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Partial Characterization of the Global Attractor of the Equation $\dot{x}(t) = -kx(t) + \beta \tanh(x(t-r))$.

Caracterización Parcial del Atractor Global de la Ecuación $\dot{x}(t) = -kx(t) + \beta \tanh(x(t-r))$.

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Abstract

The main goal of this paper is to prove that the global atractor of the equation $\dot{x}(t) = -kx(t) + \beta \tanh(x(t-r))$ is an equilibrium point for any β such that $|\beta| < k$. Furthermore, we exhibit numerical evidences that the trivial solution is global asymptotically stable in all the region of the parameters where the equilibrium is local asymptotically stable.
Key words and phrases: Global attractor, Hopf bifurcation, delay equation.

Resumen

El objetivo central de este trabajo es probar que el atractor global de la ecuación $\dot{x}(t) = -kx(t) + \beta \tanh(x(t-r))$ se reduce a un punto para cualquier valor de β tal que $|\beta| < k$. Además, se dan evidencias numéricas que el punto de equilibrio de la ecuación diferencial es global asintóticamente en toda la región de los parámetros donde el punto de equilibrio es local asintóticamente estable.
Palabras y frases clave: Atractor global, bifurcación de Hopf, ecuación con retardo.

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1 Introduction

In this paper we are going to deal with the following delay differential equation

$$\dot{x}(t) = -kx(t) + \beta \tanh(x(t-r)), \tag{1}$$

where $k > 0$ and $\beta \neq 0$. The equation (1) arises in many applications. For instance in a simplified neural network in which each neuron is represented by a linear circuit with a linear resistor and a linear capacitor. Introducing a nonlinear feedback term, we arrive to the equation (1). Here x represents the voltage of the neuron, k, the ratio of the capacitance to the resistance, and β and r, the feedback strength and time delay, respectively. For the model to make sense physically, k and r should be nonnegative, but β may take any value different of zero. See Leslie Shayer and Sue Ann Campbell [1] and the literature cited therein for more details.

The main goal of this paper is to prove that the global atractor of the equation (1) is an equilibrium point for any β such that $|\beta| < k$. In order to accomplish our goal, we use the technique developed in [3], which basically states a connection between the continuous semi-dynamical system induced by (1) and certain discrete dynamical systems. Furthermore, we explore numerically the possibility that the equilibrium point of the equation (1) remains global asymptotically stable in all the region of the parameters where the trivial solution is local asymptotically stable. This suggests that the first Hopf bifurcation is supercritical.

2 Preliminaries

In this section we will show that the equation (1) admits a global atractor for any k and β, and we will summarize some known results about the location of the roots of certain trascendental equation. It is worth noting that the only equilibrium point of the equation (1) for any $\beta < k$ is the trivial one.

Let us first introduce a few notations that we will need in the sequel. For a given $\sigma \in \mathbb{R}$ and any function $x : [\sigma - r, +\infty) \longrightarrow \mathbb{R}^n$, let us define the function $x_t : [-r, 0] \longrightarrow \mathbb{R}^n$ by $x_t(\theta) = x(t + \theta)$, for any $\theta \in [-r, 0]$ and $t \geq \sigma$. By using the step by step method, we obtain that for any $\phi \in C = C([-r, 0], \mathbb{R})$ the equation (1) has a unique solution $x(t, \phi)$ defined for any $t \geq 0$, which depends continuously on the initial data and parameters. Let us set $T(t)\phi = x_t(\phi)$ for $t \geq 0$. It is well known that $\{T(t)\}_{t \geq 0}$ is a semigroup of strongly continuous operators on C.

Let $x(t,\phi)$ be the solution of (1). Integrating this equation we obtain that

$$x(t,\phi) = x(0,\phi)e^{-kt} + \int_0^t \beta e^{-k(t-s)} \tanh(x(s-r,\phi))ds,$$

which implies that

$$|x(t,\phi)| \leq |\phi(0)|e^{-kt} + \frac{|\beta|}{k}\left[1 - e^{-kt}\right] \ , \ t \geq 0.$$

The last inequality immediately proves the dissipativeness of the equation (1). Furthermore, a straightforward application of Arzela-Ascoli's Lemma gives us that the operator $T(t)$ is completely continuous for any $t \geq r$. Thus, the existence of the global attractor of the equation (1) is an immediate consequence of theorem 3.4.8, p.40, in [4].

Now, we are going to study the local stability of the trivial solution of (1). Linearizing (1) about $x = 0$, we obtain the equation

$$\dot{x}(t) = -kx(t) + \beta x(t-r), \tag{2}$$

which characteristic equation is given by

$$P(\lambda) := \lambda + k - \beta e^{-\lambda r} = 0. \tag{3}$$

It is well known the necessary and sufficient condition in order that all roots of the equation (3) have negative real parts; see for instance the theorem A.5, pag. 416 in [5]. We are not going to apply directly those results. Instead of that, by using the so called D-partition method, we will explore geometrically the stability region of the equation (3) in the parameter plane (k,β). Hereafter, we will assume that $r > 0$ is fixed.

Setting $z := r\lambda$, the equation (3) can be rewritten as follows

$$H(z) := e^z(z + kr) - \beta r = 0. \tag{4}$$

Let us set $z = \mu + i\xi$ in (3), and splitting the equation into the real and the imaginary parts, we obtain

$$\begin{aligned} &\mu\cos\xi + kr\cos\xi - \xi\sin\xi - \beta r e^{-\xi} = 0,\\ &\xi\cos\xi + \mu\sin\xi + kr\sin\xi = 0. \end{aligned} \tag{5}$$

Let us remark that $z = 0$ is a root of (3) if and only if $k = \beta$. Setting $\mu = 0$ and solving (5) for k and β, we obtain the so called "neutral stability curves" in the parameter space (k, β) along which (4) has purely imaginary roots $z = i\xi$; namely

$$\begin{aligned} k(\xi) &= -r^{-1}\xi\cot(\xi), \\ \beta(\xi) &= -r^{-1}\xi\csc(\xi). \end{aligned} \tag{6}$$

As roots come in complex conjugate pairs, we will restrict the analysis to the region $\xi \geq 0$. Notice that the curve is well defined at $\xi = 0$ and $(k(0), \beta(0)) = (-r^{-1}, -r^{-1})$, which coincides with the parameter values at which $\xi = 0$ is a double root. For $n = 0, 1, 2, \cdots$, let us denote by C_n, the curve described by (6) with $n\pi < \xi < (n + 1)\pi$. These curves are depicted in the figure 1.

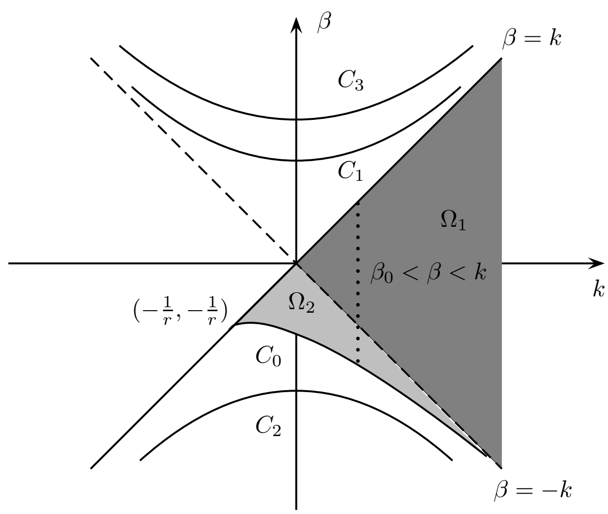


Figure 1:

The following result holds. For the proof we refer the readers to Smith [6].

Proposition 1. *All roots of the equation* (4) *have negative real parts if and only if* $(k, \beta) \in \Omega$. *Where* $\Omega = \Omega_1 \cup \Omega_2$ *is the open region bounded from above by the straightline* $k = \beta$ *and from the bottom by the curve* C_0, *see figure 1.*

3 Global stability of the equilibrium point.

In this section, we will show that for $-k < \beta < k$ the trivial solution of (1) is global asymptotically stable.

In order to carry out this study we follow basically the main ideas developed in [3]. Which states a connection between the continuous semi-dynamical system induced by (1) and certain discrete dynamical systems. More concretely, we will associate to the semi-flow generated by the solution of (1) a map $g : I \longrightarrow I$, where I is an interval, which is given by $g(x) = \beta \tanh(x)/k$. Then, the global stability of the equilibrium point of (1) is obtained from the global stability of the fixed point of g.

The function g satisfies the following properties:

1. For positive values of β (for $\beta < 0$) g is monotone increasing on the whole real line (is monotone decreasing); and g is concave (convex) for $x \geq 0$ and convex (concave) for $x \leq 0$.

2. For $|\beta| < k$, g has a unique fixed point, namely $x = 0$. Moreover, the graph of the function g lies over the line $y = x$ for $x < 0$, and under the line $y = x$ for $x > 0$.

This is enough to realize that $x = 0$ is a global attractor for the map g, for any $|\beta| < k$.

The following theorem is the main result of this paper.

Theorem 1. *If $-k < \beta < k$, then the trivial equilibrium of the equation (1) is global asymptotically stable.*

Proof. From Theorem 1, it follows that the trivial equilibrium of the equation (1) is local asymptotically stable for any β such that $-k < \beta < k$.

Let us suppose first that $-k < \beta < 0$, in this case the function g is strictly decreasing. Let $\phi \in A^*$, where A^* is the global attractor of the equation (1). From (1), we obtain that

$$x(t, \phi) = x(\tau, \phi)e^{-k(t-\tau)} + \beta \int_{\tau}^{t} e^{-k(t-s)} \tanh(x_s(\phi)))ds$$

Since solutions on the global attractor are bounded and defined on the whole real line, letting $\tau \longrightarrow -\infty$ in the previous formula, we get that any solution of (1) with initial data in A^* admits the following representation

$$x(t, \phi) = \beta \int_{-\infty}^{t} e^{-k(t-s)} \tanh(x_s(\phi)))ds$$

Let us set

$$m=\min_{\phi\in A^*}\inf_{t\in\mathbb{R}}\{x(t,\phi)\}\quad\text{and}\quad M=\max_{\phi\in A^*}\sup_{t\in\mathbb{R}}\{x(t,\phi)\}.$$

It is obvious that $0\in[m,M]$.

Let us prove that there exist ϕ_1 and t_1 such that $m=x(t_1,\phi_1)$. Indeed, since $m=\min_{\phi\in A^*}\inf_{t\in\mathbb{R}}\{x(t,\phi)\}$ and the compactness of the set A^*, there exists a ϕ^* such that $m=\inf_{t\in\mathbb{R}}\{x(t,\phi^*)\}$. We may assert the existence of a sequence $\{t_n\}$ such that $m\le x(t_n,\phi^*)<m+1/n$, for all $n\in\mathbb{N}$. Arises two possibilities: the sequence $\{t_n\}$ is bounded from above (or below), in that case we may claim that the same sequence $t_n\to t^*$, which implies that $m=x(t_1,\phi_1)$, where $t_1=t^*$ and $\phi_1=\phi^*$. If $t_n\to+\infty$, we obtain that $\lim_{n\to\infty}x(t_n,\phi^*)=m$. Henceforth $m\in\omega(\phi^*)$, and must exist a $\bar\phi\in A^*$ such that their orbit coincide with $\omega(\phi^*)$, which in turn implies the existence of a $\bar t$ such that $m=x(\bar t,\bar\phi)$. Choosing $t_1=\bar t$ and $\phi_1=\bar\phi$, we obtain our claim.

$$\begin{aligned} m=x(t_1,\phi_1) &= \beta\int_{-\infty}^{t_1} e^{-k(t_1-s)}\tanh(x_s(\phi_1))ds\\ &\ge \beta\int_{-\infty}^{t_1} e^{-k(t_1-s)}\tanh(M)ds=g(M)\end{aligned}$$

Analogously, to the above reasoning we can prove that there exist ϕ_2 and t_2 such that

$$\begin{aligned} M=x(t_2,\phi_2) &= \beta\int_{-\infty}^{t_2} e^{-k(t_2-s)}\tanh(x_s(\phi_2))ds\\ &\le \beta\int_{-\infty}^{t_2} e^{-k(t_2-s)}\tanh(m)ds=g(m).\end{aligned}$$

From the last two inequalities, we get

$$[m,M]\subset g([m,M])\subset g^2([m,M])\subset\ldots\subset g^n([m,M])\subset\ldots\quad. \tag{7}$$

Let us suppose that the trivial solution of the equation (1) is not global asymptotically stable. Henceforth $m<M$ and from (7), it follows that the fixed point of the dynamical system induced by g cannot be global asymptotically stable. Which is a contradiction. Therefore $M=m$.

In the case that $0<\beta<k$, the proof follows analogously to the former reasoning, except obvious modifications. This completes the proof of our assertion. ■

4 Local Hopf Bifurcation

Finally, let us show that the trivial solution of the equation (1) undergoes a Hopf bifurcation by crossing the curve C_0 (see figure 1), taking β as a bifurcation parameter. Actually, it is the first Hopf bifurcation.

Let us fix k and r. From Theorem 1, we obtain that the equation (4) has all the roots with negative real part if $0 < -\beta < \sqrt{\xi^2 r^{-2} + k^2}$, where ξ is the unique root of the equation $\xi = -kr\tan(\xi)$, $\pi/2 < \xi < \pi$. Moreover, a pair of pure imaginary roots arise at $\beta_0 = -\sqrt{\xi^2 r^{-2} + k^2}$.

Setting $G(z, \beta) = e^z(z + kr) - \beta r$, we obtain that

$$G(i\xi, \beta_0) = 0 \quad , \quad G_z(i\xi, \beta_0) = (i\xi + kr + 1)e^{i\xi} \neq 0 \quad , \quad G_\beta(i\xi, \beta_0) = -r.$$

Thus, the Implicit Function Theorem implies that we can solve $G(z, \beta) = 0$ for $z(\beta) = \alpha(\beta) + i\omega(\beta)$ satisfying $z(\beta_0) = i\xi$ and

$$\frac{dz}{d\beta} = -\frac{r}{(i\xi + kr + 1)e^{i\xi}}.$$

A tedious but straightforward computation gives us

$$\frac{d\alpha}{d\beta} = -\frac{r[-\xi\sin(\xi) + (kr + 1)\cos(\xi)]}{[-\xi\sin(\xi) + kr\cos(\xi) + cos(\xi)]^2 + [\xi\cos(\xi) + kr\sin(\xi) + sin(\xi)]^2} > 0.$$

Is easy to show that the root $z_0 = i\xi$ is simple and the non-resonance condition; i.e. $z_j \neq z_0$ for any root $z_j \neq mz_0, \overline{z_0}$ and any integer m, is satisfied. Thus, the equation (1) satisfies all conditions of the Hopf bifurcation theorem 1.1 p. 246 in [4]. Henceforth, the equation (1) undergoes a Hopf bifurcation at $\beta = \beta_0$.

5 Final Remarks

By using the numerical graphical interface XppAuto by Bard Ermentrout [2], we performed a thorough numerical analysis of the behavior of the solutions of the equation (1), when the parameters belong to the region Ω_2. All the numerical results suggest that the global attractor of the equation (1) is just an equilibrium point for any β and k in the region $\Omega_1 \bigcup \Omega_2$. Henceforth, the Hopf bifurcation should be supercritical. Below, we are going to exhibit some of the obtained graphics. All simulations were done by picking up the initial condition $x(t) = 0.5$, $-r \leq t \leq 0$ and $k = 1$, $r = 0.1$. With these choices the Hopf bifurcation occurs at $\beta_0 = -16.3505592$. The value of the parameter β is specified in the corresponding graphic.

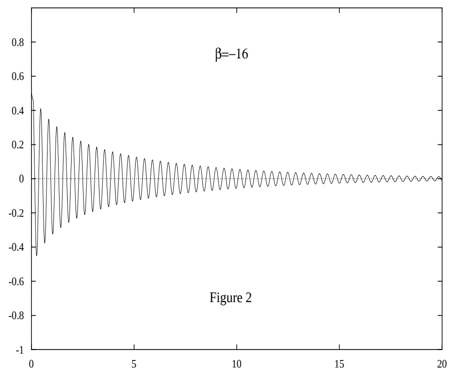


Figure 2

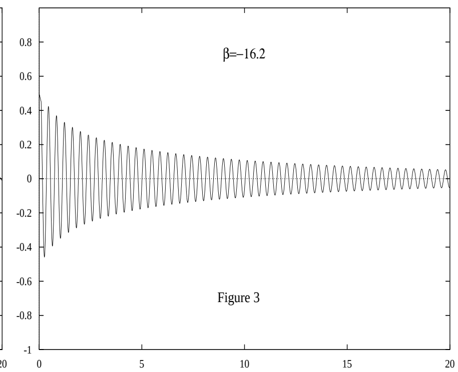


Figure 3

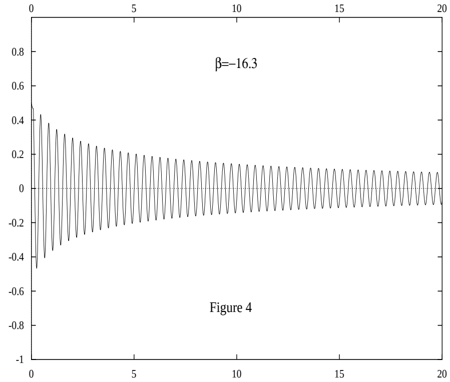


Figure 4

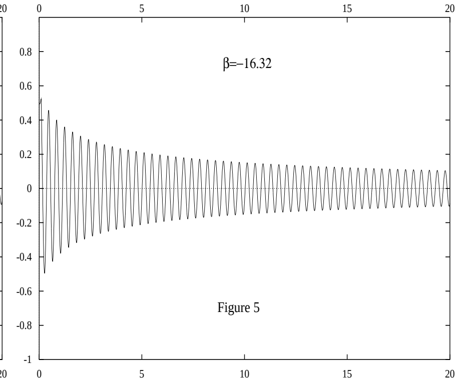


Figure 5

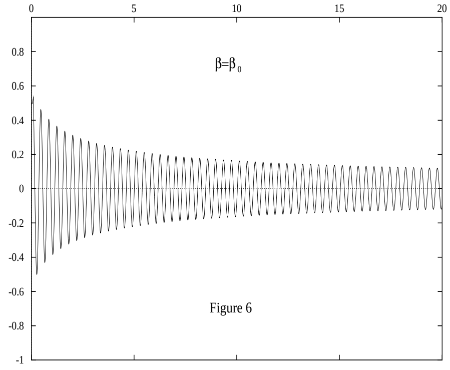


Figure 6

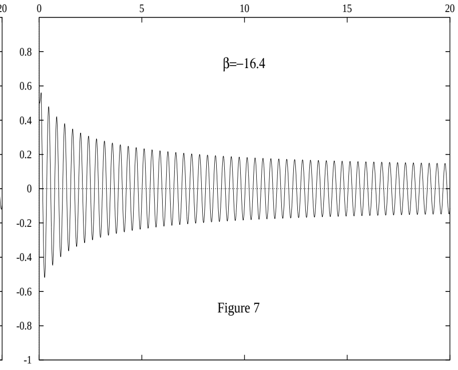
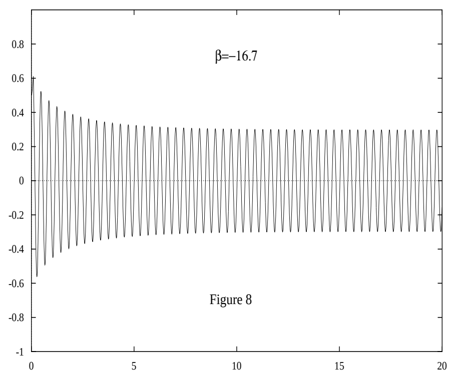
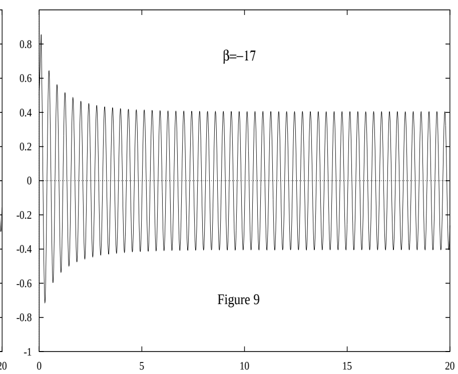


Figure 7





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