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Spec(R) y Axiomas de Separación entre T_0 y T_1

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Abstract

In this paper we characterize the rings whose prime spectrum satisfy some of the separation axioms between T_0 and T_1. Additionally, the notion of D-ring , m-ring , U-ring and Y-ring are introduced. Finally, some remarks are made on the defined rings.
Key words and phrases: ring, prime ideal, prime spectrum, low separation axioms.

Resumen

En este artículo se caracterizan los anillos cuyo espectro primo satisface algunos axiomas de separación entre T_0 y T_1. Para esto se introducen las nociones de D-anillo , m-anillo , U-anillo y Y-anillo. Finalmente, se hacen algunos comentarios sobre los anillos definidos.
Palabras y frases clave: anillo, ideal primo, espectro primo, axiomas bajos de separación.

1 Introduction

The theory of the prime spectrum of a ring has been developed from 1930. The precursor ideas are in the one to one correspondence between Boolean lattices and Boolean rings (Stone's Theorem). The modern theory was developed by Jacobson and Zariski mainly. Since the spectrum can be seem as a functor from the category of commutative rings with identity to the topological spaces, a "dictionary" of properties between the objects and morphisms

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of this categories can be constructed. Then, certain topological properties assumed in Spec(R) can be translated into algebraic properties of the ring R. Some of these properties are well known and have been included as exercises in some classic texts [2,5] or developed partially in [1] and [9].

The aim of this paper is to characterize the rings whose spectra satisfy some topological notions like T_D, T_{DD}, T_F, T_{FF}, T_Y, T_{YS} and to show that rings exist for each one of these classes. This characterizations lead to the notion of D-ring, Y-ring and m-ring.

2 Preliminaries

2.1 Zariski Topology

In this paper it is assumed that all the rings are commutative with identity. Now we present the Zariski topology defined on the set of prime ideals of a commutative ring with identity and a base for this topology. A complete exhibition of this topic can be consulted in [1], [2] and [9].

Let R be a ring and Spec(R) the set of all prime ideals of R. For $E \subseteq R$ and $V(E) = \{P \in Spec(R) \mid E \subseteq P\}$, the following properties hold:

i) $V(0) = Spec(R)$, $V(1) = \emptyset$.

ii) If $\{E_i\}_{i \in I}$ is a family of sets of R then $\bigcap_{i \in I} V(E_i) = V(\bigcup_{i \in I} E_i)$.

iii) If E_1 and E_2 are subsets of R then $V(E_1) \cup V(E_2) = V(\langle E_1 \rangle \cap \langle E_2 \rangle) = V(\langle E_1 \rangle \langle E_2 \rangle)$. $\langle E \rangle$ is the ideal generated by E.

The collection $\varsigma = \{V(E) \mid E \subseteq R\}$ satisfies the axioms for closed subsets of a topological space. Defining $\tau = \{Spec(R) - V(E) \mid E \subseteq R\}$, one has that $(Spec(R), \tau)$ is a topological space, called the prime spectrum of R and τ is known as the Zariski topology. The open sets are $X - V(E)$, where $E \subseteq R$ and $X = Spec(R)$. For $f \in R$, the collection $\beta = \{X_f \mid f \in R\}$ with $X_f = X - V(f)$, is a base for the Zariski topology.

There are interesting relations between the elements X_f and the algebraic properties of the ring R. However, they will not be presented here and can be consulted in [1], [2] and [9].

The following proposition will be used repeatedly in the section 3. Its proof can be found in [13].

Proposition 2.1.1. If X is a finite poset, then there is a ring A with $Spec(A) \cong X$ (as ordered sets).

2.2 Separation Axioms Between $\mathbf{T}_0$ and $\mathbf{T}_1$

The study of the separation axioms began with the works of Uryshon and later with Freudenthal and Van Est in 1951, however they referred to stronger axioms that T_1. The development of the axioms between T_0 and T_1 began with the work of Young, who introduced the T_Y spaces in your study on locally connected spaces [15]. Later Aull and Thron [3], based on the observation that in a T_0 space $\{x\}'$ is the union of closed sets for every x, while in a T_1 space $\{x\}' = \emptyset$, and hence closed, for all x. And that in a T_1 space $\{x\}' \cap \{y\}' = \emptyset$ for $x \neq y$, introduce the axioms T_{UD}, T_D and T_{DD}. The axioms T_F and T_{FF} arise from a consideration of weak separation of finite sets. Since given two finite sets F_1, F_2, with $F_1 \cap F_2 = \emptyset$ in a T_1 space, then $F_1 \vdash F_2$ and $F_2 \vdash F_1$. These axioms will be considered in this paper and defined next.

Remark
Remember that a set is called *degenerate* if it is the empty set or a singleton. On the other hand $A \vdash B$ indicates that there is an open G such that $A \subseteq G$ and $G \cap B = \emptyset$.

Definition 2.2.1. A topological space (X, τ) is:
a) T_{UD} if for any $x \in X$, $\{x\}'$ is the union of disjoint closed sets.
b) T_D if for any $x \in X$, $\{x\}'$ is a closed set.
c) T_{DD} if it is T_D and for any $x, y \in X$, $x \neq y$, $\{x\}' \cap \{y\}' = \emptyset$.
d) T_F if for any $x \in X$, and any finite set F of X such that $x \notin F$, either $\{x\} \vdash F$ or $F \vdash \{x\}$.
e) T_{FF} if for any finite subsets $F_1, F_2 \subseteq X$, with $F_1 \cap F_2 = \emptyset$, either $F_1 \vdash F_2$ or $F_2 \vdash F_1$.
f) T_Y if for any $x, y \in X$, with $x \neq y$, $\overline{\{x\}} \cap \overline{\{y\}}$ is degenerate.
g) T_{YS} if for any $x, y \in X$, with $x \neq y$, $\overline{\{x\}} \cap \overline{\{y\}}$ is either $\emptyset$ or $\{x\}$ or $\{y\}$.
The following diagram shows the relation between these axioms:

$$
\begin{array}{ccccccccc}
T_1 & \longrightarrow & T_{DD} & & \longrightarrow & & T_D & & \\
 & & \downarrow & & & & & & \\
\downarrow & & T_{YS} & & & & \downarrow & & \\
 & & \downarrow & & & & & & \\
T_{FF} & \longrightarrow & T_Y & \longrightarrow & T_F & \longrightarrow & T_{UD} & \longrightarrow & T_0
\end{array}
$$

Figure 1

3 Spec(R) and Separation Axioms Between $\mathbf{T}_0$ and $\mathbf{T}_1$

3.1 New Classes of Commutative Rings

Since Spec(R) is T_0 for all ring R and it is T_1 (and hence T_2) if and only if R is a zero-dimensional ring [2], the following questions arise naturally: Does Spec(R) satisfy the axioms between T_0 and T_1 for all rings R?, Does Spec(R) satisfy the axioms between T_0 and T_1 if and only if R is a zero-dimensional ring?.

Both questions have negative answer. In fact, $Spec(\mathbb{Z})$ is not a T_D space and it is a T_{UD} space. Since for $P = \langle 0 \rangle$ the set $\{P\}' = \{\langle 2 \rangle, \langle 3 \rangle, \ldots\} = \{\langle 2 \rangle\} \cup \{\langle 3 \rangle\} \cup \{\langle 5 \rangle\} \cup \ldots$, and if $P \neq \langle 0 \rangle$, then $\{P\}' = \emptyset$.

Definition 3.1.1. A ring R is D-ring if every prime non maximal ideal P, is different to the intersection of the prime ideals properly containing it.

It is clear that the zero-dimensional rings and the semilocal rings of dimension one, are $D-rings$. $\mathbb{Z}$ and all Jacobson ring R with $\dim R \geq 1$ are not $D-rings$.

Proposition 3.1.2. Spec(R) is T_D iff R is a D-ring.

Proof. For any ring R and $P \in Spec(R)$ it holds that $\{P\}' = V(P) \setminus \{P\}$. Thus, the set $\{P\}'$ is closed if and only if P does not contain the intersection of the prime ideals properly containing it. That is to say, if and only if R is a D-ring. ■

A topological space is called a space of Alexandrov, if the arbitrary union of closed sets is a closed set. We recall that in a T_0 space, the set of points of accumulation of a singleton is union of closed sets. Thus, if Spec(R) is a space of Alexandrov, then Spec(R) is T_D. This means that the class of rings whose spectrum is a space of Alexandrov, is a subclass of the $D - rings$.

Proposition 3.1.3. Spec(R) is T_F iff $\dim R \leq 1$.

Proof. Let $P, P_1, P_2, \ldots, P_n \in Spec(R)$, $P \neq P_i$ for all i. Then $\{P\} \vdash \{P_1, P_2, \ldots, P_n\}$ or $\{P_1, P_2, \ldots, P_n\} \vdash \{P\}$ if and only if $P_1 \cap P_2 \cap \ldots \cap P_n \nsubseteq P$ or $P \nsubseteq P_i$ for all i. That is to say, if and only if $\dim R \leq 1$. ■

Definition 3.1.4. A ring R is a Y-ring if any two distinct minimal prime ideals are contained in at most one maximal ideal.

Observe that this class of rings contains two subclasses: the $pm - rings$ that are rings where each prime ideal is contained in only one maximal ideal and the $m - rings$ that next are defined.

Definition 3.1.5. A ring R is an m-ring if all prime ideal contains only one minimal prime ideal.

The $pm - rings$ have been studied in [8] and examples of them are in the theory of rings of functions [10]. Rings with this property can also be built, according to the Proposition 2.2.1. The $m - rings$ don't appear in the consulted references. However, Hochster in [11] proves that given a ring R, there exists a ring T whose spectrum has the inverse order of the spectrum of R .

In fact, the class of $Y - rings$ is wide because all the domains, the local rings and the zero-dimensional rings are of this class. Also, for the Proposition 2.2.1, there are rings with finite spectrum, which are not $Y - rings$.

Proposition 3.1.6. Spec(R) is T_Y iff R is a Y-ring and $\dim R \leq 1$.

Proof. As T_Y implicate T_F, then Spec(R) is T_Y if and only if $\dim R \leq 1$ and for any minimal prime ideals P, Q, the set $\overline{\{P\}} \cap \overline{\{Q\}}$ is degenerate. That is to say, if and only if R is a Y-ring and $\dim R \leq 1$. ■

Proposition 3.1.7. Spec(R) is T_{YS} iff R is an m-ring and $\dim R \leq 1$.

Proof. As T_{YS} implicate T_Y, then Spec(R) is T_{YS} if and only if R is a Y-ring, $\dim R \leq 1$ and any two minimal prime ideals are comaximal. That is to say, R is an m-ring and $\dim R \leq 1$. ■

Remark. If P is a prime non maximal ideal, with $J_s(P)$ will be denoted the intersection of all maximal ideals that contain P. A family $\mathcal{F}$ of ideals of R, is comaximal if any $A, B \in \mathcal{F}$ are comaximal.

Proposition 3.1.8. Spec(R) is T_{DD} iff R is an m-ring, $\dim R \leq 1$ and for all prime non maximal ideal P, $J_s(P) \neq P$.

Proof. As T_{DD} implies T_{YS} and T_D, then Spec(R) is T_{DD} if and only if R is a D-ring, m-ring, $\dim R \leq 1$ and for any prime ideals P, Q, $\{P\}' \cap \{Q\}' = \emptyset$. Thus, $\{T\}'$ is a closed set for all minimal prime ideal T. The result follows.■

Proposition 3.1.9. Spec(R) is T_{FF} iff $\dim R = 0$ or $\dim R = 1$ and exist at most one minimal non maximal prime ideal, or one maximal non minimal prime ideal.

Proof. If $\dim R \neq 0$ and exists more than one minimal non maximal prime ideal and more than one maximal non minimal prime ideal. Then, there are different prime ideals $P, Q, R, S \in Spec(R)$, such that $P \subsetneqq Q$ and $R \subsetneqq S$. Now, for the sets $F_1 = \{P, S\}$ and $F_2 = \{R, Q\}$, $F_1 \nvdash F_2$ and $F_2 \nvdash F_1$. The converse is evident. ■

For the characterization of the rings whose spectrum is T_{UD}, the notion of U-ring is used. However don't characterize this property.

Definition 3.1.10. A ring R is an U-ring if for all prime non maximal ideal P, $\{P\}'$ is comaximal.

It is clear that the zero-dimensional rings, the rings of dimension 1 and the valuation domains are $U - rings$. In fact for the Proposition 2.2.1, there are $U - rings$ of any finite dimension. Also, if Spec(R) is a tree as ordered set then R is a U-ring. In [13] is proven that there are domains (Prüfer and Bézout) whose spectrum is a tree.

Observe that, if for all prime non maximal ideal P, $\{P\}'$ has minimal prime ideals (for example rings of finite dimension or rings where each prime has finite depth). Then, R is an U-ring if each pair of them is comaximal.

Proposition 3.1.11. If $\dim R$ is finite and R is an U-ring, then Spec(R) is T_{UD}.

Proof. For $P \in Spec(R)$, let M be the set of minimal prime ideals of $\{P\}'$. Then $\{P\}' = \bigcup_{Q \in M} V(Q)$ and this union is disjoint. Since, if $V(Q_1) \cap V(Q_2) \neq \emptyset$, there exists $T \in Spec(R)$ such that $Q_1 \subseteq T$ and $Q_2 \subseteq T$. Hence, $Q_1 + Q_2 \neq R$, which is false. ■

In the previous proposition, it is not required finite dimension only finite depth for all prime ideal.

In the following example it is shown that the converse of the Proposition 3.1.11, in general is false.

Example 3.1.12. For the Proposition 2.2.1 there exists a ring R whose spectrum is isomorphic at the diagram of the Figure 2. And since the spectrum is finite R is a D-ring. Thus, Spec(R) is T_D and T_{UD}. Now, $\{P_1\}' = \{P_2, P_3, P_4\}$ and $P_2 + P_3 \neq R$.. Therefore, R is not an $U - ring$.

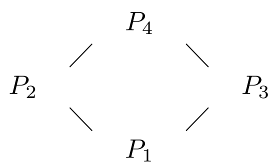

Figure 2

3.2 Final Remarks

The definition of the $D - rings$, $U - rings$, $m - rings$ and $U - rings$, can drive to the realization of new works, because one could make a study of the algebraic properties of this rings and a study of their spectra. Some of the topics that could be developed are the following ones: dimension, product ring, ring of fractions, quotient ring, ring of polynomials, ring of formal power series, extensions of ring and other characterizations of this rings. Some observations that can be made regarding these topics are: if R is local then R is a D-ring iff Spec(R) is T_{UD}, if Spec(R) is finite, then R is a $D - ring$, the quotient ring and the ring of fractions of a D-ring (U-ring) is a D-ring (U-ring), if I is a prime ideal of the ring R, then R/I is a Y-ring but the quotient ring for ideals in general is not an U-ring, the $U - rings$ and $Y - rings$ admit zero divisor.

To conclude we mention that actually the theory of the prime spectrum has been developed for modules ([4], [6], [7], [14]). This theory generalizes the ring case. Thus, all the characterizations carried out in this paper could be studied for modules.

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