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Remark on Dirichlet Series Satisfying Functional Equations

Nota sobre Series de Dirichlet que Satisfacen Ecuaciones Funcionales

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Abstract

Besides a well known example of Davenport and Heilbronn, there exist other Dirichlet series satisfying a functional equation, similar to the one satisfied by the Riemann zeta function. As in the case of the former, some of them also have zeros off the critical line.
Key words and phrases: Dirichlet series, Riemann zeta function, Functional equations.

Resumen

Además de un ejemplo bien conocido debido a Davenport y Heilbronn, existen otras series de Dirichlet que satisfacen una ecuación funcional. Como en el caso de la primera, algunas de estas series también tienen ceros fuera de la linea crítica.
Palabras y frases clave: Series de Dirichlet, función zeta de Riemann, ecuaciones funcionales.

1 Introduction

It is a well known fact that the Riemann zeta function $\zeta(s)$ is an analytic function in the entire complex plane, save for the point $s = 1$, where it has a simple pole with residue 1. Moreover, $\zeta(s)$ satisfies the following functional equation (see [6], page 13)

$$\zeta(s) = \chi(s)\,\zeta(1-s), \qquad \chi(s) = 2\,(2\pi)^{s-1}\,\Gamma(1-s)\,\sin\left(\frac{\pi s}{2}\right). \tag{FE}$$

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Let $s = \sigma + it$. Dirichlet series (with σ_0 as abscissa of absolute convergence)

$$f(s) \;=\; \sum_{n=1}^{\infty} \frac{f_n}{n^s}, \qquad\qquad \sigma > \sigma_0$$

whose meromorphic continuation to the complex plane satisfy a functional equation are not scarce. The next theorem is attributed to H. Davenport and H. Heilbronn ([3], page 212).

Theorem 1. *Let* $\xi = \dfrac{\sqrt{10-2\sqrt{5}}-2}{\sqrt{5}-1}$. *For s, a complex number with real part greater than one, let*

$$f_1(s) = 1 + \frac{\xi}{2^s} - \frac{\xi}{3^s} - \frac{1}{4^s} + \frac{0}{5^s} + \cdots$$

be a periodic Dirichlet series with period 5. Then $f_1(s)$ *defines an entire function satisfying the functional equation*

$$f_1(s) \;=\; 5^{-s+\frac{1}{2}}\,\chi_1(s)\,f_1(1-s), \qquad \chi_1(s) = 2\,(2\pi)^{s-1}\,\Gamma(1-s)\,\cos\big(\frac{\pi s}{2}\big).$$

Moreover, $f_1(s)$ *has zeros off the critical line* $\sigma = 1/2$.

This example of Dirichlet series is interesting in view of the hitherto unproved Riemann hypothesis to the effect that $\zeta(s)$ has non-trivial complex zeros only on the line $\sigma = 1/2$ and nowhere else. The Davenport-Heilbronn example has received due attention (see [4], for example). By a theorem of H. Hamburger (see [6], page 31) the zeta function of Riemann is determined by its functional equation (FE). Hence, if we want to produce other Dirichlet series satisfying a functional equation, then it is necessary to change (FE) somehow. In the above theorem the functional equation (FE) has been altered in two ways; first by introducing an extra factor $5^{-s+\frac{1}{2}}$, and also by replacing the sine with a cosine function. This last change to (FE) is unnecessary. Indeed, the examples to follow provide us with Dirichlet series satisfying functional equations which closer resembles (FE) than the Davenport-Heilbronn example does.

2 Dirichlet polynomials

It is easy to see that if $f(s)$ and $g(s)$ are two Dirichlet series, each satisfying a functional equation, then the product $f(s) \cdot g(s)$ defines a third Dirichlet series also satisfying a given functional equation. In this section we exhibit Dirichlet polynomials satisfying a simple functional equation.

Proposition 2. *Let $s = \sigma + it$. If d is natural number greater than one, then*

$$1 \pm \frac{\sqrt{d}}{d^s} \;=\; \pm\, d^{-s+\frac{1}{2}} \cdot \big(1 \pm \frac{\sqrt{d}}{d^{1-s}}\big).$$

This proposition provides us with simple Dirichlet polynomials satisfying: $f(s) = \pm\, d^{-s+\frac{1}{2}} \cdot f(1-s)$. It is clear how to produce more complex examples.

Proposition 3. *Let A be a positive integer. Let $A = a_1 a_2 \cdots a_r$ be a decomposition of A into a product of integers $a_j > 1$. For $s = \sigma + it$, define the Dirichlet polynomial*

$$p(s) = \prod_{j=1}^{r} \big(1 + \frac{\sqrt{a_j}}{a_j^s}\big). \tag{1}$$

Then $p(s)$ satisfies: $p(s) = \epsilon \cdot A^{-s+\frac{1}{2}} \cdot p(1-s)$. The sign of each $\sqrt{a_j}$ can be taken to be, either positive or negative. The term ϵ equals -1 if an odd number of signs in $\sqrt{a_j}$ have taken to be negative, otherwise $\epsilon = 1$.

Proposition 4. *Let $p(s)$ be as in Proposition 3. If $p(s) = 0$ then $\sigma = 1/2$. Thus, all zeros of $p(s)$ lie in the critical line $\sigma = 1/2$.*

Proof. Assume that $1 = \mp\sqrt{d} \cdot d^{-s}$. Now we can take absolute values: $1 = \sqrt{d}/d^{\sigma}$. Solving for σ we get: $\sigma = 1/2$. □

3 Dirichlet Series

In this section we will use Proposition 3 to produce Dirichlet series satisfying a functional equation. In section 5 we will look at a concrete example and obtain finite dimensional vector spaces of Dirichlet series, all whose elements satisfy a functional equation.

Theorem 5. *Let A be a positive integer. Let $A = a_1 a_2 \cdots a_r$ be a decomposition of A into a product of integers $a_j > 1$. Let $f(s)$ be a Dirichlet series defining a meromorphic function on the whole complex plane. Assume $f(s)$ satisfies the functional equation*

$$f(s) = \mathcal{X}(s) \cdot f(1-s). \tag{2}$$

Define a new Dirichlet series $g(s) = p(s) \cdot f(s)$, where $p(s)$ is as in (1). Then $g(s)$ satisfies the following functional equation:

$$g(s) = \pm\, A^{-s+\frac{1}{2}} \cdot \mathcal{X}(s) \cdot g(1-s). \tag{3}$$

Moreover, if we take $f(s)$ to be the Riemann zeta function, then $g(s) = p(s) \cdot \zeta(s)$ is a periodic Dirichlet series of period A.

Proof. Only the assertion about the periodicity of $g(s) = p(s) \cdot \zeta(s)$ needs to be verified. Let $\sigma > \sigma_0$. Let $\mathcal{A} = \{a_1, \ldots, a_r\}$ be the set of integers in the above decomposition of A. Let

$$p(s) \;=\; \sum_{n \le A} \frac{p_n}{n^s}, \qquad \text{and} \qquad g(s) = \sum_{n=1}^{\infty} \frac{g_n}{n^s}.$$

Notice that $p_n = 0$ unless n is a product of the elements of some subset of $\mathcal{A}$. Assume $n \equiv m \pmod{A}$. Then (see [1], Theorem 11.5, page 283)

$$g_n \;=\; \sum_{\substack{jk=n \\ j \le A}} p_j \;=\; \sum_{\substack{jk=m \\ j \le A}} p_j \;=\; g_m$$

because if $j|n$ and $p(j) \neq 0$ (so that also $j|A$) then $j|m = n + Aq$. □

4 Zeros off $\sigma = 1/2$

From Theorem 1 we know that the Davenport-Heilbronn example has zeros off the critical line $\sigma = 1/2$. If we have two linearly independent Dirichlet series, both satisfying the same functional equation, then it is easy to produce a third Dirichlet series satisfying the same functional equation and having zeros at preassigned places.

Theorem 6. *Let $f_1(s)$ and $f_2(s)$ be two periodic, linearly independent Dirichlet series. Assume that both $f_1(s)$ and $f_2(s)$ satisfy the functional equation (2). Let s_0 be any complex number. Then there exists a Dirichlet series $f(s)$ satisfying (2) and such that $f(s_0) = 0$.*

Proof. A sufficient condition for $f(s) := \alpha f_1(s) - \beta f_2(s) = 0$ is that

$$\frac{\alpha}{\beta} \;=\; \frac{f_2(s)}{f_1(s)}.$$

□

Thus for example, the functional equation

$$f(s) = 5^{-s+\frac{1}{2}} \cdot \chi(s) \cdot f(1-s), \tag{4}$$

with $\chi(s)$ as in (FE), is satisfied by both (see [1], Teorema 12.11, page 326 for the case of the L-function $L(s, \chi_2^{(5)})$)

$$\begin{aligned} \left(1 + \frac{\sqrt{5}}{5^s}\right) \cdot \zeta(s) &= 1 + \tfrac{1}{2^s} + \tfrac{1}{3^s} + \tfrac{1}{4^s} + \tfrac{1+\sqrt{5}}{5^s} + \cdots, \\ L(s, \chi_2^{(5)}) &= 1 - \tfrac{1}{2^s} - \tfrac{1}{3^s} + \tfrac{1}{4^s} + \tfrac{0}{5^s} + \cdots \end{aligned}$$

Since these are linearly independent Dirichlet series, then we have examples of 5-periodic Dirichlet series satisfying (4) and having zeros off the critical line $\sigma = 1/2$.

5 Case $A = 6$

Now we produce as many as we can, essentially distinct (i.e., linearly independent) periodic Dirichlet series of period 6. These will arise from the distinct factorizations of the period $A = 6$, via producing a Dirichlet polynomial $p(s)$ and then forming the product $p(s) \cdot \zeta(s)$. Since we are dealing with periodic Dirichlet series, we have only to specify the first $A = 6$ coefficients. We write these as 6 dimensional vectors.

Thus, corresponding to the factorizations $6 = 2 \cdot 3$ and $6 = (-2) \cdot (-3)$, we obtain

$$\begin{pmatrix} 1 & 1+\sqrt{2} & 1+\sqrt{3} & 1+\sqrt{2} & 1 & 1+\sqrt{2}+\sqrt{3}+\sqrt{6} \\ 1 & 1-\sqrt{2} & 1-\sqrt{3} & 1-\sqrt{2} & 1 & 1-\sqrt{2}-\sqrt{3}+\sqrt{6} \end{pmatrix} \tag{5}$$

where we use each row in the above matrix as the first coefficients of a periodic Dirichlet series. Each of these series satisfies:

$$f(s) = 6^{-s+\frac{1}{2}} \cdot \chi(s) \cdot f(1-s), \tag{6}$$

where $\chi(s)$ is as in (FE). Any linear combination of these two series also satisfies (6). We notice from (5) that $(1\ 1\ 1\ 1\ 1\ 1+\sqrt{6})$ defines a Dirichlet series satisfying (6). This corresponds to the trivial factorization $6 = 6$.

Also, corresponding to the factorizations $-6 = (-2) \cdot 3$ and $-6 = 2 \cdot (-3)$, we obtain that each Dirichlet series in the two dimensional space generated by the rows of

$$\begin{pmatrix} 1 & 1-\sqrt{2} & 1+\sqrt{3} & 1-\sqrt{2} & 1 & 1-\sqrt{2}+\sqrt{3}-\sqrt{6} \\ 1 & 1+\sqrt{2} & 1-\sqrt{3} & 1+\sqrt{2} & 1 & 1+\sqrt{2}-\sqrt{3}-\sqrt{6} \end{pmatrix} \tag{7}$$

satisfies:

$$f(s) = -6^{-s+\frac{1}{2}} \cdot \chi(s) \cdot f(1-s). \tag{8}$$

From (7) it follows that $(1\ 1\ 1\ 1\ 1\ 1 - \sqrt{6})$ defines a Dirichlet series satisfying (8). This corresponds to $-6 = -6$.

6 Pairs of Dirichlet Series

Let us say that two Dirichlet series $f(s)$ and $f^*(s)$ are the one dual of the other, if there exists a function $\mathcal{X}(s)$ such that

$$f(s) \;=\; \mathcal{X}(s) \cdot f^*(1-s). \tag{9}$$

As a continuation of the example in §5, we now produce a pair of such Dirichlet series. Let $f_1(s)$, $f_2(s)$ and $f_3(s)$ be three linearly independent Dirichlet series such that the first two satisfy (6) while the third satisfies (8). Let

$$f(s) \;=\; \alpha_1 f_1(s) + \alpha_2 f_2(s) + \alpha_3 f_3(s).$$

Then we have

$$f(s) \;=\; 6^{-s+\frac{1}{2}} \cdot \chi(s) \cdot \big\{\alpha_1 f_1(1-s) + \alpha_2 f_2(1-s) - \alpha_3 f_3(1-s)\big\}.$$

Notice the change of sign in the third term. By considering $s = \sigma + it$ such that $1-\sigma > \sigma_0$, one can determine the Dirichlet series which equals the last linear combination. Thus for example, let $f_1(s)$ and $f_2(s)$ be the two Dirichlet series obtained from matrix (5) and let $f_3(s)$ be the Dirichlet series obtained from $(1\ 1\ 1\ 1\ 1\ 1 - \sqrt{6})$. Now put $\alpha_1 = \alpha_2 = (6-\sqrt{6})/24$ and $\alpha_3 = (6+\sqrt{6})/24$. Then we have that

$$\begin{aligned} f(s) &= 1 + \tfrac{1}{2^s} + \tfrac{1}{3^s} + \tfrac{1}{4^s} + \tfrac{1}{5^s} + \tfrac{0}{6^s} + \cdots \\ f^*(s) &= -\eta - \tfrac{\eta}{2^s} - \tfrac{\eta}{3^s} - \tfrac{\eta}{4^s} - \tfrac{\eta}{5^s} + \tfrac{5\eta}{6^s} + \cdots \end{aligned}$$

where $\eta = 1/6^{\frac{1}{2}}$, is a dual pair of 6-periodic Dirichlet series.

7 Final Remark

The examples presented here can also be obtained from a theorem in [5], by solving some elementary eigenvalue problems. The Davenport-Heilbronn example can also be obtained in this manner.

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