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Testing Block Sphericity of a Covariance Matrix

Prueba de Esfericidad por Bloques para una Matriz de Covarianza

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Abstract

This article deals with the problem of testing the hypothesis that q p-variate normal distributions are independent and that their covariance matrices are equal. The exact null distribution of the likelihood ratio statistic when $q = 2$ is obtained using inverse Mellin transform and the definition of Meijer's G-function. Results for $p = 2$, 3, 4 and 5 are given in computable series forms.

Key words and phrases: block sphericity, distribution, inverse Mellin transform, Meijer's G-function, residue theorem.

Resumen

Este artículo trata el problema de probar la hiptesis de que q distribuciones normales p-variadas son independientes y con matrices de covarianza son iguales. La distribución exacta nula del estadístico de razón de verosimilitudes cuando $q = 2$ se obtiene usando la transformada inversa de Mellin y la definicin de la G-funcin de Meijer. Los resultados para $p = 2$, 3, 4 y 5 se dan en forma de series calculables.
Palabras y frases clave: esfericidad por bloques, distribución, transformada inversa de Mellin, función G de Meijer, teorema del residuo.

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1 Introduction

Suppose that the $pq \times 1$ random vector $\boldsymbol{X}$ has a multivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix Σ and that $\boldsymbol{X}$, $\boldsymbol{\mu}$ and Σ are partitioned as

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{X}_1' & \boldsymbol{X}_2' & \cdots & \boldsymbol{X}_q' \end{pmatrix}', \qquad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1' & \boldsymbol{\mu}_2' & \cdots & \boldsymbol{\mu}_q' \end{pmatrix}'$$

and

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1q} \\ \Sigma_{21} & \Sigma_{22} & \cdots & \Sigma_{2q} \\ \vdots & & & \\ \Sigma_{q1} & \Sigma_{q2} & \cdots & \Sigma_{qq} \end{pmatrix}$$

where $\boldsymbol{X}_i$ and $\boldsymbol{\mu}_i$ are $p \times 1$ and Σ_{ij} is $p \times p$, $i, j = 1, \ldots, q$. Consider testing the null hypothesis H that the subvectors $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_q$ are independent and that the covariance matrices of these subvectors are equal, i.e.,

$$H : \Sigma = \begin{pmatrix} \Delta & 0 & \cdots & 0 \\ 0 & \Delta & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \Delta \end{pmatrix}$$

against the alternative H_a that H is not true. In H the common covariance matrix Δ is unspecified. Olkin, while extending the circular symmetric model to the case where the symmetries are exhibited in blocks, defined H and called it *block sphericity hypothesis* (see [13]). Let A be the sample sum of squares and product matrix formed from a sample of size $N = n + 1$. Partition A as $A = (A_{ij})$ where A_{ij} is $p \times p$, $i, j = 1, \ldots, q$. The likelihood ratio statistic for testing H (see [17]) is given by

$$\Lambda = \frac{\det(A)^{\frac{1}{2}N}}{\det\left(\frac{1}{q}\sum_{i=1}^{q} A_{ii}\right)^{\frac{1}{2}qN}}.$$

The null hypothesis is rejected if $\Lambda \leq \Lambda_0$ where Λ_0 is determined by the null distribution and level of significance. When $p = 1$, the hypothesis of block sphericity reduces to the Mauchly sphericity hypothesis $H : \Sigma = \sigma^2 I_q$ which has been studied extensively, e.g., see Sugira [15, 16], Khatri and Srivastava

[6, 7], Muirhead [10], Gupta [3], Nagar, Jain and Gupta [11] and Amey and Gupta [1]. The hypothesis of block sphericity is useful in multivariate repeated measures designs (see [17]). Since, $A \sim W_{pq}(n, \Sigma)$, the h^{th} null moment of Λ can be obtained by integrating over Wishart density. We have

$$E(\Lambda^h) = q^{\frac{1}{2}pqNh} c_{pq,n} \int\limits_{A>0} \frac{\det(A)^{\frac{1}{2}Nh}}{\det\big(\sum_{i=1}^{q} A_{ii}\big)^{\frac{1}{2}qNh}} \det(A)^{\frac{1}{2}(n-pq-1)} \operatorname{etr}\big(-\tfrac{1}{2}\Sigma^{-1}A\big)\, dA$$

$$= q^{\frac{1}{2}pqNh} c_{pq,n} \int\limits_{A>0} \det\left(\sum_{i=1}^{q} A_{ii}\right)^{-\frac{1}{2}qNh} \det(A)^{\frac{1}{2}(Nh+n-pq-1)} \operatorname{etr}\big(-\tfrac{1}{2}\Sigma^{-1}A\big)\, dA$$

where

$$c_{m,n} = \left[2^{\frac{1}{2}nm} \pi^{\frac{1}{4}m(m-1)} \prod_{j=1}^{m} \left[\frac{1}{2}(n-j+1)\right] \det(\Sigma)^{\frac{1}{2}n} \right]^{-1}. \tag{1.1}$$

Consequently,

$$E(\Lambda^h) = q^{\frac{1}{2}pqNh} \frac{c_{pq,n}}{c_{pq,Nh+n}} E\left[\det\left(\sum_{i=1}^{q} A_{ii}\right)^{-\frac{1}{2}qNh} \right] \tag{1.2}$$

where $A \sim W_{pq}(n+Nh, \Sigma)$. Under H, $A_{11}, \dots, A_{qq}$ are independent Wishart matrices, $A_{ii} \sim W_p(n+Nh, \Delta)$. Hence, $\sum_{i=1}^{q} A_{ii} \sim W_p((n+Nh)q, \Delta)$. Using Theorem 3.3.22 of [5], we obtain

$$\begin{aligned} & E\left[\det\left(\sum_{i=1}^{q} A_{ii}\right)^{-\frac{1}{2}qNh} \right] \\ & = 2^{-\frac{1}{2}pqNh} \det(\Delta)^{-\frac{1}{2}qNh} \prod_{i=1}^{p} \frac{\Gamma[\frac{1}{2}(nq-i+1)]}{\Gamma[\frac{1}{2}(nq+Nqh-i+1)]}. \end{aligned} \tag{1.3}$$

Substituting (1.3) in (1.2) and using (1.1) we obtain the h^{th} null moment of Λ as

$$E(\Lambda^h) = q^{\frac{1}{2}pqNh} \prod_{j=1}^{pq} \frac{\Gamma[\frac{1}{2}(n+Nh-j+1)]}{\Gamma[\frac{1}{2}(n-j+1)]} \prod_{i=1}^{p} \frac{\Gamma[\frac{1}{2}(nq-i+1)]}{\Gamma[\frac{1}{2}(nq+Nqh-i+1)]}. \tag{1.4}$$

The exact non-null distribution of Λ, for $q = 2$ and under two specific alternatives, has been derived by Gupta and Chao [4]. The asymptotic null distribution of $-2\ln\Lambda$ is chi-square with $\frac{1}{2}p(q-1)(pq+p+1)$ d.f. The null distribution of $\Lambda^{\frac{2}{N}}$, in series involving Bernoulli polynomials, is available in [**?**, CG]

In this article we will derive the exact null distribution of $V = \Lambda^{\frac{1}{N}}$ by using inverse Mellin transform and residue theorem (see [14, 12]).

2 Exact density of V

In order to obtain the exact density function $f(v)$ of $V = \Lambda^{\frac{1}{N}}$ we start with the null moment expression. Substituting $q = 2$ in (1.4) the h^{th} moment of V is simplified as

$$E(V^h) = 2^{ph}\prod_{j=1}^{2p}\left\{\frac{\Gamma[\frac{1}{2}h+\frac{1}{2}(n-j+1)]}{\Gamma[\frac{1}{2}(n-j+1)]}\right\}\prod_{j=1}^{p}\left\{\frac{\Gamma[n-\frac{1}{2}(j-1)]}{\Gamma[h+n-\frac{1}{2}(j-1)]}\right\}$$

This will be rewritten in a slightly different form by using duplication formula for gamma function

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right).$$

Then

$$E(V^h) = K(n,p)\prod_{j=1}^{p}\left\{\frac{\Gamma(h+n-2p-1+2j)}{\Gamma[h+n-\frac{1}{2}(j-1)]}\right\} \tag{2.1}$$

where

$$K(n,p) = \prod_{j=1}^{p}\left\{\frac{\Gamma[n-\frac{1}{2}(j-1)]}{\Gamma(n-2p-1+2j)}\right\}. \tag{2.2}$$

Now the density function $f(v)$ of $V = \Lambda^{\frac{1}{N}}$ is obtained by taking inverse Mellin transform of $E(V^h)$ as

$$f(v) = (2\pi\iota)^{-1}\int_L E(V^h)v^{-h-1}\,dh,\ 0 < v < 1, \tag{2.3}$$

where $\iota = \sqrt{-1}$ and L is a suitable contour. Substituting (2.1) in (2.3) and applying the transformation $h + n - 2p = t$, one obtains

$$f(v) = K(n,p)v^{n-2p-1}(2\pi\iota)^{-1}\int_{L_1}\prod_{j=1}^{p}\left\{\frac{\Gamma(t+2j-1)}{\Gamma[t+2p-\frac{1}{2}(j-1)]}\right\}v^{-t}\,dt,$$
$$0 < v < 1, \tag{2.4}$$

where L_1 is the changed contour and $K(n,p)$ is defined in (2.2). Now, using the definition of Meijer's G-function (see [8]), the density of V is written as

$$f(v) = K(n,p)v^{n-2p-1}G^{p,0}_{p,p}\left[v\middle|\begin{matrix}2p-\frac{1}{2}(j-1), j=1,\ldots,p\\ 2j-1, j=1,\ldots,p\end{matrix}\right], 0<v<1,$$

where $G^{p,0}_{p,p}[\,]$ is the Meijer's G-function.

Explicit expressions for the density of V for particular values of p will be obtained by evaluating the integral in (2.4) with the help of of residue theorem.

From (2.4), the density for $p=2$ is obtained as

$$f(v) = K(n,2)v^{n-5}(2\pi\iota)^{-1}\int_{L_1}\frac{\Gamma(t+1)}{(t+3)\Gamma(t+\frac{7}{2})}v^{-t}\,dt,\ 0<v<1. \qquad (2.5)$$

The integrand has simple poles at $t=-r-1$, $r=0,1,3,4,\ldots$, and a pole of order two at $t=-3$. The residue at $t=-r-1$, $r\neq 2$, is

$$\begin{aligned}
&\lim_{t\to -r-1}\left[\frac{(t+1+r)\Gamma(t+1)}{(t+3)\Gamma(t+\frac{7}{2})}v^{-t}\right]\\
&=\lim_{t\to -r-1}\left[\frac{(t+1+r)(t+1+r-1)\cdots(t+1)\Gamma(t+1)}{(t+1+r-1)\cdots(t+1)(t+3)\Gamma(t+\frac{7}{2})}v^{-t}\right]\\
&=\lim_{t\to -r-1}\left[\frac{\Gamma(t+1+r+1)}{(t+1+r-1)\cdots(t+1)(t+3)\Gamma(t+\frac{7}{2})}v^{-t}\right]\\
&=\frac{1}{(-1)^{r+1}\,r!\,(r-2)\Gamma(\frac{5}{2}-r)}v^{r+1}.
\end{aligned}$$

Simplifying $\Gamma(\frac{5}{2}-r)$ in the above expression using the result $\Gamma(\beta-j)=\frac{(-1)^j\Gamma(\beta)\Gamma(1-\beta)}{\Gamma(1-\beta+j)}$ we obtain the residue at $t=-r-1$ as

$$-\frac{1}{\pi}\frac{\Gamma(r-\frac{3}{2})}{r!\,(r-2)}v^{r+1},\ r\neq 2.$$

The residue at $t=-3$ is derived as

$$\lim_{t\to-3}\frac{\partial}{\partial t}\left[\frac{(t+3)^2\Gamma(t+1)}{(t+3)\Gamma(t+\frac{7}{2})}v^{-t}\right]=\lim_{t\to-3}\frac{\partial}{\partial t}\left[\frac{\Gamma(t+4)}{(t+1)(t+2)\Gamma(t+\frac{7}{2})}v^{-t}\right]$$

$$=\lim_{t\to-3}\left\{\frac{\partial}{\partial t}\ln\left[\frac{\Gamma(t+4)}{(t+1)(t+2)\Gamma(t+\frac{7}{2})}\right]-\ln v\right\}\frac{\Gamma(t+4)}{(t+1)(t+2)\Gamma(t+\frac{7}{2})}v^{-t}$$

$$=\lim_{t\to-3}\left\{\psi(t+4)-\sum_{i=1,2}(t+i)^{-1}-\psi(t+\tfrac{7}{2})-\ln v\right\}\frac{\Gamma(t+4)}{(t+1)(t+2)\Gamma(t+\frac{7}{2})}v^{-t}$$

$$=\left\{\psi(1)+\frac{1}{2}+1-\psi(\tfrac{1}{2})-\ln v\right\}\frac{1}{(-2)(-1)\Gamma(\frac{1}{2})}v^3$$

$$=\left\{\frac{3}{2}+2\ln 2-\ln v\right\}\frac{1}{2\sqrt{\pi}}v^3$$

where $\psi(\ )$ is the psi function (ver [9]). Now applying the residue theorem to the right hand side of (2.5), the density $f(v)$ is obtained as

$$f(v)=K(n,2)v^{n-5}\left[-\frac{1}{\pi}\sum_{r=0 r\neq 2}^{\infty}\frac{\Gamma(r-\frac{3}{2})}{r!\,(r-2)}v^{r+1}+\left\{\frac{3}{2}-\ln\left(\frac{v}{4}\right)\right\}\frac{1}{2\sqrt{\pi}}v^3\right]$$

for $0<v<1$. For $p=3$, (2.4) simplifies to

$$f(v)=K(n,3)v^{n-7}(2\pi\iota)^{-1}\int_{L_1}\frac{\Gamma(t+1)}{(t+5)(t+4)(t+3)\Gamma(t+\frac{11}{2})}v^{-t}\,dt \quad (2.6)$$

The integrand has simple poles at $t=-r-1$, $r=0,1,5,6,\ldots$, and poles of order two at $t=-r-1$, $r=2,3,4$. Evaluating residues at these poles and using residue theorem, the density in this case is obtained (for $0<v<1$) as

$$\begin{aligned} f(v) &= \frac{K(n,3)}{\sqrt{\pi}}v^{n-7}\left[\frac{2}{315}v-\frac{4}{45}v^2-\frac{1}{\sqrt{\pi}}\sum_{r=5}^{\infty}\frac{\Gamma(r-\frac{7}{2})}{r!\,(r-2)(r-3)(r-4)}v^{r+1}\right.\\ &\left.+\frac{1}{3}\Big(\frac{8}{3}-\ln\frac{v}{4}\Big)v^3-\frac{1}{3}\Big(\frac{1}{6}+\ln\frac{v}{4}\Big)v^4+\frac{1}{48}\Big(\frac{43}{12}-\ln\frac{v}{4}\Big)v^5\right]. \end{aligned}$$

For $p=4$, (2.4) reduces to

$$f(v)=K(n,4)v^{n-9}(2\pi\iota)^{-1}\int_{L_1}\frac{\Gamma(t+1)\Gamma(t+3)}{\prod_{j=5}^{7}(t+j)\Gamma(t+\frac{15}{2})\Gamma(t+\frac{13}{2})}v^{-t}\,dt. \quad (2.7)$$

The integrand has simple poles at $t = -1, -2$, poles of order two at $t = -1-r$, $r = 2, 3, 7, 8, \ldots$, and poles of order three at $t = -1-r$, $r = 4, 5, 6$. Evaluating residues at these poles and using the residue theorem, we get the density for $p = 4$ as

$$\begin{aligned} f(v) = K(n,4)v^{n-9}\Bigg[&\frac{1}{660\,\Gamma^2(\frac{11}{2})}v - \frac{1}{270\,\Gamma^2(\frac{9}{2})}v^2 \\ &- \frac{1}{168\,\Gamma^2(\frac{7}{2})}\left(\tfrac{2521}{420} + \ln\frac{v}{16}\right)v^3 - \frac{1}{90\,\Gamma^2(\frac{5}{2})}\left(\frac{71}{15} + \ln\frac{v}{16}\right)v^4 \\ &+ \frac{1}{\pi^2}\sum_{r=7}^{\infty}\left\{\psi(r+1) + \psi(r-1) + \sum_{j=4}^{7}\frac{1}{r-j} - \psi(r-\tfrac{11}{2}) - \psi(r-\tfrac{9}{2}) - \ln v\right\} \\ &\frac{\Gamma(r-\frac{9}{2})\Gamma(r-\frac{11}{2})}{r!(r-2)!(r-4)(r-5)(r-6)}v^{r+1} + \frac{1}{12\pi}\left\{\left(\ln\frac{v}{16} + \frac{13}{4}\right)^2 + \frac{1781}{144} - \frac{2\pi^2}{3}\right\}\frac{v^5}{2} \\ &- \frac{1}{360\pi}\left\{\left(\frac{127}{60} - \ln\frac{v}{16}\right)^2 + \frac{31769}{3600} - \frac{2\pi^2}{3}\right\}\frac{v^6}{2} \\ &- \frac{15}{2(6!)^2\pi}\left\{\left(-\ln\frac{v}{16} + \frac{121}{30}\right)^2 + \frac{33}{200} - \frac{2\pi^2}{3}\right\}\frac{v^7}{2}\Bigg],\ 0 < v < 1. \end{aligned}$$

For $p = 5$, (2.4) slides to

$$f(v) = K(n,5)v^{n-11}(2\pi\iota)^{-1}\int_{L_1}\frac{\Gamma(t+1)\Gamma(t+3)}{\prod_{j=5}^{9}(t+j)^{a_j}\Gamma(t+\frac{17}{2})\Gamma(t+\frac{19}{2})}v^{-t}\,dt,$$

$$0 < v < 1, \quad (2.8)$$

where $a_5 = a_6 = a_8 = a_9 = 1$ and $a_7 = 2$. The integrand has simple poles at $t = -1, -2$, poles of order 2 at $t = -1 - r$, $r = 2, 3, 9, 10, \ldots$, poles of order 3 at $t = -5, -6, -8, -9$ and a pole of order 4 at $t = -7$. Evaluating residues at these poles and using residue theorem, the density is derived as

$$f(v) = K(n,5)v^{n-11}\Big[A_1^{(0)}v + A_2^{(0)}v^2 + (-\ln v + B_3^{(0)})A_3^{(0)}v^3$$
$$+ (-\ln v + B_4^{(0)})A_4^{(0)}v^4 + \sum_{j=5,6,8}\left\{(-\ln v + B_j^{(0)})^2 + B_j^{(1)}\right\}A_j^{(0)}\frac{v^j}{2}$$
$$+\left\{(-\ln v + B_7^{(0)})^3 + 3(-\ln v)B_7^{(1)} + 3B_7^{(0)}B_7^{(1)} + B_7^{(2)}\right\}A_7^{(0)}\frac{v^7}{3!}$$
$$+\sum_{r=9}^{\infty}(-\ln v + B_r^{(0)})A_r^{(0)}v^r\Big],\ 0 < v < 1.$$

where

$$A_1^{(0)} = \frac{12}{(10)!\Gamma^2(\frac{15}{2})},\qquad A_2^{(0)} = \frac{4}{5(7)!\Gamma^2(\frac{13}{2})},\qquad A_3^{(0)} = \frac{1}{44(6)!\Gamma^2(\frac{11}{2})},$$

$$B_3^{(0)} = -\frac{5219}{693} + 2\ln 4,\quad A_4^{(0)} = \frac{2}{3(6)!\Gamma^2(\frac{9}{2})},\qquad B_4^{(0)} = -\frac{1691}{252} + 2\ln 4$$

$$A_5^{(0)} = \frac{5}{(8)!\Gamma^2(\frac{7}{2})},\qquad B_5^{(0)} = -\frac{569}{105} + 2\ln 4,\quad B_5^{(1)} = \frac{1202849}{88200} - \frac{2\pi^2}{3},$$

$$A_6^{(0)} = \frac{2}{15(6)!\Gamma^2(\frac{5}{2})},\qquad B_6^{(0)} = -\frac{69}{20} + 2\ln 4,\quad B_6^{(1)} = \frac{10969}{720} - \frac{2\pi^2}{3},$$

$$A_8^{(0)} = -\frac{1}{3(5)!(7)!\pi},\qquad B_8^{(0)} = \frac{989}{210} + 2\ln 4,\quad B_8^{(1)} = \frac{911681}{88200} - \frac{2\pi^2}{3},$$

$$A_7^{(0)} = \frac{7}{2(9)!\Gamma^2(\frac{3}{2})},\qquad B_7^{(0)} = -\frac{2}{15} + 2\ln 4,\quad B_7^{(1)} = \frac{24947}{1800} - \frac{2\pi^2}{3},$$

$$B_7^{(2)} = -\frac{656261}{54000} + 2\pi^2 - 4\zeta(3),$$
$$A_r^{(0)} = -\frac{\Gamma(r-\frac{13}{2})\Gamma(r-\frac{15}{2})}{\pi^2 r!(r-2)!\prod_{j=5}^{9}(r+1-j)^{a_j}}$$

and

$$B_r^{(0)} = \psi(r+1) + \psi(r-1) + \sum_{j=5}^{9}a_j(r+1-j)^{-1} - \psi(r-\frac{13}{2}) - \psi(r-\frac{15}{2}).$$

The method employed above can also be used to derive the exact density for $p \geq 6$. However, because of the higher order of poles, the expressions for the density will involve generalized Riemann zeta function (see [9]).

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