{n+1}}^k \\ &= \sum_{j=1}^{n+1} X_{i_1, \dots, i_m}^j (b_{i_1, \dots, i_m}^k)_{jt} = X_{i_1, \dots, i_m}^t (b_{i_1, \dots, i_m}^k)_{tt} \\ &= \begin{cases} -\varepsilon_{tt} X_{i_1, \dots, i_m}^t & \text{if } k \in \{i_1, \dots, i_m\} \\ \varepsilon_{tt} X_{i_1, \dots, i_m}^t & \text{if } k \notin \{i_1, \dots, i_m\} \end{cases}. \end{aligned}$$

The following theorem relates the Laplacian and the D'Alembertian operators, which are defined in (5) and (6).

Theorem 8. *Let M be an n -dimensional non null hypersurface in \mathbb{R}_1^{n+1} with induced connection $\overline{\nabla}$. Let E_1, \dots, E_{n+1} be an orthonormal basis of $\overline{\Xi}(M)$. Then, the Laplacian Δ and the D'Alembertian \square operators on vector fields of $\overline{\Xi}(M)$ are related by:*

$$\square = \sum_{t=1}^{n+1} \langle \Delta, E_t \rangle_{B_{n+1}}^1 E_t \quad (8)$$

and

$$\Delta = \sum_{t=1}^{n+1} \langle \square, E_t \rangle_{B_{n+1}}^1 E_t. \quad (9)$$

Proof. Let $X \in \overline{\Xi}(M)$ and let $\overline{\nabla}_{E_i} \overline{\nabla}_{E_i} X = \sum_{j=1}^{n+1} X_{ii}^j E_j$. By (5) and (6),

$$\begin{aligned} \sum_{t=1}^{n+1} \langle \Delta X, E_t \rangle_{B_{n+1}}^1 E_t &= \sum_{t=1}^{n+1} \langle \sum_{i=1}^n \varepsilon_i \overline{\nabla}_{E_i} \overline{\nabla}_{E_i} X, E_t \rangle_{B_{n+1}}^1 E_t \\ &= \sum_{t=1}^{n+1} \left\{ \sum_{i=1}^n \varepsilon_i \langle \overline{\nabla}_{E_i} \overline{\nabla}_{E_i} X, E_t \rangle_{B_{n+1}}^1 \right\} E_t \\ &= \sum_{t=1}^{n+1} \left\{ \sum_{i=1}^n \varepsilon_i \sum_{j=1}^{n+1} X_{ii}^j \langle E_j, E_t \rangle_{B_{n+1}}^1 \right\} E_t. \end{aligned}$$

From the orthonormality condition of the basis of $\overline{\Xi}(M)$,

$$\begin{aligned}
 & \sum_{t=1}^{n+1} \left\{ \sum_{i=1}^n \varepsilon_i \sum_{j=1}^{n+1} X_{ii}^j \langle E_j, E_t \rangle_{B_{n+1}}^1 \right\} E_t \\
 &= \sum_{t=1}^{n+1} \left\{ -\varepsilon_1 X_{11}^t \langle E_t, E_t \rangle + \sum_{i=2}^{n+1} \varepsilon_i X_{ii}^t \langle E_t, E_t \rangle \right\} E_t \\
 &= -\sum_{t=1}^{n+1} \varepsilon_1 \langle \sum_{j=1}^{n+1} X_{11}^j E_j, E_t \rangle E_t + \sum_{t=1}^{n+1} \sum_{i=2}^{n+1} \varepsilon_i \langle \sum_{j=1}^{n+1} X_{ii}^j E_j, E_t \rangle E_t \\
 &= -\varepsilon_1 \sum_{t=1}^{n+1} \langle \bar{\nabla}_{E_1} \bar{\nabla}_{E_1} X, E_t \rangle E_t + \sum_{i=2}^{n+1} \varepsilon_i \sum_{t=1}^{n+1} \langle \bar{\nabla}_{E_i} \bar{\nabla}_{E_i} X, E_t \rangle E_t \\
 &= -\varepsilon_1 \bar{\nabla}_{E_1} \bar{\nabla}_{E_1} X + \sum_{i=2}^{n+1} \varepsilon_i \bar{\nabla}_{E_i} \bar{\nabla}_{E_i} X = \square X.
 \end{aligned}$$

Therefore, $\square = \sum_{t=1}^{n+1} \langle \Delta, E_t \rangle_{B_{n+1}}^1 E_t$. Analogously,

$$\begin{aligned}
 & \sum_{t=1}^{n+1} \langle \square X, E_t \rangle_{B_{n+1}}^1 E_t \\
 &= \sum_{t=1}^{n+1} \left\{ -\varepsilon_1 \langle \bar{\nabla}_{E_1} \bar{\nabla}_{E_1} X, E_t \rangle_{B_{n+1}}^1 + \sum_{i=2}^n \varepsilon_i \langle \bar{\nabla}_{E_i} \bar{\nabla}_{E_i} X, E_t \rangle_{B_{n+1}}^1 \right\} E_t \\
 &= \sum_{t=1}^{n+1} \left\{ -\varepsilon_1 \sum_{j=1}^{n+1} X_{11}^j \langle E_j, E_t \rangle_{B_{n+1}}^1 + \sum_{i=2}^n \varepsilon_i \sum_{j=1}^{n+1} X_{ii}^j \langle E_j, E_t \rangle_{B_{n+1}}^1 \right\} E_t \\
 &= \sum_{t=1}^{n+1} \left\{ \sum_{i=1}^n \varepsilon_i \langle \sum_{j=1}^{n+1} X_{ii}^j E_j, E_t \rangle \right\} E_t \\
 &= \sum_{i=1}^n \varepsilon_i \left\{ \sum_{t=1}^{n+1} \langle \bar{\nabla}_{E_i} \bar{\nabla}_{E_i} X, E_t \rangle E_t \right\} \\
 &= \sum_{i=1}^n \varepsilon_i \bar{\nabla}_{E_i} \bar{\nabla}_{E_i} X = \Delta X. \quad \square
 \end{aligned}$$

From now onwards, we will extend Definition 5 and Theorem 8 to general, not necessary orthonormal basis. In order to do that we first define the Laplacian and D'Alembertian operators on vector fields when M is a n -dimensional non null hypersurface in \mathbb{R}_1^{n+1} . In a classical way, we denote $g_{ij} = \langle E_i, E_j \rangle$, $1 \leq i, j \leq n+1$, and $(g^{ij}) = (g_{ij})^{-1}$.

Definition 9. Let M be an n -dimensional non null hypersurface in \mathbb{R}_1^{n+1} with induced connection $\bar{\nabla}$. Let E_1, \dots, E_n be a basis of $\Xi(M)$.

a) The Laplacian Δ on vector fields of $\Xi(M)$ is given by:

$$\Delta = \sum_{i,j=1}^n g^{ij} \bar{\nabla}_{E_i} \bar{\nabla}_{E_j}. \quad (10)$$

b) The D'Alembertian \square on vector fields of $\Xi(M)$ is given by:

$$\square = -g^{11} \bar{\nabla}_{E_1} \bar{\nabla}_{E_1} - \sum_{i=2}^n g^{i1} (\bar{\nabla}_{E_i} \bar{\nabla}_{E_1} + \bar{\nabla}_{E_1} \bar{\nabla}_{E_i}) + \sum_{i,j=2}^n g^{ij} \bar{\nabla}_{E_i} \bar{\nabla}_{E_j}. \quad (11)$$

Naturally, the B_{n+1}^k -product must also be extended to general basis.

Definition 10. Let M be an n -dimensional non null hypersurface in \mathbb{R}_1^{n+1} with induced connection $\bar{\nabla}$. Let E_1, \dots, E_{n+1} be an orthonormal basis of

$\bar{\Xi}(M)$. For $k : 0, \dots, n$, the B_{n+1}^k -product is an application on $\mathcal{P}(M)$ to $\mathcal{P}(M)$ which is characterized by:

$$(b_{i_1, \dots, i_m}^k)_{jt} = \begin{cases} -g_{jt} & \text{if } k \in \{i_1, \dots, i_m\} \\ g_{jt} & \text{if } k \notin \{i_1, \dots, i_m\} \end{cases} . \quad (12)$$

We denote the B_{n+1}^k -product with $\langle, \rangle_{B_{n+1}}^k$.

Remark 11. Since \langle, \rangle is $\mathcal{F}(M)$ -bilinear, the B_{n+1}^k -product is $\mathcal{F}(M)$ -bilinear too.

Remark 12. If $\bar{\nabla}_{i_1, \dots, i_m}^m X = \sum_{j=1}^{n+1} X_{i_1, \dots, i_m}^j E_j$ then we have

$$\begin{aligned} \left\langle \bar{\nabla}_{i_1, \dots, i_m}^m X, E_t \right\rangle_{B_{n+1}}^k &= \sum_{j=1}^{n+1} X_{i_1, \dots, i_m}^j \langle E_j, E_t \rangle_{B_{n+1}}^k \\ &= \sum_{j=1}^{n+1} X_{i_1, \dots, i_m}^j (b_{i_1, \dots, i_m}^k)_{jt} \\ &= \begin{cases} -\sum_{j=1}^{n+1} g_{jt} X_{i_1, \dots, i_m}^j & \text{if } k \in \{i_1, \dots, i_m\} \\ \sum_{j=1}^{n+1} g_{jt} X_{i_1, \dots, i_m}^j & \text{if } k \notin \{i_1, \dots, i_m\} \end{cases} \\ &= \begin{cases} -\left\langle \bar{\nabla}_{i_1, \dots, i_m}^m X, E_t \right\rangle & \text{if } k \in \{i_1, \dots, i_m\} \\ \left\langle \bar{\nabla}_{i_1, \dots, i_m}^m X, E_t \right\rangle & \text{if } k \notin \{i_1, \dots, i_m\} \end{cases} . \end{aligned}$$

Theorem 13. Let M be an n -dimensional non null hypersurface in \mathbb{R}_1^{n+1} with induced connection $\bar{\nabla}$. Let E_1, \dots, E_n be a basis of $\Xi(M)$ and let E_{n+1} be the unit normal vector field. Then,

$$\square = \sum_{t=1}^{n+1} \langle \Delta, E_t \rangle_{B_{n+1}}^1 E_t \quad (13)$$

and

$$\Delta = \sum_{t=1}^{n+1} \langle \square, E_t \rangle_{B_{n+1}}^1 E_t. \quad (14)$$

Proof. Clearly, the Laplacian Δ and the D'Alembertian \square are two operators of $\mathcal{P}(M)$.

Let $X \in \bar{\Xi}(M)$, then $\bar{\nabla}_{E_i} \bar{\nabla}_{E_j} X = \sum_{r=1}^{n+1} X_{ij}^r E_r$, where

$X_{ij}^r = \sum_{s=1}^{n+1} g^{sr} \langle \bar{\nabla}_{E_i} \bar{\nabla}_{E_j} X, E_s \rangle$ By (10) and (11),

$$\begin{aligned} \sum_{t=1}^{n+1} \langle \Delta X, E_t \rangle_{B_{n+1}}^1 E_t &= \sum_{t=1}^{n+1} \left\langle \sum_{i,j=1}^n g^{ij} \bar{\nabla}_{E_i} \bar{\nabla}_{E_j} X, E_t \right\rangle_{B_{n+1}}^1 E_t \\ &= \sum_{t=1}^{n+1} \left\{ \sum_{i,j=1}^n g^{ij} \langle \bar{\nabla}_{E_i} \bar{\nabla}_{E_j} X, E_t \rangle_{B_{n+1}}^1 \right\} E_t \\ &= \sum_{t=1}^{n+1} \left\{ \sum_{i,j=1}^n g^{ij} \sum_{r=1}^{n+1} X_{ij}^r \langle E_r, E_t \rangle_{B_{n+1}}^1 \right\} E_t \end{aligned}$$

$$\begin{aligned}
 &= \sum_{t=1}^{n+1} \left\{ \sum_{i,j=1}^n g^{ij} \sum_{r=1}^{n+1} X_{ij}^r (b_{ij}^1)_{rt} \right\} E_t \\
 &= \sum_{t,r=1}^{n+1} \left\{ - \sum_{j=1}^n g^{1j} X_{1j}^r g_{rt} - \sum_{i=2}^n g^{i1} X_{i1}^r g_{rt} + \sum_{i,j=2}^n g^{ij} X_{ij}^r g_{rt} \right\} E_t \\
 &= - \sum_{t=1}^{n+1} \left\{ \sum_{j=1}^n g^{1j} \langle \bar{\nabla}_{E_1} \bar{\nabla}_{E_j} X, E_t \rangle + \sum_{i=2}^n g^{i1} \langle \bar{\nabla}_{E_i} \bar{\nabla}_{E_1} X, E_t \rangle \right\} E_t \\
 &\quad + \sum_{t=1}^{n+1} \left\{ \sum_{i,j=2}^n g^{ij} \langle \bar{\nabla}_{E_i} \bar{\nabla}_{E_j} X, E_t \rangle \right\} E_t \\
 &= - \sum_{j=1}^n g^{1j} \bar{\nabla}_{E_1} \bar{\nabla}_{E_j} X - \sum_{i=2}^n g^{i1} \bar{\nabla}_{E_i} \bar{\nabla}_{E_1} X + \sum_{t=1}^{n+1} \sum_{i,j=2}^n g^{ij} \bar{\nabla}_{E_i} \bar{\nabla}_{E_j} X \\
 &= \square X.
 \end{aligned}$$

Therefore, $\square = \sum_{t=1}^{n+1} \langle \Delta, E_t \rangle_{B_{n+1}}^1 E_t$.

In similar way, $\sum_{t=1}^{n+1} \langle \square X, E_j \rangle_{B_{n+1}}^1 E_t$

$$\begin{aligned}
 &= - \sum_{t,r=1}^{n+1} \left\{ \sum_{j=1}^n g^{1j} X_{1j}^r (b_{1j}^1)_{rt} + \sum_{i=2}^n g^{i1} X_{i1}^r (b_{i1}^1)_{rt} \right\} E_t \\
 &\quad + \sum_{t,r=1}^{n+1} \left\{ \sum_{i,j=2}^n g^{ij} X_{ij}^r (b_{ij}^1)_{rt} \right\} E_t \\
 &= - \sum_{t,r=1}^{n+1} \left\{ - \sum_{j=1}^n g^{1j} X_{1j}^r g_{rt} - \sum_{i=2}^n g^{i1} X_{i1}^r g_{rt} \right\} E_t \\
 &\quad + \sum_{t,r=1}^{n+1} \left\{ \sum_{i,j=2}^n g^{ij} X_{ij}^r g_{rt} \right\} E_t \\
 &= \sum_{t=1}^{n+1} \left\{ \sum_{i,j=1}^n g^{ij} \langle \bar{\nabla}_{E_i} \bar{\nabla}_{E_j} X, E_t \rangle \right\} E_t = \sum_{i,j=1}^n g^{ij} \bar{\nabla}_{E_i} \bar{\nabla}_{E_j} X = \Delta X.
 \end{aligned}$$

Therefore, $\Delta = \sum_{t=1}^{n+1} \langle \square, E_t \rangle_{B_{n+1}}^1 E_t$. □

4 B_{n+1}-congruence

Let M be an n -dimensional non null hypersurface in \mathbb{R}_1^{n+1} with induced connection $\bar{\nabla}$. From now onwards, we consider E_1, \dots, E_{n+1} vector fields such that $E_{n+1}(p)$ is the unit normal vector at p and $\{E_1(p), \dots, E_n(p)\}$ is a basis of $T_p(M)$, at all $p \in M$.

Definition 14. Let k_1, \dots, k_l be integer numbers such that $0 \leq k_1 < \dots < k_l \leq n$. The $B_{n+1}^{k_1, \dots, k_l}$ -product is characterized by:

$$\left(b_{i_1, \dots, i_m}^{k_1, \dots, k_l} \right)_{jt} = (-1)^c g_{jt}, \tag{15}$$

with $c = |\{k_t; k_t \in \{i_1, \dots, i_m\} \text{ and } 1 \leq t \leq l\}|$.

In classical way, we consider $|\emptyset| = 0$. It is obvious that $0 \leq c \leq \min \{l, m\} \leq n$.

We denote the $B_{n+1}^{k_1, \dots, k_l}$ -product with $\langle \cdot, \cdot \rangle_{B_{n+1}}^{k_1, \dots, k_l}$. Since $\langle \cdot, \cdot \rangle$ is $\mathcal{F}(M)$ -bilinear, the $B_{n+1}^{k_1, \dots, k_l}$ -product is too $\mathcal{F}(M)$ -bilinear.

If $Q \in \mathcal{P}(M)$, we denote $\sum_{t=1}^{n+1} \langle Q, E_t \rangle_{B_{n+1}}^{k_1, \dots, k_l} E_t$ with $B_{n+1}^{k_1, \dots, k_l}(Q)$.

From Definition 14, we get $B_{n+1}^{k_1, \dots, k_l}(Q)$ is a differential operator of $\mathcal{P}(M)$.

Remark 15. Let us note that $(b_{i_1, \dots, i_m}^0)_{jt} = g_{jt}$, at all $j, t : 1, \dots, n+1$ and $\{i_1, \dots, i_m\} \subset \{1, \dots, n\}$. Thus $B_{n+1}^0(Q) = Q$.

Definition 16. Let P, Q be two differential operators of $\mathcal{P}(M)$. We say that P is B_{n+1} -congruent to Q if $P = B_{n+1}^k(Q)$.

Lemma 17. Let $(b_{i_1, \dots, i_m}^k)_{uv}, (b_{j_1, \dots, j_s}^h)_{rt}$ be as in (15), then

$$(b_{i_1, \dots, i_m}^k)_{uv} (b_{j_1, \dots, j_s}^h)_{rt} = \begin{cases} -g_{uv}g_{rt} & \text{if } k \in \{i_1, \dots, i_m\} \wedge h \notin \{j_1, \dots, j_s\}, \\ -g_{uv}g_{rt} & \text{if } k \notin \{i_1, \dots, i_m\} \wedge h \in \{j_1, \dots, j_s\}, \\ g_{uv}g_{rt} & \text{in other case.} \end{cases}$$

Proof. It follows from the table:

	$k = i_1$...	$k = i_m$	$k \notin \{i_1, \dots, i_m\}$
$h = j_1$	$(-g_{uv})(-g_{rt})$...	$(-g_{uv})(-g_{rt})$	$g_{uv}(-g_{rt})$
$h = j_2$	$(-g_{uv})(-g_{rt})$...	$(-g_{uv})(-g_{rt})$	$g_{uv}(-g_{rt})$
...
$h = j_s$	$(-g_{uv})(-g_{rt})$...	$(-g_{uv})(-g_{rt})$	$g_{uv}(-g_{rt})$
$h \notin \{j_1, \dots, j_s\}$	$(-g_{uv})g_{rt}$...	$(-g_{uv})g_{rt}$	$g_{uv}g_{rt}$

□

We denote the set of all $B_{n+1}^{k_1, \dots, k_l}$ -products with \mathcal{B}_{n+1} , where $0 \leq k_1 < \dots < k_l \leq n$.

Consecutive application of products in \mathcal{B}_{n+1} result in another product in \mathcal{B}_{n+1} . Its proof is more dull than the idea itself. So we have developed it in steps.

Proposition 18. Let $P, Q, R \in \mathcal{P}(M)$ such that $P = B_{n+1}^k(R)$ and $R = B_{n+1}^h(Q)$, then

$$P = B_{n+1}^k(B_{n+1}^h(Q)) = \begin{cases} B_{n+1}^{k,h}(Q) & \text{if } k < h, \\ Q & \text{if } k = h, \\ B_{n+1}^{h,k}(Q) & \text{if } k > h, \end{cases} \quad (16)$$

where $0 \leq k, h \leq n$.

Proof. We will explicitly show that $B_{n+1}^k (B_{n+1}^h (Q))$. Let

$$Q = \sum_{i_1=1}^n q_{i_1} \bar{\nabla}_{i_1}^1 + \cdots + \sum_{i_1, \dots, i_m=1}^n q_{i_1, \dots, i_m} \bar{\nabla}_{i_1, \dots, i_m}^m. \text{ Since}$$

$P = B_{n+1}^k (R) = \sum_{t=1}^{n+1} \langle R, E_t \rangle_{B_{n+1}}^k E_t$ and $R = \sum_{j=1}^{n+1} \langle Q, E_j \rangle_{B_{n+1}}^h E_j$, then

$$\begin{aligned} P &= B_{n+1}^k \left(\sum_{j=1}^{n+1} \langle Q, E_j \rangle_{B_{n+1}}^h E_j \right) \\ &= B_{n+1}^k \left(\sum_{j=1}^{n+1} \left\{ \sum_{i_1=1}^n q_{i_1} \langle \bar{\nabla}_{i_1}^1, E_j \rangle_{B_{n+1}}^h \right\} E_j \right) + \cdots \\ &\quad + B_{n+1}^k \left(\sum_{j=1}^{n+1} \left\{ \sum_{i_1, \dots, i_m=1}^n q_{i_1, \dots, i_m} \langle \bar{\nabla}_{i_1, \dots, i_m}^m, E_j \rangle_{B_{n+1}}^h \right\} E_j \right). \end{aligned}$$

Let us note that

$$\sum_{i_1=1}^n q_{i_1} \langle \bar{\nabla}_{i_1}^1, E_j \rangle_{B_{n+1}}^h = \sum_{\substack{i_1=1 \\ i_1 \neq h}}^n q_{i_1} \langle \bar{\nabla}_{i_1}^1, E_j \rangle - q_h \langle \bar{\nabla}_h^1, E_j \rangle,$$

$$\begin{aligned} \sum_{i_1, i_2=1}^n q_{i_1, i_2} \langle \bar{\nabla}_{i_1, i_2}^2, E_j \rangle_{B_{n+1}}^h &= \sum_{\substack{i_1, i_2=1 \\ i_1, i_2 \neq h}}^n q_{i_1, i_2} \langle \bar{\nabla}_{i_1, i_2}^2, E_j \rangle \\ &- \sum_{i_2=1}^n q_{h, i_2} \langle \bar{\nabla}_{h, i_2}^2, E_j \rangle - \sum_{i_1=1}^n q_{i_1, h} \langle \bar{\nabla}_{i_1, h}^2, E_j \rangle + q_{h, h} \langle \bar{\nabla}_{h, h}^2, E_j \rangle, \end{aligned}$$

and in the same way,

$$\begin{aligned} &\sum_{i_1, \dots, i_m=1}^n q_{i_1, \dots, i_m} \langle \bar{\nabla}_{i_1, \dots, i_m}^m, E_j \rangle_{B_{n+1}}^h \\ &= \sum_{\substack{i_1, \dots, i_m=1 \\ i_1, \dots, i_m \neq h}}^n q_{i_1, \dots, i_m} \langle \bar{\nabla}_{i_1, \dots, i_m}^m, E_j \rangle \\ &- \sum_{i_2, \dots, i_m=1}^n q_{h, i_2, \dots, i_m} \langle \bar{\nabla}_{h, i_2, \dots, i_m}^m, E_j \rangle \\ &- \sum_{i_1, i_3, \dots, i_m=1}^n q_{i_1, h, i_3, \dots, i_m} \langle \bar{\nabla}_{i_1, h, i_3, \dots, i_m}^m, E_j \rangle \\ &- \cdots - \sum_{i_1, \dots, i_{m-1}=1}^n q_{i_1, \dots, i_{m-1}, h} \langle \bar{\nabla}_{i_1, \dots, i_{m-1}, h}^m, E_j \rangle \\ &+ \sum_{i_3, \dots, i_m=1}^n q_{h, h, i_3, \dots, i_m} \langle \bar{\nabla}_{h, h, i_3, \dots, i_m}^m, E_j \rangle \\ &+ \sum_{i_2, i_4, \dots, i_m=1}^n q_{h, i_2, h, i_4, \dots, i_m} \langle \bar{\nabla}_{h, i_2, h, i_4, \dots, i_m}^m, E_j \rangle \\ &+ \cdots + \sum_{i_1, \dots, i_{m-2}=1}^n q_{i_1, \dots, i_{m-2}, h, h} \langle \bar{\nabla}_{i_1, \dots, i_{m-2}, h, h}^m, E_j \rangle \\ &- \sum_{i_4, \dots, i_m=1}^n q_{h, h, h, i_4, \dots, i_m} \langle \bar{\nabla}_{h, h, h, i_4, \dots, i_m}^m, E_j \rangle \\ &- \cdots - \sum_{i_1, \dots, i_{m-3}=1}^n q_{i_1, \dots, i_{m-3}, h, h, h} \langle \bar{\nabla}_{i_1, \dots, i_{m-3}, h, h, h}^m, E_j \rangle \\ &+ \cdots + (-1)^m q_{h, \dots, h} \langle \bar{\nabla}_{h, \dots, h}^m, E_j \rangle. \end{aligned}$$

We distinguish two cases:

a) If $k \neq h$,

$$\begin{aligned}
P &= \sum_{j=1}^{n+1} \left(\sum_{\substack{i_1=1 \\ i_1 \neq h}}^n q_{i_1} \langle \bar{\nabla}_{i_1}^1, E_j \rangle_{B_{n+1}}^k \right) E_j - \sum_{j=1}^{n+1} q_h \langle \bar{\nabla}_h^1, E_j \rangle_{B_{n+1}}^k E_j \\
&+ \cdots + \sum_{j=1}^{n+1} \left(\sum_{\substack{i_1, \dots, i_m=1 \\ i_1, \dots, i_m \neq h}}^n q_{i_1, \dots, i_m} \langle \bar{\nabla}_{i_1, \dots, i_m}^m, E_j \rangle_{B_{n+1}}^k \right) E_j \\
&- \sum_{j=1}^{n+1} \left(\sum_{i_2, \dots, i_m=1}^n q_{h, i_2, \dots, i_m} \langle \bar{\nabla}_{h, i_2, \dots, i_m}^m, E_j \rangle_{B_{n+1}}^k \right) E_j \\
&- \cdots - \sum_{j=1}^{n+1} \left(\sum_{i_1, \dots, i_{m-1}=1}^n q_{i_1, \dots, i_{m-1}, h} \langle \bar{\nabla}_{i_1, \dots, i_{m-1}, h}^m, E_j \rangle_{B_{n+1}}^k \right) E_j \\
&+ \cdots + \sum_{j=1}^{n+1} (-1)^m q_{h, \dots, h} \langle \bar{\nabla}_{h, \dots, h}^m, E_j \rangle_{B_{n+1}}^k E_j \\
&= \sum_{j=1}^{n+1} \left(\sum_{\substack{i_1=1 \\ i_1 \neq h, k}}^n q_{i_1} \langle \bar{\nabla}_{i_1}^1, E_j \rangle \right) E_j \\
&- \sum_{j=1}^{n+1} q_k \langle \bar{\nabla}_k^1, E_j \rangle E_j - \sum_{j=1}^{n+1} q_h \langle \bar{\nabla}_h^1, E_j \rangle E_j \\
&+ \cdots + \sum_{j=1}^{n+1} \left(\sum_{\substack{i_1, \dots, i_m=1 \\ i_1, \dots, i_m \neq h, k}}^n q_{i_1, \dots, i_m} \langle \bar{\nabla}_{i_1, \dots, i_m}^m, E_j \rangle \right) E_j \\
&- \sum_{j=1}^{n+1} \left(\sum_{\substack{i_2, \dots, i_m=1 \\ i_2, \dots, i_m \neq h}}^n q_{k, i_2, \dots, i_m} \langle \bar{\nabla}_{k, i_2, \dots, i_m}^m, E_j \rangle \right) E_j \\
&- \sum_{j=1}^{n+1} \left(\sum_{\substack{i_1, \dots, i_{m-1}=1 \\ i_1, \dots, i_{m-1} \neq h}}^n q_{i_1, \dots, i_{m-1}, k} \langle \bar{\nabla}_{i_1, \dots, i_{m-1}, k}^m, E_j \rangle \right) E_j \\
&+ \cdots + \sum_{j=1}^{n+1} (-1)^m q_{k, \dots, k} \langle \bar{\nabla}_{k, \dots, k}^m, E_j \rangle E_j \\
&- \sum_{j=1}^{n+1} \left(\sum_{i_2, \dots, i_m=1}^n q_{h, i_2, \dots, i_m} \langle \bar{\nabla}_{h, i_2, \dots, i_m}^m, E_j \rangle \right) E_j \\
&+ \cdots + \sum_{j=1}^{n+1} (-1)^m q_{h, k, \dots, k} \langle \bar{\nabla}_{h, k, \dots, k}^m, E_j \rangle E_j \\
&- \cdots - \sum_{j=1}^{n+1} \left(\sum_{\substack{i_1, \dots, i_{m-1}=1 \\ i_1, \dots, i_{m-1} \neq k}}^n q_{i_1, \dots, i_{m-1}, h} \langle \bar{\nabla}_{i_1, \dots, i_{m-1}, h}^m, E_j \rangle \right) E_j \\
&+ \cdots + \sum_{j=1}^{n+1} (-1)^m q_{k, \dots, k, h} \langle \bar{\nabla}_{k, \dots, k, h}^m, E_j \rangle E_j \\
&+ \cdots + \sum_{j=1}^{n+1} (-1)^m q_{h, \dots, h} \langle \bar{\nabla}_{h, \dots, h}^m, E_j \rangle E_j \\
&= \sum_{\substack{i_1=1 \\ i_1 \neq h, k}}^n q_{i_1} \bar{\nabla}_{i_1}^1 - q_k \bar{\nabla}_k^1 - q_h \bar{\nabla}_h^1, E_j + \cdots \\
&+ \sum_{\substack{i_1, \dots, i_m=1 \\ i_1, \dots, i_m \neq h, k}}^n q_{i_1, \dots, i_m} \bar{\nabla}_{i_1, \dots, i_m}^m - \sum_{\substack{i_2, \dots, i_m=1 \\ i_2, \dots, i_m \neq h}}^n q_{k, i_2, \dots, i_m} \bar{\nabla}_{k, i_2, \dots, i_m}^m
\end{aligned}$$

$$\begin{aligned}
 & - \sum_{\substack{i_1, \dots, i_{m-1}=1 \\ i_1, \dots, i_{m-1} \neq h}}^n q_{i_1, \dots, i_{m-1}, k} \bar{\nabla}_{i_1, \dots, i_{m-1}, k}^m + \dots + (-1)^m q_{k, \dots, k} \bar{\nabla}_{k, \dots, k}^m \\
 & - \sum_{\substack{i_2, \dots, i_m=1 \\ i_2, \dots, i_m \neq k}}^n q_{h, i_2, \dots, i_m} \bar{\nabla}_{h, i_2, \dots, i_m}^m + \dots + (-1)^m q_{h, k, \dots, k} \bar{\nabla}_{h, k, \dots, k}^m - \dots \\
 & - \sum_{\substack{i_1, \dots, i_{m-1}=1 \\ i_1, \dots, i_{m-1} \neq k}}^n q_{i_1, \dots, i_{m-1}, h} \bar{\nabla}_{i_1, \dots, i_{m-1}, h}^m + \dots + (-1)^m q_{k, \dots, k, h} \bar{\nabla}_{k, \dots, k, h}^m \\
 & + \dots + (-1)^m q_{h, \dots, h} \bar{\nabla}_{h, \dots, h}^m \\
 & = \underbrace{\sum_{i_1=1}^n q_{i_1} \bar{\nabla}_{i_1}^1 + \dots + \sum_{i_1, \dots, i_m=1}^n q_{i_1, \dots, i_m} \bar{\nabla}_{i_1, \dots, i_m}^m}_{\forall j: i_j \neq h \wedge i_j \neq k} \\
 & - \underbrace{\sum_{i_1=1}^n q_{i_1} \bar{\nabla}_{i_1}^1 + \dots + \sum_{i_1, \dots, i_m=1}^n q_{i_1, \dots, i_m} \bar{\nabla}_{i_1, \dots, i_m}^m}_{\exists j: i_j = h \wedge \forall t: i_t \neq k} \\
 & - \underbrace{\sum_{i_1=1}^n q_{i_1} \bar{\nabla}_{i_1}^1 + \dots + \sum_{i_1, \dots, i_m=1}^n q_{i_1, \dots, i_m} \bar{\nabla}_{i_1, \dots, i_m}^m}_{\exists j: i_j = k \wedge \forall t: i_t \neq h} \\
 & + \underbrace{\sum_{i_1=1}^n q_{i_1} \bar{\nabla}_{i_1}^1 + \dots + \sum_{i_1, \dots, i_m=1}^n q_{i_1, \dots, i_m} \bar{\nabla}_{i_1, \dots, i_m}^m}_{\exists j: i_j = h \wedge \exists t: i_t = k}.
 \end{aligned}$$

From Definition 14, if $k < h$ then the above expression is the same as $B_{n+1}^{k,h}(Q)$, thus we write $P = B_{n+1}^{k,h}(Q)$. If $k > h$ then we get $P = B_{n+1}^{h,k}(Q)$.

b) If $k = h$,

$$\begin{aligned}
 P & = \sum_{j=1}^{n+1} \left(\sum_{\substack{i_1=1 \\ i_1 \neq h}}^n q_{i_1} \langle \bar{\nabla}_{i_1}^1, E_j \rangle_{B_{n+1}}^h \right) E_j - \sum_{j=1}^{n+1} q_h \langle \bar{\nabla}_h^1, E_j \rangle_{B_{n+1}}^h E_j \\
 & + \dots + \sum_{j=1}^{n+1} \left(\sum_{\substack{i_1, \dots, i_m=1 \\ i_1, \dots, i_m \neq h}}^n q_{i_1, \dots, i_m} \langle \bar{\nabla}_{i_1, \dots, i_m}^m, E_j \rangle_{B_{n+1}}^h \right) E_j \\
 & - \sum_{j=1}^{n+1} \left(\sum_{i_2, \dots, i_m=1}^n q_{h, i_2, \dots, i_m} \langle \bar{\nabla}_{h, i_2, \dots, i_m}^m, E_j \rangle_{B_{n+1}}^h \right) E_j \\
 & - \dots - \sum_{j=1}^{n+1} \left(\sum_{i_1, \dots, i_{m-1}=1}^n q_{i_1, \dots, i_{m-1}, h} \langle \bar{\nabla}_{i_1, \dots, i_{m-1}, h}^m, E_j \rangle_{B_{n+1}}^h \right) E_j \\
 & + \dots + \sum_{j=1}^{n+1} (-1)^m q_{h, \dots, h} \langle \bar{\nabla}_{h, \dots, h}^m, E_j \rangle_{B_{n+1}}^h E_j
 \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{n+1} \left(\sum_{\substack{i_1=1 \\ i_1 \neq h}}^n q_{i_1} \langle \bar{\nabla}_{i_1}^1, E_j \rangle \right) E_j + \sum_{j=1}^{n+1} q_h \langle \bar{\nabla}_h^1, E_j \rangle E_j \\
&+ \cdots + \sum_{j=1}^{n+1} \left(\sum_{\substack{i_1, \dots, i_m=1 \\ i_1, \dots, i_m \neq h}}^n q_{i_1, \dots, i_m} \langle \bar{\nabla}_{i_1, \dots, i_m}^m, E_j \rangle \right) E_j \\
&+ \sum_{j=1}^{n+1} \left(\sum_{\substack{i_2, \dots, i_m=1 \\ i_2, \dots, i_m \neq h}}^n q_{h, i_2, \dots, i_m} \langle \bar{\nabla}_{h, i_2, \dots, i_m}^m, E_j \rangle \right) E_j \\
&+ \cdots + \sum_{j=1}^{n+1} \left(\sum_{\substack{i_1, \dots, i_{m-1}=1 \\ i_1, \dots, i_{m-1} \neq h}}^n q_{i_1, \dots, i_{m-1}, h} \langle \bar{\nabla}_{i_1, \dots, i_{m-1}, h}^m, E_j \rangle \right) E_j \\
&+ \cdots + \sum_{j=1}^{n+1} (-1)^{m+1} q_{h, \dots, h} \langle \bar{\nabla}_{h, \dots, h}^m, E_j \rangle E_j \\
&= \sum_{j=1}^{n+1} \left(\left\langle \sum_{i_1=1}^n q_{i_1} \bar{\nabla}_{i_1}^1 + \cdots + \sum_{i_1, \dots, i_m=1}^n q_{i_1, \dots, i_m} \bar{\nabla}_{i_1, \dots, i_m}^m, E_j \right\rangle \right) E_j \\
&= Q.
\end{aligned}$$

By Remark 15, $P = B_{n+1}^0(Q) = Q$. \square

Corollary 19. *Let $Q \in \mathcal{P}(M)$ and $0 \leq k_1, k_2, k_3 \leq n$, then*

$$B_{n+1}^{k_1} \left(B_{n+1}^{k_2, k_3}(Q) \right) = \begin{cases} B_{n+1}^{k_3}(Q) & \text{if } k_1 = k_2 \\ B_{n+1}^{k_2}(Q) & \text{if } k_1 = k_3 \\ B_{n+1}^{k_1, k_2, k_3}(Q) & \text{if } k_1 < k_2 < k_3 \\ B_{n+1}^{k_2, k_1, k_3}(Q) & \text{if } k_2 < k_1 < k_3 \\ B_{n+1}^{k_2, k_3, k_1}(Q) & \text{if } k_2 < k_3 < k_1 \end{cases} . \quad (17)$$

Proof. Let $Q = \sum_{i_1=1}^n q_{i_1} \bar{\nabla}_{i_1}^1 + \cdots + \sum_{i_1, \dots, i_m=1}^n q_{i_1, \dots, i_m} \bar{\nabla}_{i_1, \dots, i_m}^m$.

In similar way to Proposition 18,

$$\begin{aligned}
B_{n+1}^{k_1} \left(B_{n+1}^{k_2, k_3}(Q) \right) &= \underbrace{\sum_{i_1=1}^n q_{i_1} \bar{\nabla}_{i_1}^1 + \cdots + \sum_{i_1, \dots, i_m=1}^n q_{i_1, \dots, i_m} \bar{\nabla}_{i_1, \dots, i_m}^m}_{\forall j: i_j \neq k_1 \wedge i_j \neq k_2 \wedge i_j \neq k_3} \\
&- \underbrace{\sum_{i_1=1}^n q_{i_1} \bar{\nabla}_{i_1}^1 + \cdots + \sum_{i_1, \dots, i_m=1}^n q_{i_1, \dots, i_m} \bar{\nabla}_{i_1, \dots, i_m}^m}_{\exists j: i_j = k_1 \wedge \forall t: i_t \neq k_2 \text{ or } \exists j: i_j = k_1 \wedge \forall t: i_t \neq k_3} \\
&- \underbrace{\sum_{i_1=1}^n q_{i_1} \bar{\nabla}_{i_1}^1 + \cdots + \sum_{i_1, \dots, i_m=1}^n q_{i_1, \dots, i_m} \bar{\nabla}_{i_1, \dots, i_m}^m}_{\exists j: i_j = k_2 \wedge \forall t: i_t \neq k_1 \text{ or } \exists j: i_j = k_3 \wedge \forall t: i_t \neq k_1}
\end{aligned}$$

$$+ \underbrace{\sum_{i_1=1}^n q_{i_1} \nabla_{i_1}^1 + \cdots + \sum_{i_1, \dots, i_m=1}^n q_{i_1, \dots, i_m} \nabla_{i_1, \dots, i_m}^m}_{\exists j: i_j=k_1 \wedge \exists t: i_t=k_2 \wedge \exists r: i_r=k_3}$$

Equality (17) follows from Definition 14. □

Corollary 20. Let $B_{n+1}^{k_1, \dots, k_l} \in \mathcal{B}_{n+1}$. Then

$$B_{n+1}^{k_1, \dots, k_l} = B_{n+1}^{k_1} \left(B_{n+1}^{k_2} \left(\cdots \left(B_{n+1}^{k_l} \right) \right) \cdots \right). \tag{18}$$

Proof. It follows from Proposition 18 and Corollary 19. □

Now, we define an operation on $\mathcal{B}_{n+1} \times \mathcal{B}_{n+1}$.

Definition 21. Define $\circ : \mathcal{B}_{n+1} \times \mathcal{B}_{n+1} \rightarrow \mathcal{B}_{n+1}$ by $(B_{n+1}^{k_1, \dots, k_l} \circ B_{n+1}^{h_1, \dots, h_t})(P) = B_{n+1}^{k_1, \dots, k_l} (B_{n+1}^{h_1, \dots, h_t}(P))$, for all $P \in \mathcal{P}(M)$.

Proposition 22. $(\mathcal{B}_{n+1}, \circ)$ is an abelian group.

Proof. By Proposition 18 and corollaries 19 and 20, \circ is a well-defined operation. By Proposition 18 and Corollary 20, \circ is a commutative operation. From Remark 15 and Proposition 18, the identity of $(\mathcal{B}_{n+1}, \circ)$ is the B_{n+1}^0 -product. By Proposition 18 and Corollary 19,

$$B_{n+1}^{k_1} \circ (B_{n+1}^{k_2} \circ B_{n+1}^{k_3}) = \begin{cases} B_{n+1}^{k_1} \circ B_{n+1}^0 & \text{if } k_2 = k_3 \\ B_{n+1}^{k_1} \circ B_{n+1}^{k_2, k_3} & \text{if } k_2 < k_3 \\ B_{n+1}^{k_1} \circ B_{n+1}^{k_3, k_2} & \text{if } k_2 > k_3 \end{cases}$$

$$= \begin{cases} B_{n+1}^{k_1} & \text{if } k_2 = k_3 \\ B_{n+1}^{k_2} & \text{if } k_1 = k_3 \\ B_{n+1}^{k_3} & \text{if } k_1 = k_2 \\ B_{n+1}^{k_1, k_2, k_3} & \text{if } k_1 < k_2 < k_3 \\ B_{n+1}^{k_2, k_1, k_3} & \text{if } k_2 < k_1 < k_3 \\ B_{n+1}^{k_2, k_3, k_1} & \text{if } k_2 < k_3 < k_1 \\ B_{n+1}^{k_1, k_3, k_2} & \text{if } k_1 < k_3 < k_2 \\ B_{n+1}^{k_3, k_1, k_2} & \text{if } k_3 < k_1 < k_2 \\ B_{n+1}^{k_3, k_2, k_1} & \text{if } k_3 < k_2 < k_1 \end{cases} = B_{n+1}^{k_1, k_2} \circ B_{n+1}^{k_3}$$

$$= (B_{n+1}^{k_1} \circ B_{n+1}^{k_2}) \circ B_{n+1}^{k_3}.$$

According to Corollary 20, \circ is an associative operation. From the above properties,

$$B_{n+1}^{k_1, \dots, k_l} \circ B_{n+1}^{k_1, \dots, k_l} = B_{n+1}^{k_1} \circ \dots \circ B_{n+1}^{k_l} \circ B_{n+1}^{k_1} \circ \dots \circ B_{n+1}^{k_l}$$

$$= \left(B_{n+1}^{k_1} \circ B_{n+1}^{k_1} \right) \circ \dots \circ \left(B_{n+1}^{k_l} \circ B_{n+1}^{k_l} \right) = B_{n+1}^0 \circ \dots \circ B_{n+1}^0 = B_{n+1}^0.$$

Therefore $\left(B_{n+1}^{k_1, \dots, k_l} \right)^{-1} = B_{n+1}^{k_1, \dots, k_l}$. \square

Remark 23. The order of $B_{n+1}^{k_1, \dots, k_l}$ is $\begin{cases} 1 & \text{if } B_{n+1}^{k_1, \dots, k_l} = B_{n+1}^0, \\ 2 & \text{in other case.} \end{cases}$

Theorem 24. *Let P and Q be two B_{n+1} -congruent operators. There exists $B_{n+1}^{k_1, \dots, k_l} \in \mathcal{B}_{n+1}$ such that $P = B_{n+1}^{k_1, \dots, k_l}(Q)$, and this is an equivalence relationship.*

Proof. Let $P, Q, R \in \mathcal{P}(M)$. By Proposition 22, for all $P \in \mathcal{P}(M)$ there exists $B_{n+1}^0 \in \mathcal{B}_{n+1}$ such that $P = B_{n+1}^0(P)$. Therefore, P is B_{n+1} -congruent to P , for all $P \in \mathcal{P}(M)$.

If P is B_{n+1} -congruent to Q then there exists $B_{n+1}^k \in \mathcal{B}_{n+1}$ such that $P = B_{n+1}^k(Q)$. From Proposition 22 $Q = B_{n+1}^k(B_{n+1}^k(Q)) = B_{n+1}^k(P)$, that is, there exists $B_{n+1}^k \in \mathcal{B}_{n+1}$ such that $Q = B_{n+1}^k(P)$. Therefore, Q is B_{n+1} -congruent to P .

If $P = B_{n+1}^k(R)$ and $R = B_{n+1}^h(Q)$, it follows from Proposition 18 that $P = B_{n+1}^{k,h}(Q)$, with $B_{n+1}^{k,h} \in \mathcal{B}_{n+1}$. \square

5 Examples

Lastly, we show some examples of the Laplacian and D'Alembertian operators on vector fields and B_{n+1}^k -products.

In [3] the reader can find more information about non null surfaces of constant curvature in \mathbb{R}_1^3 , mean curvature vector fields, and Laplacian on mean curvature vector fields on non null surfaces in \mathbb{R}_1^3 .

Example 25. Let x_1, x_2, x_3 be a coordinate system in \mathbb{R}_1^3 such that $\{\partial_1, \partial_2, \partial_3\}$ is an orthonormal basis for \mathbb{R}_1^3 , where $\partial_i = \frac{\partial}{\partial x_i}$. The pseudosphere S_1^2 in \mathbb{R}_1^3 is the surface defined by $S_1^2 = \{(x_1, x_2, x_3) \in \mathbb{R}_1^3 : -x_1^2 + x_2^2 + x_3^2 = 1\}$.

S_1^2 can be parametrized as $x_1 = \sinh \omega$, $x_2 = \cos \theta \cosh \omega$, $x_3 = \sin \theta \cosh \omega$, where $\omega \in \mathbb{R}$ and $0 \leq \theta < 2\pi$.

The tangent vectors are expressed as follows:

$$\partial_\omega = \frac{\partial}{\partial \omega} = \cosh \omega \partial_1 + \sinh \omega \cos \theta \partial_2 + \sinh \omega \sin \theta \partial_3,$$

$$\partial_\theta = \frac{\partial}{\partial \theta} = -\cosh \omega \sin \theta \partial_2 + \cosh \omega \cos \theta \partial_3.$$

The unit normal vector to the surface S_1^2 at (ω, θ) is

$$N = (\sinh \omega, \cosh \omega \cos \theta, \cosh \omega \sin \theta).$$

Hence $\langle \partial_\omega, \partial_\omega \rangle = -1$, $\langle \partial_\omega, \partial_\theta \rangle = 0$, $\langle \partial_\theta, \partial_\theta \rangle = \cosh^2 \omega$ and $\langle N, N \rangle = 1$.

According to Definition 9, the Laplacian operator on vector fields, Δ , for S_1^2 is given by:

$$\Delta = -\bar{\nabla}_{\partial_\omega} \bar{\nabla}_{\partial_\omega} + \frac{1}{\cosh^2 \omega} \bar{\nabla}_{\partial_\theta} \bar{\nabla}_{\partial_\theta}.$$

The mean curvature vector field H for S_1^2 is given by $H = -N$ (cf. [3]).

By applying to H the Laplacian operator on vector fields for S_1^2 , we obtain $\Delta H = 2N - \tanh \omega \partial_\omega = -2H - \tanh \omega \partial_\omega$, (cf. [3]).

Example 26. Let x_1, x_2, x_3 be a coordinate system in \mathbb{R}_1^3 such that $\{\partial_1, \partial_2, \partial_3\}$ is an orthonormal basis for \mathbb{R}_1^3 , where $\partial_i = \frac{\partial}{\partial x_i}$. The cylinder $\mathbb{R}_1^1 \times S^1$ in \mathbb{R}_1^3 is the surface defined by $\mathbb{R}_1^1 \times S^1 = \{(x_1, x_2, x_3) \in L^3 : x_2^2 + x_3^2 = 1\}$.

$\mathbb{R}_1^1 \times S^1$ can be parametrized as $x_1 = t$, $x_2 = \cos \theta$, $x_3 = \sin \theta$, where $t \in \mathbb{R}$ and $0 < \theta < 2\pi$.

The tangent vectors are expressed as $\partial_t = \partial_1$, $\partial_\theta = -\sin \theta \partial_2 + \cos \theta \partial_3$.

The unit normal vector to the surface $\mathbb{R}_1^1 \times S^1$ at (t, θ) is $N = (0, \cos \theta, \sin \theta)$.

Hence, $\langle \partial_t, \partial_t \rangle = -1$, $\langle \partial_\theta, \partial_t \rangle = 0$, $\langle \partial_\theta, \partial_\theta \rangle = 1$ and $\langle N, N \rangle = 1$.

According to Definition 9, the D'Alembertian operator on vector fields for $\mathbb{R}_1^1 \times S^1$ is given by $\square = \bar{\nabla}_{\partial_t} \bar{\nabla}_{\partial_t} + \bar{\nabla}_{\partial_\theta} \bar{\nabla}_{\partial_\theta}$.

The mean curvature vector field H is given by $H = -\frac{1}{2}N$ (cf. [3]).

Since $\bar{\nabla}_{\partial_\omega} \bar{\nabla}_{\partial_\omega} H = 0$ and $\bar{\nabla}_{\partial_\theta} \bar{\nabla}_{\partial_\theta} H = -\frac{1}{2} \bar{\nabla}_{\partial_\theta} (\partial_\theta) = \frac{1}{2}N$, applying to H the D'Alembertian operator on vector fields for $\mathbb{R}_1^1 \times S^1$ we obtain $\square H = \frac{1}{2}N = -H$.

Example 27. Let Δ be the Laplacian operator on vector fields for a surface M in \mathbb{R}_1^3 . We denote the B_{n+1} -equivalence class of Δ with $[\Delta]$. If Δ is defined by

$$\Delta = g^{11} \bar{\nabla}_{\partial_1} \bar{\nabla}_{\partial_1} + g^{12} \bar{\nabla}_{\partial_1} \bar{\nabla}_{\partial_2} + g^{21} \bar{\nabla}_{\partial_2} \bar{\nabla}_{\partial_1} + g^{22} \bar{\nabla}_{\partial_2} \bar{\nabla}_{\partial_2},$$

then,

$$\begin{aligned} B_{n+1}^0(\Delta) &= \Delta, \\ B_{n+1}^1(\Delta) &= -(g^{11} \bar{\nabla}_{\partial_1} \bar{\nabla}_{\partial_1} + g^{12} \bar{\nabla}_{\partial_1} \bar{\nabla}_{\partial_2} + g^{21} \bar{\nabla}_{\partial_2} \bar{\nabla}_{\partial_1}) + g^{22} \bar{\nabla}_{\partial_2} \bar{\nabla}_{\partial_2} \\ &= -\Delta + 2g^{22} \bar{\nabla}_{\partial_2} \bar{\nabla}_{\partial_2}, \\ B_{n+1}^2(\Delta) &= g^{11} \bar{\nabla}_{\partial_1} \bar{\nabla}_{\partial_1} - (g^{12} \bar{\nabla}_{\partial_1} \bar{\nabla}_{\partial_2} + g^{21} \bar{\nabla}_{\partial_2} \bar{\nabla}_{\partial_1} + g^{22} \bar{\nabla}_{\partial_2} \bar{\nabla}_{\partial_2}) \\ &= -\Delta + 2g^{11} \bar{\nabla}_{\partial_1} \bar{\nabla}_{\partial_1}, \end{aligned}$$

and

$$\begin{aligned} B_{n+1}^{12}(\Delta) &= -g^{11} \bar{\nabla}_{\partial_1} \bar{\nabla}_{\partial_1} + g^{12} \bar{\nabla}_{\partial_1} \bar{\nabla}_{\partial_2} + g^{21} \bar{\nabla}_{\partial_2} \bar{\nabla}_{\partial_1} - g^{22} \bar{\nabla}_{\partial_2} \bar{\nabla}_{\partial_2} \\ &= \Delta - 2(g^{11} \bar{\nabla}_{\partial_1} \bar{\nabla}_{\partial_1} + g^{22} \bar{\nabla}_{\partial_2} \bar{\nabla}_{\partial_2}) = -\frac{B_{n+1}^1(\Delta) + B_{n+1}^2(\Delta)}{2}. \end{aligned}$$

are B_{n+1} -congruent operators. Therefore,

$$[\Delta] = \left\{ \Delta, \square, B_{n+1}^2(\Delta), -\frac{\square + B_{n+1}^2(\Delta)}{2} \right\}.$$

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