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Controllability of the Benjamin-Bona-Mahony Equation

Controlabilidad de la Ecuación de Benjamin-Bona-Mahony

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Abstract

In this note we study the controllability of the Generalized Benjamin-Bona-Mahony equation (BBM) with homogeneous Dirichlet boundary conditions. Under some conditions we shall prove the system is approximately controllable on $[0, t_1]$ if and only if the following algebraic condition holds $Rank[B_j] = \gamma_j$, where B_j acts from $\mathbb{R}^m$ to $R(E_j)$, λ_j's are the eigenvalues of $-\Delta$ with Dirichlet boundary condition and γ_j the corresponding multiplicity, E_j's are the projections on the corresponding eigenspace and $R(E_j)$ denotes the range of E_j.

Key words and phrases: BBM- equation, algebraic condition, approximate controllability.

Resumen

En este articulo estudiaremos la contrabilidad de la forma generalizada de la ecuación de Benjamin-Bona-Mahony (BBM) con condiciones de borde de Dirichet homogéneas en un dominio Ω acotado. Sobre ciertas condiciones en las funciones de controles $u_i \in L^2(0, t_1, \mathbb{R})$, $i = 1, \cdots, m$ y en las constantes a, b y $b_i \in L^2(\Omega, \mathbb{R}^n)$ que aparecen en la ecuación BBM demostraremos que el sistema es aproximadamente controlable en $[0, t_1]$ si y solo si la siguiente condición algebraica es válida $Rang[B_j] = \gamma_j$ donde B_j actúa de $\mathbb{R}^m$ en $R(E_j)$ y γ_j es la multiplicidad del autovalor λ_j (λ_j es el autovalor de $-\Delta$).

Palabras y frases clave: Controlabilidad, Ecuación BBM.

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1 Introduction

In this paper we give a necessary and sufficient algebraic condition for the approximate controllability of the following Benjamin-Bona-Mahony equation (BBM) with homogeneous Dirichlet boundary conditions

$$\begin{cases} z_t - a\Delta z_t - b\Delta z = b_1(x)u_1 + \ldots + b_m(x)u_m, & t \geq 0, \quad x \in \Omega, \\ z(t,x) = 0, \quad t \geq 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where a and b are positive numbers, $b_i \in L^2(\Omega; IR^n)$, the control functions $u_i \in L^2(0, t_1; I\!R)$; $i = 1, 2, \ldots, m$, Ω is a bounded domain in $I\!R^N (N \geq 1)$.
One of the goal in this work is to prove the following statement:
System (1) is approximately controllable on $[0, t_1]$, $t_1 > 0$ iff each of the following finite dimensional systems are controllable on $[0, t_1]$

$$y' = -\frac{b\lambda_j}{1 + a\lambda_j} y + B_j u, \quad y \in R(E_j), \quad j = 1, 2, \ldots, \infty, \tag{2}$$

where

$$B_j : I\!R^m \to R(E_j), \quad B_j U = \sum_{i=1}^{\gamma_j} \frac{1}{1 + a\lambda_j} E_j b_i U_i,$$

λ_j' s are the eigenvalues of $-\Delta$ with Dirichlet boundary condition and γ_j the corresponding multiplicity, E_j' s are the projections on the corresponding eigenspace and $R(E_j)$ denotes the range of E_j .
Since $dim R(E_j) = \gamma_j < \infty$, the controllability of (2) is equivalente to the following algebraic condition:

$$Rank[B_j] = \gamma_j, \quad j = 1, 2, \ldots, \infty. \tag{3}$$

Here, we will not make distinction between the operator B_j and its corresponding matrix representation.
The original Benjamin-Bona-Mahony equation was proposed in [2] for the case $N = 1$ as a model for the propagation of long waves. This equation and related types of Pseudo-Parabolic equations have been studied by many authors. Results about existence and uniqueness of solutions can be found in [1] and [12]. The long time behavior of solutions and the existence of attractors were studied by many authors to mention [3], [4] and [5] and the controllability for the case $N = 1$ with control in the boundary has been studied in [13].

2 Abstract Formulation of the Problem

In this section we choose the space in which this problem will be set as an abstract ordinary differential equation.
Let $Z = L^2(\Omega) = L^2(\Omega, I\!R)$ and consider the linear unbounded operator $A : D(A) \subset Z \to Z$ defined by $A\phi = -\Delta\phi$, where

$$D(A) = H^2(\Omega, I\!R) \cap H_0^1(\Omega, I\!R).$$

The operator A has the following very well known properties: the spectrum of A consists of only eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \qquad \text{with} \qquad \lambda_n \to \infty, \tag{4}$$

each one with finite multiplicity γ_n equal to the dimension of the corresponding eigenspace. Therefore:
a) there exists a complete orthonormal set $\{\phi_{n,k}\}$ of eigenvectors of A.
b) for all $z \in D(A)$ we have

$$Az = \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{\gamma_n} < z, \phi_{n,k} > \phi_{n,k} = \sum_{n=1}^{\infty} \lambda_n E_n z, \tag{5}$$

where $< \, , \, >$ is the inner product in Z and

$$E_n z = \sum_{k=1}^{\gamma_n} < z, \phi_{n,k} > \phi_{n,k}. \tag{6}$$

So, $\{E_n\}$ is a family of complete orthogonal projections in Z and

$$z = \sum_{n=1}^{\infty} E_n z, \quad z \in Z. \tag{7}$$

c) $-A$ generates the analytic semigroup $\left\{e^{-At}\right\}$ given by

$$e^{-At} z = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n z. \tag{8}$$

Hence, the equation (1) can be written as an abstract ordinary differential equation in $D(A)$ as follows

$$z' + aAz' + bAz = b_1 u_1 + ... + b_m u_m, \quad t \geq 0, \tag{9}$$

Since $(I + aA) = a(A - (-\frac{1}{a})I)$ and $-\frac{1}{a} \in \rho(A)$ (the resolvent set of A), then the operator:

$$I + aA : D(A) \to Z$$

is invertible with bounded inverse

$$(I + aA)^{-1} : Z \to D(A).$$

Therefore, the equation (9) also can be written as follows

$$z' + b(I + aA)^{-1}Az = (I + aA)^{-1}\sum_{i=1}^{m} b_i u_i, \quad t \geq 0. \tag{10}$$

Moreover, $(I+aA)$ and $(I+aA)^{-1}$ can be written in terms of the eigenvalues of A:

$$(I + aA)z = \sum_{n=1}^{\infty}(1 + \lambda_n)E_n z$$

$$(I + aA)^{-1}z = \sum_{n=1}^{\infty} \frac{1}{1 + a\lambda_n} E_n z.$$

Therefore, if we put $B = (I + aA)^{-1}$, the equation (10) can be written as follows

$$z' + bBAz = B\sum_{i=1}^{m} b_i u_i, \quad t \geq 0, \tag{11}$$

Now, we formulate a simple proposition.

Proposition 2.1. *The operators bAB and $T(t) = e^{-bABt}$ are given by the following expression*

$$bABz = \sum_{n=1}^{\infty} \frac{b\lambda_n}{1 + a\lambda_n} E_n z \tag{12}$$

$$T(t)z = e^{-bABt}z = \sum_{n=1}^{\infty} e^{\frac{-b\lambda_n}{1+a\lambda_n}t} E_n z, \tag{13}$$

and

$$\| T(t) \| \leq e^{-\beta t}, \quad t \geq 0, \tag{14}$$

where

$$\beta = \inf_{n\geq 1}\left\{\frac{b\lambda_n}{1 + a\lambda_n}\right\} = \frac{b\lambda_1}{1 + a\lambda_1}. \tag{15}$$

With this notation the system (11) can be written as follows

$$z' = -Az + Bu, \quad t > 0, \tag{16}$$

where $A = bBA$ *and* $B : I\!R^m \to Z$ *is a linear bounded operator given by*

$$BU = \sum_{i=1}^{m} Bb_i U_i, \quad U = (U_1, U_2, , U_m) \in I\!R^m. \tag{17}$$

So, the control $u \in L^2(0, t_1; I\!R^m)$.

Now, we shall give the definition of approximate controllability in terms of system (16). To this end, for all $z_0 \in D(A)$ and a control $u \in L^2(0, t_1; I\!R^m)$ the equation (16) with $z(0) = z_0$ has a unique mild solution given by

$$z(t) = T(t)z_0 + \int_0^t T(t-s)Bu(s)ds, \quad 0 \le t \le t_1. \tag{18}$$

Definition 2.1. *We say that (16) is approximately controllable in* $[0, t_1]$ *if for all* z_0, $z_1 \in Z$ *and* $\epsilon > 0$, *there exists a control* $u \in L^2(0, t_1; I\!R^m)$ *such that the solution* $z(t)$ *given by (18) satisfies*

$$\|z(t_1) - z_0\| \le \epsilon. \tag{19}$$

The following theorem holds in general and can be found in [6].

Theorem 2.2. *(16) is approximately controllable on* $[0, t_1]$ *iff*

$$B^*T^*(t)z = 0, \quad \forall t \in [0, t_1], \quad \Rightarrow z = 0. \tag{20}$$

3 Main Theorem

Now, we are ready to formulate the main result of this work. Under the above conditions we will prove:

Theorem 3.1. *(16) is approximately controllable on* $[0, t_1]$ *iff the following finite dimensional systems are controllable on* $[0, t_1]$

$$y' = -\frac{b\lambda_j}{1 + a\lambda_j} y + E_j B_u, \quad y \in R(E_j), \quad j = 1, 2, ..., \infty. \tag{21}$$

The next theorem can be proved in the same way as Lemma 1 from [11].

Theorem 3.2. *The following statements are equivalent:*
(a) system (21) is controllable on $[0, t_1]$*,*
(b) $(E_j B)^* = B_j^*$ *is one to one,*
(c) $Rank[B_j] = \gamma_j$*.*

For the proof of Theorem 3.1 we will use the following lemma from [6] and [7].

Lemma 3.3. *Let* $\{\alpha_j\}_{j\geq 1}$ *and* $\{\beta_{i,j} : i = 1, 2, ..., m\}_{j\geq 1}$ *be two sequences of complex numbers such that:* $\alpha_1 > \alpha_2 > \alpha_3 \cdots$*.*
Then

$$\sum_{j=1}^{\infty} e^{\alpha_j t}\beta_{i,j} = 0, \quad \forall t \in [0, t_1], \quad i = 1, 2, \cdots, m$$

iff

$$\beta_{i,j} = 0, \quad i = 1, 2, \cdots, m; \quad j \geq 1.$$

Proof of Theorem 3.2. Suppose that each system (21) is controllable in $[0, t_1]$. Now, we compute $B^*T^*(t)$.

$$B^* : Z \to I\!R^m, \quad B^*z = (< Bb_1, z >, \cdots, < Bb_m, z >),$$

and

$$T^*(t)z = \sum_{j=1}^{\infty} e^{-\rho_j t} E_j z, \quad z \in Z, \quad t \geq 0,$$

where

$$\rho_j = \frac{b\lambda_j}{1 + a\lambda_j}, \qquad j = 1, 2, \ldots.$$

Therefore,

$$B^*T^*(t)z = (< Bb_1, T^*(t)z >, \cdots, < Bb_m, T^*(t)z >).$$

Hence, system (16) is approximately controllable on $[0, t_1]$ iff

$$< Bb_i, T^*(t)z >= 0, \quad \forall t \in [0, t_1], \quad i = 1, 2, \cdots, m, \Rightarrow z = 0. \tag{22}$$

Now, we shall check condition (22):

$$< Bb_i, T^*(t)z >= \sum_{j=1}^{\infty} e^{-\rho_j t} < Bb_i, E_j z >= 0, \quad i = 1, 2, \cdots, m; \quad t \in [0, t_1]. \tag{23}$$

Applying Lemma 3.3, we conclude that

$$< Bb_i, E_j z >=< b_i, (E_j B)^* z >= \frac{1}{1 + a\lambda_j} < b_i, E_j z >= 0, \quad i = 1, 2, \cdots, m.$$

On the other hand, we have

$$B_j^* E_j z = \frac{1}{1 + a\lambda_j}(< b_1, E_j z >, \cdots, < b_m, E_j z >).$$

Therefore, $B_j^* E_j z = 0$, $j \geq 1$. Since B_j^* is one to one, then $E_j z = 0$.
Since $\{E_j\}_{j \geq 1}$ is complete, then $z = 0$.
Conversely, assume that system (16) is approximately controllable on $[0, t_1]$ and there exists J such that the system

$$y' = -\frac{b\lambda_J}{1 + a\lambda_J} y + E_J B_u, \quad y \in R(E_J),$$

is not controllable on $[0, t_1]$. Then, there exists $V_J \in R(E_J)$ such that

$$(E_J B)^* e^{-\rho_J t} V_J = 0, \quad t \in [0, t_1] \quad \text{and} \quad V_J \neq 0.$$

Then,

$$(E_J B)^* V_J = 0, \quad \text{and} \quad V_J \neq 0.$$

Letting $z = E_J^* V_J$, we obtain

$$\begin{aligned} B^* T^*(t) z &= (< Bb_1, e^{-\rho_J t} V_J >, \cdots, < Bb_m, e^{-\rho_J t} V_J >) \\ &= e^{-\rho_J t}(< b_1, (E_J B)^* V_J >, \cdots, < b_m, (E_J B)^* V_J >) = 0, \end{aligned}$$

which contradicts the assumption.

Proposition 3.4. *The matrix representation of the operator B_j is given by*

$$B_j = \frac{1}{1 + a\lambda_j} \begin{pmatrix} < b_1, \phi_{j,1} > & < b_2, \phi_{j,1} > & . & . & < b_m, \phi_{j,1} > \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ < b_1, \phi_{j,\gamma_j} > & < b_2, \phi_{j,\gamma_j} > & . & . & < b_m, \phi_{j,\gamma_j} > \end{pmatrix}_{\gamma_j \times m}$$

Proof. We know that $\left\{\phi_{j,1}, \ldots, \phi_{j,\gamma_j}\right\}$ is an orthonormal base of $R(E_j)$. Now, consider the canonical base $\{e_1, \ldots, e_m\}$ of $I\!R^m$. Then

$$B_j e_i = \frac{1}{1 + a\lambda_j} \sum_{i=1}^{\gamma_j} < b_i, \phi_{j,k} > \phi_{j,k}$$

Therefore, the above matrix representation of B_j hold.

Remark 3.1. *From proposition (3.4) we can see that the number of controls requered for the approximate controllability of (16) must be at least that of the highest multiplicity of the eigenvalues i.e.,* $m \geq \gamma_j$, $j = 1, 2, \ldots, \infty$.

3.1 The Scalar BBM Equation

The controlled BBM equation for the case $N = 1$ is give by

$$\begin{cases} z_t - az_{xxt} - bz_{xx} = b(x)u, \quad t \geq 0, \qquad 0 \leq x \leq 1, \\ z(t,1) = z(t,0) = 0. \end{cases}$$

Corollary 3.1. *The system is approximately controllable on* $[0, t_1]$ *iff*

$$\int_0^1 b(x) sin(j\pi x) dx \neq 0, \quad j = 1, 2, \ldots, \infty.$$

Proof. In this case $\lambda_j = -j^2\pi^2$ and

$$\phi_{jk}(x) = \phi_j(x) = sin(j\pi x), \quad \gamma_j = 1.$$

Therefore, from proposition (3.2). We get that

$$B_j = \frac{1}{1 + a\lambda_j}[< b_i, \phi_j >],$$

and

$$Rank[B_j] = 1, \Leftrightarrow < b_i, \phi_j > \neq 0.$$

This completes the proof.

4 Conclusion

The original Benjamin -Bona-Mohany Equation is a non-linear one, here we have proved the aproximate controllability of the linear part of this equation, which is the fundamental base for the study of the controllability of the non linear BBME. So, our next work concern with the controllability of non linear BBME.

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