Divulgaciones Matemáticas Vol. 14 No. 2(2006), pp. 95–99

# A Note on Best Approximation and Quasiconvex Multimaps

*Una Nota sobre Mejor Aproximación y Aplicaciones Cuasiconvexas*

Zoran D. Mitrović
Faculty of Electrical Engineering
University of Banja Luka
78000 Banja Luka, Patre 5
Bosnia and Herzegovina
E-mail address: zmitrovic@etfbl.net

**Abstract**

In this paper, using the methods of the KKM theory and the new notion of the measure of quasiconvexity, we prove a result on the best approximation for multimaps. As an application, a coincidence point result is also given.
**Key words and phrases**: best approximation, KKM map, coincidence point.

**Resumen**

En este artículo, usando los métodos de la teoría KKM y la nueva noción de la medida de cuasiconvexidad, probamos un resultado sobre la mejor aproximación para multiaplicaciones. Como aplicación, también se da un resultado sobre punto de coincidencia.
**Palabras y frases clave:** mejor aproximación, aplicación KKM, punto de coincidencia.

## 1 Introduction and Preliminaries

Using the methods of the KKM theory, see for example [3, 4], and the notion of the measure of quasiconvexity, we prove in this short paper a result on the

Received 2005/06/13. Revised 2006/08/13. Accepted 2006/08/20.
MSC (2000): Primary 47H10; Secondary 54H25.

best approximations for multimaps.
Let $F : X \to 2^Y$ be a multimap or map, where $2^Y$ denotes the set of all nonempty subsets of $Y$. For $A \subset X$, let

$$F(A) = \cup\{F(x) : x \in A\}.$$

For any $B \subset Y$, the lower inverse of $B$ under $F$ defined by

$$F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}.$$

Let $X$ be a normed space with norm $||\cdot||$. For any nonnegative real number $r$ and any subset $A$ of $X$, we define the $r-$parallel set of $A$ as

$$A + r = \cup\{B[a, r] : a \in A\},$$

where

$$B[a, r] = \{x \in X : ||a - x|| \leq r\}.$$

If $A$ is a nonempty subset of $X$ we define

$$||A|| = \inf\{||a|| : a \in A\}.$$

For bounded and closed subsets $A$ and $B$ of $X$, the Hausdorff distance, denoted by $H(A, B)$, is defined by

$$H(A, B) = \max\{D(A, B), D(B, A)\},$$

where

$$D(A, B) = \sup_{y \in A} \inf_{x \in B} ||x - y||.$$

Let $C$ be a subset of $X$, a map $F : C \to 2^X$ is called quasiconvex (see for example K. Nikodem [2]) if and only if it satisfies the condition

$$F(x_i) \cap S \neq \emptyset,\ i = 1, 2 \Rightarrow F(\lambda x_1 + (1 - \lambda)x_2) \cap S \neq \emptyset,$$

for all convex sets $S \subset Y$, $x_1,\ x_2 \in C$ and $\lambda \in [0, 1]$.
*Remark* 1. A map $F : C \to 2^X$ is quasiconvex if and only if the set $F^-(S)$ is convex for each convex set $S \subseteq X$.

**Definition 1.** Let $X$ and $Y$ be normed spaces and $F : X \to 2^Y$. The real number $mq(F)$, defined by

$$mq(F) = \inf\{r > 0 : co(F^-(S)) \subseteq F^-(S + r) \text{ for all convex } S \subseteq Y\}$$

is called a measure of quasiconvexity for map $F$.

*Remark* 2. 1. If $F$ is quasiconvex map then $mq(F) = 0$.
2. If $\alpha$ is a real number then $mq(\alpha F) = |\alpha| mq(F)$.

A map $F : C \to 2^X$ is called a KKM-map if $co(A) \subset F(A)$ for each finite subset $A$ of $C$.
The following KKM-theorem [1], will be used to prove the main result of this paper.

**Theorem 1.** *Let $X$ be a vector topological space, $C$ a nonempty subset of $X$ and $T : C \to 2^X$ a KKM-map with closed values. If $T(x)$ is compact for at least one $x \in C$ then $\bigcap_{x\in C} T(x) \neq \emptyset$.*

# 2 A Best Approximation Theorem

**Theorem 2.** *Let $X$ be a normed space, $C$ a nonempty convex compact subset of $X$, $F : C \to 2^X$, $G : C \to 2^X$ continuous maps with convex compact values. Then there exists $y_0 \in C$ such that*

$$||G(y_0) - F(y_0)|| \leq \inf_{x\in C} ||G(x) - F(y_0)|| + mq(G).$$

Proof. Let for every $x \in C$, $T : C \to 2^C$ be defined by

$$T(x) = \{y \in C : ||G(y) - F(y)|| \leq ||G(x) - F(y)|| + mq(G)\}.$$

The mappings $F$ and $G$ are continuous, hence they are continuous in Hausdorff distance too. From inequality

$$|\ ||A|| - ||B||\ | \leq H(A, B),$$

for each bounded and closed subsets $A, B$ and $C$ of $X$, it follows that $T(x)$ is closed. Since $C$ is compact we have that $T(x)$ is compact for each $x \in C$. We can prove that $T$ is a KKM mapping, i. e. that for every $\{x_1, \ldots, x_n\} \subset C$

$$co\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^{n} T(x_i) \tag{1}$$

If (1) does not hold, there exists $y = \sum_{i=1}^{n} \lambda_i x_i$, where $\lambda_i \geq 0,\ i = 1, \ldots, n$ and $\sum_{i=1}^{n} \lambda_i = 1$ so that $y \notin \bigcup_{i=1}^{n} T(x_i)$. Then there is

$$||G(y) - F(y)|| > ||G(x_i) - F(y)|| + mq(G) \text{ for every } i = 1, \ldots, n.$$

Sets $G(x)$ and $F(x)$ are compact, then there exist $u_i^0 \in G(x_i) - F(y), \quad i = 1, \ldots, n$, such that
$$||u_i^0|| = ||G(x_i) - F(y)||.$$
Let $S = co\{u_1^0, \ldots, u_n^0\}$. Then we have

$$(G(x_i) - F(y)) \cap S \neq \emptyset \text{ and } x_i \in G^-(F(y) + S),$$

for every $i = 1, \ldots, n$. Since the set $F(y)+S$ is convex and $mq(G)$ is a measure of quasiconvexity, we have

$$y \in G^-(F(y) + S + mq(G) + \epsilon) \text{ for each } \epsilon > 0,$$

therefore
$$G(y) \cap (F(y) + S + mq(G) + \epsilon) \neq \emptyset.$$
We obtain that there exists

$$v \in (G(y) - F(y)) \cap (S + mq(G) + \epsilon),$$

hence there $s \in S$ and $b \in X$ such that $||b|| \leq mq(G) + \epsilon$ and $v = s + b$. Since $s \in S$ there exist $\mu_i \geq 0, \ i = 1, \ldots, n$ and $\sum_{i=1}^{n} \mu_i = 1$ such that $s = \sum_{i=1}^{n} \mu_i u_i^0$. We have

$$||G(y) - F(y)|| \leq ||v|| \leq ||s|| + ||b|| = ||\sum_{i=1}^{n} \mu_i u_i^0|| + ||b|| \leq$$

$$\leq \sum_{i=1}^{n} \mu_i ||u_i^0|| + mq(G) + \epsilon \leq \max_{1 \leq i \leq n} ||G(x_i) - F(y)|| + mq(G) + \epsilon.$$

This contradicts

$$||G(y) - F(y)|| > ||G(x_i) - F(y)|| + mq(G) \text{ for every } i = 1, \ldots, n,$$

and so $T$ is a KKM mapping. From Theorem 1. $\bigcap_{x \in C} T(x) \neq \emptyset$ and so there exists $y_0 \in C$ such that

$$||G(y_0) - F(y_0)|| \leq \inf_{x \in C} ||G(x) - F(y_0)|| + mq(G).$$

**Corollary 1.** *Let $C$ be a nonempty convex compact subset of normed space $X$ and $F, G : C \to 2^X$ continuous maps with convex compact values.*

1. *If $G$ is quasiconvex then there exists $y_0 \in C$ such that*

$$||G(y_0) - F(y_0)|| = \inf_{x \in C} ||G(x) - F(y_0)||.$$

2. *If for every $x \in C$, $F(x) \cap G(C) \neq \emptyset$ then there exists $y_0 \in C$ such that*

$$||G(y_0) - F(y_0)|| \leq mq(G).$$

3. *If $G$ is a quasiconvex mapping and for every $x \in C$, $F(x) \cap G(C) \neq \emptyset$ then there exists $y_0 \in C$ such that*

$$G(y_0) \cap F(y_0) \neq \emptyset.$$

*Remark* 3. If $G(x) = \{x\}$ and $F(x) = \{f(x)\}, x \in C$, where $f$ continuous function Theorem 2. reduces to well-known best approximations theorem of Ky Fan [1].

# References


[1] Fan K. *A Generalization of Tychonoff's Fixed Point Theorem*, Math. Annalen, **142** (1961), 305–310.

[2] Nikodem, K. *K-Convex and K-Concave Set-Valued Functions*, Politechnika, Lodzka, 1989.

[3] Singh S., Watson B. and Srivastava P. *Fixed Point Theory and Best Approximation: The KKM-map Principle*, Kluwer Academic Press, 1997.

[4] Yuan G. X. Z. *KKM Theory and Applications in Nonlinear Analysis*, Marcel Dekker, New York, 1999.