Divulgaciones Matemáticas Vol. 16 No. 1(2008), pp. 231–247

A survey on the Weierstrass approximation theorem

Un estudio sobre el teorema de aproximación de Weierstrass

Dedicated to the 30th Anniversary of Postgrade in Mathematics of UCV

Dilcia Pérez (dperez@ucla.edu.ve)
Departamento de Matemáticas
Universidad Centro Occidental Lisandro Alvarado
Barquisimeto-Venezuela.
Yamilet Quintana (yquintana@usb.ve)
Departamento de Matemáticas Puras y Aplicadas
Universidad Simón Bolívar
Apartado Postal: 89000, Caracas 1080 A, Venezuela.

Abstract

The celebrated and famous Weierstrass approximation theorem characterizes the set of continuous functions on a compact interval via uniform approximation by algebraic polynomials. This theorem is the first significant result in Approximation Theory of one real variable and plays a key role in the development of General Approximation Theory. Our aim is to investigate some new results relative to such theorem, to present a history of the subject, and to introduce some open problems.
Key words and phrases: Approximation Theory, Weierstrass' theorem, Sobolev spaces, weighted Sobolev spaces, $\mathcal{G}$-valued polynomials, $\mathcal{G}$- valued smooth functions.

Resumen

El celebrado y famoso teorema de aproximación de Weierstrass caracteriza al conjunto de las funciones continuas sobre un intervalo compacto vía aproximación uniforme por polinomios algebraicos. Este teorema es el primer resultado significativo en Teoría de Aproximación de una variable real y juega un rol clave en el desarrollo de la Teoría de

Received 2006/07/06. Revised 2007/04/14. Accepted 2007/05/25.
MSC (2000): Primary 41-02, 41A10; Secondary 43A32, 47A56.
Research partially supported by DID-USB under Grant DI-CB-015-04.

Aproximación General. Nuestro propósito es investigar algunos de los nuevos resultados relativos a tal teorema, presentar una historia del tema, e introducir algunos problemas abiertos.
Palabras y frases clave: Teoría de Aproximación, Teorema de Weierstrass, espacios de Sobolev, espacios de Sobolev con peso, polinomios a valores en $\mathcal{G}$, funciones suaves a valores en $\mathcal{G}$.

1 Introduction

In its most general form, we can say that Approximation Theory is dedicated to the description of elements in a topological space X, which can be approximated by elements in a subset A of X, that is to say, Approximation Theory allows to characterize the closure of A in X.

The first significant results of previous type were those of Karl Weierstrass (1815-1897), who proved in 1885 (when he was 70 years old) the density of algebraic polynomials in the class of continuous real-valued functions on a compact interval, and the density of trigonometric polynomials in the class of 2π-periodic continuous real-valued functions. Such results were -in a sense-a counterbalance to Weierstrass' famous example of 1861 on the existence of a continuous nowhere differentiable function. The existence of such functions accentuated the need for analytic rigor in Mathematics, for a further understanding of the nature of the set of continuous functions, and substantially influenced the development of analysis. This example represented for some mathematicians a 'lamentable plague' (as Hermite wrote to Stieltjes on May 20, 1893, see [3]), so that, we can say the approximation theorems were a panacea. While on the one hand the set of continuous functions contains deficient functions, on the other hand every continuous function can be approximated arbitrarily well by the ultimate in smooth functions, the polynomials.

Weierstrass was interested in Complex Function Theory and in the ability to represent functions by power series. The result obtained in his paper in 1885 should be viewed from that perspective, moreover, the title of the paper emphasizes such viewpoint (his paper was titled *On the possibility of giving an analytic representation to an arbitrary function of real variable*, see [16]). Weierstrass' perception on analytic functions was of functions that could be represented by power series.

The paper of Weierstrass (1885) was reprinted in Weierstrass' Mathematische Werke (collected works) with some notable additions, for example a short 'introduction'. This reprint appeared in 1903 and contains the following statement:

The main result of this paper, restricted to the one variable case, can be summarized as follows:

Let $f \in C(\mathbb{R})$. Then there exists a sequence $f_1, f_2, \ldots$ of entire functions for which

$$f(x) = \sum_{i=1}^{\infty} f_i(x),$$

for each $x \in \mathbb{R}$. In addition the convergence of the above sum is uniform on every finite interval.

Notice that there isn't mention of the fact that the f_i's may be assumed to be polynomials.

We state the Weierstrass approximation theorem, not as given in his paper, but as it is currently stated and understood.

Theorem 1.1. *(K. Weierstrass).*
Given $f : [a, b] \to \mathbb{R}$ continuous and an arbitrary $\epsilon > 0$, there exists an algebraic polynomial p such that

$$|f(x) - p(x)| \leq \epsilon, \ \forall \ x \in [a, b]. \tag{1.1}$$

Over the next twenty-five or so years numerous alternative proofs were given to this result by a roster of some the best analysts of the period. Two papers of Runge published almost at the same time, gave another proof of the theorem, but unfortunately, the theorem was not titled Weierstrass-Runge theorem. The impact of the theorem of Weierstrass in the world of the Mathematics was immediate. There were later proofs of famous mathematicians such as Picard (1891), Volterra (1897), Lebesgue (1898), Mittag-Leffler (1900), Landau (1908), de la Valleé Poussin (1912). The proofs more commonly taken at level of undergraduate courses in Mathematics are those of Fejér (1900) and Bernstein (1912) (see, for example [4], [7], [16] or [20]).

We only present a limited sampling of the many results that has been related with Weierstrass' approximation theorem, which can be found in Approximation Theory and others areas. We would like to include all results but the length of this paper would not be suffice enough. In addition, we don't prove most of the results we quote. We hope, nonetheless, that the readers will find something here of interest.

In regards to the outline of the paper, it is as follows: Section 2 is dedicated to describe some improvements, generalizations and ramifications of Weierstrass approximation theorem. Section 3 presents some new results on the subject and to introduce some open problems.

2 Improvements, generalizations and ramifications of Weierstrass approximation theorem

There are many improvements, generalizations and ramifications of Weierstrass approximation theorem. For instance, we could let f be a complex-valued, or take values in a real or complex vector space of finite dimension; that is easy. We could let f be a function of several real variables; then it is also easy to formulate the result. And we could let f be a function of several complex variables. That would require a more profound study, with skillful adaptations of both hypothesis and conclusion. Finally, of course, we may try with functions which take theirs values in an infinite-dimensional space (see Section 3 below).

Among the results of this kind we can find:

THEOREM 2.1. *(Bernstein).*
Let $f : [0,1] \to \mathbb{R}$ bounded, then

$$\lim_{n\to\infty} B_n(f;x) = f(x),$$

for each $x \in [0,1]$ where f is continuous. Furthermore, if $f \in C([0,1])$ then $B_n(f;x)$ converges to f uniformly.

Notice that if $f \in C([a,b])$, we can take $y = \frac{x-a}{b-a}$ for translate the approximation problem to interval $[0,1]$ and to use the Bernstein polynomials $B_n(f;y)$.

The theorem of Bernstein besides showing Weierstrass' result, gives a bonus: an explicit expression for the polynomial approximants to the function.

Simultaneously with the study of the approximation for algebraic polynomials, also the approximation by trigonometrical polynomials was investigated - beginning with Fourier series-. The problem can generally be outlined: Given $\{f_i\}_i$ a sequence of a normed space X, when $\{f_i\}_i$ is fundamental in X?, i.e. when holds that for every $g \in X$ and $\epsilon > 0$, exist n and a sequence of scalars $\{c_i\}_i$ such that

$$\left\| g - \sum_{i=1}^{n} c_i f_i \right\| < \epsilon \ ?$$

This question occupied to many mathematicians during the past century. Maybe the oldest and simpler example was proposed by Bernstein in 1912, approximately: Let $\{\lambda_i\}_{i=1}^{\infty}$ a sequence of distinct positive numbers. When the functions $1, x^{\lambda_1}, x^{\lambda_2}, x^{\lambda_3} \ldots$ are fundamental in $C([0,1])$? Also, the answer was surmised by Bernstein and it is given in the following theorems.

THEOREM 2.2. *(Müntz's first theorem).*

Consider the set of functions $\left\{x^{\lambda_1}, x^{\lambda_2}, \ldots\right\}$ *where* $-\frac{1}{2} < \lambda_i \to \infty$. *It is fundamental in the least-squares norm on* $[0,1]$, *if and only if,* $\sum_{\lambda_i \neq 0} \frac{1}{\lambda_i} = \infty$.

THEOREM 2.3. *(Müntz's second theorem, 1914).*
Let $\{\lambda_i\}_{i=1}^{\infty}$ *a sequence of distinct positive numbers, such that* $1 \leq \lambda_i \to \infty$. *Then the set of functions* $\left\{1, x^{\lambda_1}, x^{\lambda_2}, x^{\lambda_3} \ldots\right\}$ *is fundamental in* $C([0,1])$, *if and only if,* $\sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty$.

By some time this last theorem was called Müntz-Szasz theorem, because Szasz published an independent proof of it in 1916. Seemingly, Bernstein solved part of the problem. In fact, taking $\lambda_i = i$ we can recover the Weierstrass approximation theorem for $[0,1]$.

Furthermore, the Müntz's second theorem is all the more interesting because it traces a logical connection between two apparently unrelated facts: the fundamentality of $\left\{1, x, x^2, x^3 \ldots\right\}$ and the divergence of the series of reciprocal exponents, $\sum_{n=1}^{\infty} \frac{1}{n}$. In fact, if we wish to delete functions from the set while maintaining its fundamentality, this divergence is precisely the property that must be preserved. The reader is referred to [7] for the proofs of the both theorems.

The combination of Dual Spaces Theory and Complex Analysis are very usual in the study of fundamental sequences. Another application of these techniques is given to solve the problem of Wiener on traslation of fundamental sequences, such problem dates from the thirty's decade of the past century and is related with the problems of invariant spaces. This problem has generated almost so much investigation as Müntz theorem. The following theorem is a version of Wiener's theorem.

THEOREM 2.4. *(Wiener, approximately 1933).*
Let $f \in L^1(\mathbb{R})$. *For all* $g \in L^1(\mathbb{R})$ *and every* $\epsilon > 0$ *there are* $\{c_j\}_j, \{\lambda_j\}_j \subset \mathbb{R}$, *such that*

$$\left\| g(x) - \sum_{j=1}^{n} c_j f(x - \lambda_j) \right\|_{L^1(\mathbb{R})} < \epsilon,$$

if and only if, the Fourier transform f *doesn't have real zeros, that is to say,*

$$\int_{-\infty}^{\infty} e^{-itx} f(t) dt \neq 0, \ \forall x \in \mathbb{R}.$$

We shall enunciate two additional theorems on fundamental sequences which was published in the 30's of last century and appear in the Russian edition of the book of Akhiezer [2], and also in the translation of this book to German, but not in the translation to English.

THEOREM 2.5. *(Akhiezer-Krein. Fundamental sequences of rational functions with simple poles).*
If $\{a_j\}_{j=1}^{\infty} \subset \mathbb{C} \setminus [-1,1]$, *is such that* $a_j \neq a_k$, $\forall j \neq k$, *then the sequence* $\left\{\frac{1}{x-a_j}\right\}_{j=1}^{\infty}$ *is fundamental in* $L^p([-1,1])$, $(1 \leq p < \infty)$ *and in* $C([-1,1])$, *iff*

$$\sum_{k=1}^{\infty}\left(1-\left|a_k-\sqrt{a_k^2-1}\right|\right)=\infty,$$

where the branch of the function $f(z)=\sqrt{z}$ *considered is the usual, is to say,* $\sqrt{z}>0$ *for* $z \in (0,\infty)$.

THEOREM 2.6. *(Paley-Wiener, 1934. Trigonometric series non harmonic).*
If $\{\sigma_j\}_{j=1}^{\infty} \subset \mathbb{C}$, *is such that* $\sigma_j \neq \sigma_k$, $\forall\ j \neq k$ *and* $|\Im(\sigma_j)| < \frac{\pi}{2}$, $j \geq 1$; *then the sequence* $\{e^{-\frac{\pi}{2}|x|-ix\sigma_j}\}_{j=1}^{\infty}$ *is fundamental in* $L^2(\mathbb{R})$ *iff,*

$$\sum_{j=1}^{\infty}\frac{\cos(\Im(\sigma_j))}{\coth(\Re(\sigma_j))}=\infty,$$

where $\Re(z)$, $\Im(z)$ *denote, respectively, the real and imaginary parts of the complex number* z.

Other result which despite its recent discovery has already become "classical" and should be a part of the background of every student of Mathematics is the following.

THEOREM 2.7. *(M. H. Stone, approximately 1947).*
Let K *be a compact subset of* $\mathbb{R}^n$ *and let* $\mathcal{L}$ *be a collection of continuous functions on* K *to* $\mathbb{R}$ *with the properties:*

i) If f, g *belong to* $\mathcal{L}$, *then* $\sup\{f,g\}$ *and* $\inf\{f,g\}$ *belong to* $\mathcal{L}$.

ii) If $a, b \in \mathbb{R}$ *and* $x \neq y \in K$, *then there exists a function* f *in* $\mathcal{L}$ *such that* $f(x)=a$, $f(y)=b$.

Then any continuous function on K *to* $\mathbb{R}$ *can be uniformly approximated on* K *by functions in* $\mathcal{L}$.

The reader is referred to the paper [28], in which the original proof of Stone appears.

Another known result, but not very noted, is due to E. Hewitt, [12]. This work appeared published in 1947 and it presents certain generalizations of the Stone-Weierstrass theorem which are valid in all completely regular spaces.

There exist another three problems on approximation which have had a substantial influence on the analysis of last century:

1. The problem on polynomial approximation of Bernstein on whole real line (between 1912 and 50's).

2. To characterize the closure of polynomials in families of functions defined on compact sets $K \subset \mathbb{C}$ (between 1885 and 50's). It is also necessary to mention here the Kakutani-Stone theorem on the closure of a lattice of functions real-valued and Stone-Weierstrass theorem on the closure of an algebra of functions continuous complex-valued (see [28], [18]).

3. The Szegö's extremum problem (between 1920 and 40's).

With respect the first problem, it must be treated for non bounded polynomials in $\pm\infty$. If let us consider a weight $w : \mathbb{R} \to [0, 1]$ (i.e. a non-negative, measurable function) and we define

$$C_w := \left\{ f : \mathbb{R} \to \mathbb{R}, \text{ continuous with } \lim_{|x|\to\infty} (fw)(x) = 0 \right\},$$

with norm

$$\|f\|_{C_w} := \|fw\|_{L^\infty(\mathbb{R})}.$$

Bernstein wondered when holds that for all $f \in C_w$ and every $\epsilon > 0$, there exists a polynomial p, such that

$$\|(f - p)w\|_{L^\infty(\mathbb{R})} < \epsilon \ ? \tag{2.2}$$

The condition fw null in $\pm\infty$ is necessary to give sense to the problem: we would like $\|pw\|_{L^\infty(\mathbb{R})} < \infty$, $\forall p \in \mathbb{P}$, and in particular $\|x^n w\|_{L^\infty(\mathbb{R})} < \infty$, $\forall n \geq 0$. Notice that this condition necessarily makes $\lim_{|x|\to\infty} x^n w(x) = 0$, then using (2.2) we would have $\|fw\|_{L^\infty(\mathbb{R})} < \epsilon$, therefore fw is null in $\pm\infty$.

Among those that contributed to the solution of this problem are Bernstein (1912, 1924), T. S. Hall (1939, 1950), Dzrbasjan (1947) and Videnskii (1953). In the case w continuous weight the solution solution was given by H. Pollard (1953); other solutions were given by Akhiezer (1954), Mergelyan (1956) and Carleson (1951).

The techniques used in the solution of Bernstein's problem includes Dual Spaces Theory, other problems of approximation, Entire Functions Theory, analytic on the half-plane and a lot of hard work.

For the approximation problem on compact subset of the complex plane, the situation is as follows:

Let K a compact set of complex plane, we define

$$A(K) := \{f : K \to \mathbb{C}, \text{ continuous in } K \text{ and analytic in } K^\circ\},$$

where K° denotes the interior of K and the norm is given by

$$\|f\| := \|f\|_{L^\infty(K)}.$$

If we consider

$$P(K) := \Big\{f : K \to \mathbb{C}, \text{ such that there exist polynomials } \{p_n\} \text{ with } \lim_{n\to\infty} \|f - p_n\|_{L^\infty(K)} = 0\Big\},$$

the interesting question is, when $P(K) = A(K)$?

May be this problem was never assigned a name, but in spirit it should be called the Runge problem (1885), whose fundamental contributions to Approximation Theory still can be seen in the course of Complex Analysis. Among the interested on the solution of this problem are J. L. Walsh (1926), Hartogs-Rosenthal (1931), Lavrentiev (1936) and Keldysh (1945).

Theorem 2.8. *(Mergelyan, 1950's).*
Let $K \subset \mathbb{C}$ compact, the following statements are equivalent:

1. $P(K) = A(K)$.
2. $\mathbb{C} \setminus K$ *is connected.*

With respect to Szegö's extremum problem, it can be expressed as follows: Let us consider a non-negative and finite Borel measure μ on the unit circle $\Pi := \{z \in \mathbb{C} : |z| = 1\}$ and $p > 0$. The Szegö's extremum problem consists on determining the value

$$\begin{aligned}\delta_p(\mu) &:= \inf\left\{\frac{1}{2\pi}\int_\Pi |P(z)|^p \, d\mu(z) : P \text{ is a polynomial with } P(0) = 1\right\} \\ &= \inf_{\{c_j\}}\left\{\frac{1}{2\pi}\int_\Pi \Big|1 - \sum_{j\geq 1} c_j z^j\Big|^p d\mu(z)\right\}.\end{aligned}$$

Since every polynomial $P(z)$ of degree $\leq n$ with $P(0) = 1$,

$$Q(z) := z^n \overline{P\left(\frac{1}{\bar{z}}\right)}$$

is a monic polynomial of degree n, $|Q(z)| = |P(z)|$ whenever $|z| = 1$, and we have

$$\delta_p(\mu) = \inf\left\{\frac{1}{2\pi}\int_{\Pi} |Q(z)|^p \, d\mu(z) : Q \text{ is monic polynomial}\right\}.$$

This problem, has had more impact in Approximation Theory than Bernstein's problem or the Mergelyan theorem. Its solution and the techniques developed from it have had a great influence in the Spaces Hardy Theory and also on the Functions Theory in the 20th. century.

Theorem 2.9. *(Szegö, 1921).*
Let μ absolutely continuous measure with respect to Lebesgue measure on the unit circle Π, with $d\mu(z) = w(e^{i\theta})d\theta$. Then

$$\delta_p(\mu) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log(w(e^{i\theta}))d\theta\right).$$

First Szegö showed this result with w a trigonometric polynomial and then he used such approximation for extending the result for w continuous function and also for general case (see [29]). The generalization to not necessarily absolutely continuous measures came later twenty years and is the merit of A. N. Kolmogorov (1941) for $p = 2$, and M. G. Krein (1945) for $p > 0$.

Theorem 2.10. *(Kolmogorov-Krein).*
If μ is a non-negative Borel measure on unit circle Π, such that

$$d\mu(z) = w(z)d\theta + d\mu_s(z), \quad z = e^{i\theta},$$

then

$$\delta_p(\mu) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log(w(e^{i\theta}))d\theta\right).$$

So the singular component μ_s of μ plays no role at all.

Corollary 2.1. *(Kolmogorov-Krein).*
Given $p > 0$ and a non-negative Borel measure μ on the unit circle Π, such that $d\mu(z) = w(z)d\theta + d\mu_s(z)$, $z = e^{i\theta}$, the following statements are equivalent:

1. *The polynomials are dense in* $L^p(\mu)$.
2. $\int_0^{2\pi} \log(w(e^{i\theta}))d\theta = -\infty$.

Therefore, the polynomials are dense in $L^p(\mu)$ only in really exceptional cases. In Orthogonal Polynomials Theory, the condition

$$\int_0^{2\pi} \log(w(e^{i\theta}))d\theta > -\infty$$

is called the Szegö's condition.

In recent years it has arisen a new focus on the generalizations of Weierstrass approximation theorem, which uses the weighted approximation or approximation by weighted polynomials. More precisely, if I is a compact interval, the approximation problem is studied with the norm $L^\infty(I, w)$ defined by

$$\|f\|_{L^\infty(I,\, w)} := \operatorname{ess\,sup}_{x\in I}|f(x)|w(x)\,, \tag{2.3}$$

where w is a weight, i.e. a non-negative measurable function and the convention $0 \cdot \infty = 0$ is used. Notice that (1.1) is not the usual definition of the L^∞ norm in the context of measure theory, although it is the correct definition when we work with weights (see e.g. [5] and [10]).

Considering weighted norms $L^\infty(w)$ has been proved to be interesting mainly because of two reasons: first, it allows to wider the set of approximable functions (since the functions in $L^\infty(w)$ can have singularities where the weight tends to zero); and, second, it is possible to find functions which approximate f whose qualitative behavior is similar to the one of f at those points where the weight tends to infinity.

In this case, it is easy to see that $L^\infty(I, w)$ and $L^\infty(I)$ are isomorphic, since the map $\Psi_w : L^\infty(I, w) \to L^\infty(I)$ given by $\Psi_w(f) = fw$ is a linear and bijective isometry, and therefore, Ψ_w is also homeomorphism, or equivalently, $Y \subseteq L^\infty(I, w)$, $\Psi_w(\overline{Y}) = \overline{\Psi_w(Y)}$, where we take each closure with respect to the norms $L^\infty(I, w)$ and $L^\infty(I)$, in each case. Analogously, for all $A \subseteq L^\infty(I)$, $\Psi_w^{-1}(\overline{A}) = \overline{\Psi_w^{-1}(A)}$ and $\Psi_w^{-1} = \Psi_{w^{-1}}$. Then using Weierstrass' theorem we have,

$$\Psi_w^{-1}(\overline{\mathbb{P}}) = \overline{\Psi_w^{-1}(\mathbb{P})} \quad = \quad \{f \in L^\infty(I, \omega) : fw \in C(I)\}. \tag{2.4}$$

Unfortunately, in many applications it is not just the isomorphism type is of interest, since the last equality in (2.4) doesn't allow to obtain information on local behavior of the functions $f \in L^\infty(I, w)$ which can be approximated. Furthermore, if $f \in L^\infty(I, w)$, then in general fw is not continuous function, since its continuity depends of the nature of weight w. So, arises the necessary research of the family of weights for which have sense the approximation problem by continuous functions. The reader can find in [22] and [27] a recent and detailed study of such problem. It is precisely in [22] where appears one of the most general results known at the present time:

THEOREM 2.11. *([25], Theorem 2.1).*
Let w be any weight and

$$\begin{aligned} H_0 := \big\{ & f \in L^\infty(w) : f \textit{ is continuous to the right at every point of } R^+(w), \\ & f \textit{ is continuous to the left at every point of } R^-(w), \\ & \textit{for each } a \in S^+(w), \quad \operatorname{ess\,lim}_{x\to a^+} |f(x) - f(a)|\, w(x) = 0, \\ & \textit{for each } a \in S^-(w), \quad \operatorname{ess\,lim}_{x\to a^-} |f(x) - f(a)|\, w(x) = 0 \big\}. \end{aligned}$$

Then:

(a) The closure of $C(\mathbb{R}) \cap L^\infty(w)$ in $L^\infty(w)$ is H_0.

(b) If $w \in L^\infty_{loc}(\mathbb{R})$, then the closure of $C^\infty(\mathbb{R}) \cap L^\infty(w)$ in $L^\infty(w)$ is also H_0.

(c) If $\operatorname{supp} w$ *is compact and $w \in L^\infty(\mathbb{R})$, then the closure of the space of polynomials is H_0 as well.*

Another special kind of approximation problems arise when we consider simultaneous approximation which includes to derivatives of certain functions; this is the case of versions of Weierstrass' theorem in weighted Sobolev spaces. In our opinion, very interesting results in such direction appears in [24]. Under enough general conditions concerning the vector weight (the so called type 1) defined in a bounded interval I, the authors characterize to the closure in the weighted Sobolev space $W^{(k,\infty)}(I,w)$ of the spaces of polynomials, k-differentiable functions in the real line, and infinite differentiable functions in the real line, respectively.

THEOREM 2.12. *([24], Theorem 4.1).*
Let us consider a vectorial weight $w = (w_0, \dots, w_k)$ of type 1 in a compact interval $I = [a,b]$. Then the closure of $\mathbb{P} \cap W^{k,\infty}(I,w)$, $C^\infty(\mathbb{R}) \cap W^{k,\infty}(I,w)$ and
$C^k(\mathbb{R}) \cap W^{k,\infty}(I,w)$ in $W^{k,\infty}(I,w)$ are, respectively,

$$\begin{aligned}
H_1 &:= \left\{ f \in W^{k,\infty}(I,w) :\ f^{(k)} \in \overline{\mathbb{P} \cap L^\infty(I,w_k)}^{L^\infty(I,w_k)} \right\}, \\
H_2 &:= \left\{ f \in W^{k,\infty}(I,w) :\ f^{(k)} \in \overline{C^\infty(I) \cap L^\infty(I,w_k)}^{L^\infty(I,w_k)} \right\}, \\
H_3 &:= \left\{ f \in W^{k,\infty}(I,w) :\ f^{(k)} \in \overline{C(I) \cap L^\infty(I,w_k)}^{L^\infty(I,w_k)} \right\}.
\end{aligned}$$

The reader is referred to [24] for the proof of this theorem.

3 Hilbert-extension of the Weierstrass approximation theorem

Throughout this section, let us consider a real and separable Hilbert space $\mathcal{G}$, a compact interval $I \subset \mathbb{R}$, the space $L^\infty_{\mathcal{G}}(I)$ of all $\mathcal{G}$-valued essentially bounded functions, a weakly measurable function $w : I \to \mathcal{G}$, the space $\mathbb{P}(\mathcal{G})$ of all $\mathcal{G}$-valued polynomials and the space $L^\infty_{\mathcal{G}}(I,w)$ of all $\mathcal{G}$-valued functions which are bounded whit respect to the norm defined by

$$\|f\|_{L^\infty_{\mathcal{G}}(I,w)} := \operatorname{ess\,sup}_{t\in I} \|(fw)(t)\|_{\mathcal{G}}, \tag{3.5}$$

where $fw : I \longrightarrow \mathcal{G}$ is defined as follows: If $\dim \mathcal{G} < \infty$, we have the functions f and w can be expressed as $f = (f_1, \dots, f_{n_0})$ and $w = (w_1, \dots, w_{n_0})$, respectively, where $f_k,\ w_j : I \longrightarrow \mathbb{R}$, for $j = 1, \dots, n_0$, with $n_0 = \dim \mathcal{G}$. Then

$$(fw)(t) := (f_1(t)w_1(t), \dots, f_{n_0}(t)w_{n_0}(t)), \text{ for } t \in I.$$

If $\dim \mathcal{G} = \infty$, let $\{\tau_j\}_{j\in\mathbb{Z}_+}$ be a complete orthonormal system, then

$$(fw)(t) := \sum_{j=0}^{\infty} \langle f(t), \tau_j \rangle_{\mathcal{G}} \, \langle w(t), \tau_j \rangle_{\mathcal{G}} \, \tau_j, \text{ for } t \in I.$$

It is known that every $\mathcal{G}$ is a real separable Hilbert space isomorphic either to $\mathbb{R}^n$ for some $n \in \mathbb{N}$ or to $l^2(\mathbb{R})$. And in both cases $\mathcal{G}$ has a structure of commutative Banach algebra, with the coordinatewise operations and identity in the first case and without identity in the second case (see [14]). We use this fact in what follows.

3.1 $\mathcal{G}$-valued functions

Let $\mathcal{G}$ a real separable Hilbert space. A $\mathcal{G}$-valued polynomial on I is a function $\phi : I \to \mathcal{G}$, such that
$$\phi(t) = \sum_{n\in\mathbb{N}} \xi_n t^n,$$
where $(\xi)_{n\in\mathbb{N}} \subset \mathcal{G}$ has finite support.

Let $\mathbb{P}(\mathcal{G})$ be the space of all $\mathcal{G}$-valued polynomials on I. It is well known that $\mathbb{P}(\mathcal{G})$ is a subalgebra of the space of all continuous $\mathcal{G}$-valued functions on I. We denote by $L^\infty_\mathcal{G}(I)$ the set of all weakly measurable essentially bounded functions $f : I \to \mathcal{G}$. For $1 \le p < \infty$ let $L^p_\mathcal{G}(I)$ denote the set of all weakly measurable functions $f : I \to \mathcal{G}$ such that
$$\int_I \|f(t)\|^p_\mathcal{G}\, dt < \infty.$$

Then $L^2_\mathcal{G}(I)$ is a Hilbert space with respect to inner product
$$\langle f, g\rangle_{L^2_\mathcal{G}(I)} = \int_I \langle f(t), g(t)\rangle_\mathcal{G}\, dt.$$

$\mathbb{P}(\mathcal{G})$ is also dense in $L^p_\mathcal{G}(I)$, for $1 \le p < \infty$. More details about these spaces can be found in [30].

In [26] the author studies the set of functions $\mathcal{G}$-valued which can be approximated by $\mathcal{G}$-valued continuous functions in the norm $L^\infty_\mathcal{G}(I, w)$, where I is a compact interval, $\mathcal{G}$ is a real and separable Hilbert space and w is certain $\mathcal{G}$-valued weakly measurable weight, and using the previous results of [24] the author obtains a new extension of Weierstrass' approximation theorem in the context of real separable Hilbert spaces.

3.2 Approximation in $W^{1,\infty}_\mathcal{G}(I)$

For dealing with definition of Sobolev space $W^{1,\infty}_\mathcal{G}(I)$. First of all, notice that we need a definition of derivative f' of a function $f \in L^\infty_\mathcal{G}(I)$, such definition can be introduced by means of the properties of $\mathcal{G}$ and the definition of classical Sobolev spaces as follows.

Definition 3.1.
Given $\mathcal{G}$ a real separable Hilbert space and $f : I \to \mathcal{G}$, we say that a function $g : I \to \mathcal{G}$ is the derivative of f if

$$g \sim \begin{cases} (f_1', \dots, f_{n_0}'), \text{ if } \dim \mathcal{G} = n_0. \\ \\ \{f_k'\}_{k\in\mathbb{N}} \text{ with } \sum_{k\in\mathbb{N}} |f_k'(t)|^2 < \infty, \text{ if } \mathcal{G} \text{ is infinite-dimesional.} \end{cases}$$

So that, we can define the Sobolev space $W^{1,\infty}_{\mathcal{G}}(I)$ of the space of all the $\mathcal{G}$-valued essentially bounded functions by

$$W^{1,\infty}_{\mathcal{G}}(I) := \big\{ f \in L^\infty_{\mathcal{G}}(I) : \text{there exists } f' \text{ in the sense of the Definition 3.1} \\ \text{and } f' \in L^\infty_{\mathcal{G}}(I) \big\},$$

with norm

$$\|f\|_{W^{1,\infty}_{\mathcal{G}}(I)} := \|f\|_{L^\infty_{\mathcal{G}}(I)} + \|f'\|_{L^\infty_{\mathcal{G}}(I)},$$

3.3 Interesting problems

The following problem in this area arises:
If we consider the norm

$$\|f\|_{W^{1,\infty}_{\mathcal{G}}(I,w)} := \|f\|_{L^\infty_{\mathcal{G}}(I,w)} + \|f'\|_{L^\infty_{\mathcal{G}}(I,w)},$$

and we define the weighted Sobolev space of $\mathcal{G}$-valued functions by means of the relation

$$f \in W^{1,\infty}_{\mathcal{G}}(I,w) \Leftrightarrow \begin{cases} f \in L^\infty_{\mathcal{G}}(I,w), \\ \text{there exists } f' \text{ in the sense of the Definition 3.1} \\ \text{and } f' \in L^\infty_{\mathcal{G}}(I,w), \end{cases}$$

then we can ask

1. What is $\overline{\mathbb{P}(\mathcal{G})}^{W^{1,\infty}(I,w)}$?

2. Which are the most general conditions on the weight w which allow to characterize $\overline{\mathbb{P}(\mathcal{G})}^{W^{1,\infty}(I,w)}$?

Remark 3.1. Some pieces of information are taken from Internet-based resources without the URL's.

Acknowledgements

We would like to thank Professors Neptali Romero and Ramón Vivas by the offered hospitality while one of the authors was in UCLA, concluding this paper.

References

[1] R. A. Adams, *Sobolev Spaces,* Academic Press Inc., San Diego, (1978).

[2] N. I. Akhiezer, *Theory of Approximation,* translated by C. J. Hyman, Dover, New York, (1992).

[3] B. Baillaud and H. Bourget, *Correspondence d'Hermite et de Stieltjes,* Gauthier–Villars, Paris, Tome II (1905).

[4] R. G. Bartle, *The elements of Real Analysis,* John Wiley & Sons, Inc. (2nd ed.), New York, (1976).

[5] R. C. Brown, B. Opic, *Embeddings of weighted Sobolev spaces into spaces of continuous functions,* Proc. R. Soc. Lond. A. **439** (1992), 279–296.

[6] R. Bruzual, M. Domínguez, *Operator-valued extension of the theorem of Helson and Szegö,* Operator Theory Advances and Applications **149** (2004), 139–152.

[7] E. W. Cheney, *Introduction to Approximation Theory,* McGraw-Hill Book Company, New York, (1966).

[8] T. S. Chihara, *An Introduction to Orthogonal Polynomials,* Gordon and Breach, Science Publishers,Inc. Paris, (1978).

[9] P. J. Davis, *Interpolation and Approximation,* Dover, New York, (1975).

[10] B. Della Vecchia, G. Mastroianni, J. Szabados, *Approximation with exponential weights in* $[-1, 1]$, J. Math. Anal. Appl. **272** (2002), 1–18.

[11] N. Dinculeanu, *Vector Measures,* Pergamon Press Ltd., Headington Hill Hall, Oxford, (1967).

[12] E. Hewitt, *Certain generalizations of the Weierstrass approximation theorem,* Duke Math. J. **14** (2) (1947), 419–427.

[13] E. Hewitt, H. S. Zuckerman, *Approximation by polynomials with integral coefficients, a reformulation of the Stone-Weierstrass theorem,* Duke Math. J. **26** (2) (1959), 305–324.

[14] T. Husain, Orthogonal Schauder Bases. Marcel Dekker, Inc. (1st ed.), New York, 1990.

[15] Kufner, A., Opic, B., *How to define reasonably Weighted Sobolev Spaces,* Commentationes Mathematicae Universitatis Caroline **25**(3) (1984), 537–554.

[16] D. S. Lubinsky, *Weierstrass' Theorem in the twentieth century: a selection,* Quaestiones Mathematicae **18** (1995), 91–130.

[17] D. S. Lubinsky, *Bernstein's weighted approximation on* $\mathbb{R}$ *still has problems.* Preprint.

[18] L. Nachbin; *Elements of approximation theory*, D. Van Nostrand Company Inc., New York, (1967).

[19] H. Pijeira, Y. Quintana, W. Urbina, *Zero localization and asymptotic behavior of orthogonal polynomials of Jabobi-Sobolev,* Rev. Col. Mat. **35**(2) (2001), 77–97.

[20] A. Pinkus, *Weierstrass and Approximation Theory,* J. Approx. Theory **107** (2000), 1–66.

[21] A. Pinkus, *Density Methods and results in Approximation Theory*, Banach Center Publications **64** (2004).

[22] A. Portilla, Y. Quintana, J. M. Rodríguez, E. Tourís, *Weierstrass' Theorem with weights,* J. Approx. Theory **127** (2004), 83–107.

[23] A. Portilla, Y. Quintana, J. M. Rodríguez, E. Tourís, *Weierstrass' theorem in weighted sobolev spaces with* k *derivatives: announcement of results*, Electronic Transactions on Numerical Analysis, **24** (2006), 103–107.

[24] A. Portilla, Y. Quintana, J. M. Rodríguez, E. Tourís, *Weierstrass' theorem in weighted Sobolev spaces with* k *derivatives*, Rocky Mountain J. Math., **37** (2007), 1989–2024.

[25] A. Portilla, Y. Quintana, J. M. Rodríguez, E. Tourís, *Weighted Weierstrass' theorem with first derivatives*, J. Math. Anal. Appl. **334** (2007), 1667–1698.

[26] Y. Quintana, *On Hilbert-extensions of Weierstrass' theorem with weights.* Preprint.

[27] J. M. Rodríguez, *Weierstrass' Theorem in weighted Sobolev spaces,* J. Approx. Theory **108** (2001), 119–160.

[28] M. H. Stone, *The Generalized Weierstrass Approximation Theorem,* Mathematics Magazine **21** (1947 /1948), 167-184/237-254. Reprinted in MAA Studies in Mathematics, **1**, R. C. Buck, editor, Math. Assn. America, (1962).

[29] G. Szegö, *Orthogonal Polynomials,* Coll. Publ. Amer. Math. Soc., **23**, (4th ed.), Providence, R.I., (1975).

[30] B. Sz.-Nagy, C. Foias, *Analyse Harmonique des Opérateurs de L'espace de Hilbert,* Masson et Cie. Akadémiai Kiadó. Hongrie, 1966.