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Approximations of Fixed Points for Mappings in the Class A(T,α)

Aproximaciones de Puntos Fijos para Aplicaciones en la Clase A(T,α)

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Abstract

Let (M,d) be a complete metric space, let $0 \leq \alpha < 1$, and let S,T be two selfmappings of M. Supposing that S belongs to the class $A(T,\alpha)$ (i.e. condition (A) below is satisfied) we prove in Theorem 1.1 that S and T have a unique common fixed point. Although we do not use any continuity requirement neither for T nor for S, we conclude some regularity properties. Indeed, we show that S and TS must be continuous at the unique common fixed point. In Theorem 1.2, when $\alpha < \frac{1}{2}$, we provide four equivalent properties characterizing the existence and uniqueness of the common fixed point for S,T, and give sequences which approximate this fixed point. In particular, we show that all the Picard sequences defined by S converge to this common fixed point.
Key words and phrases: Common fixed points in complete metric spaces, approximations, Picard sequences.

Resumen

Sean (M,d) un espacio métrico completo, $0 \leq \alpha < 1$, S y T dos aplicaciones de M en sí mismo. Suponiendo que S pertenece a la clase $A(T,\alpha)$ (i.e., que se satisface la condición (A) de más abajo) se prueba en el Teorema 1.1 que S y T tienen un punto fijo común único. Aunque no se hace ningún requerimiento de continuidad para S ni para T se concluyen algunas propiedades de regularidad. En efecto se muestra que S y TS deben ser continuos en el único punto fijo común. En el Teorema 1.2, para $\alpha < \frac{1}{2}$, se proveen cuatro propiedades equivalentes

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que caracterizan la existencia y unicidad del punto fijo común para S y T, y se dan sucesiones que aproximan este punto fijo. En particular se muestra que todas las sucesiones de Picard definidas por S convergen a este punto fijo común.
Palabras y frases clave: puntos fijos comunes en espacios métricos completos, aproximaciones, sucesiones de Picard.

1 Introduction and statement of the results

The study of common fixed points has started in the year 1936 by the well known result of Markov and Kakutani. Since this year, many works were devoted to Fixed point theory. The literature on the subject is now very rich. Many authors have studied the existence of common fixed points. Once the problem of existence of fixed or common fixed points is solved, a practical problem arises. It is the problem of determining or at least approximating them. In many situations, the proofs given for the existence of fixed or common fixed points provide effective methods of approximation and computation, but this is not the general case. The aim of this note is to contribute to this area of investigation in metric fixed point theory and approximations.

Let (M,d) be a complete metric space. Let T be a fixed selfmapping and let $\alpha \in [0,1[$. We define $A(T,\alpha)$ as the set of selfmappings S of M such that for all $x,y \in M$, the following condition is satisfied:

$$d(Sx,TSy) \leq \alpha \max\left\{d(x,Sy), d(x,Sx), d(Sy,TSy), \frac{1}{2}\left[d(x,TSy)+d(Sx,Sy)\right]\right\}. \tag{A}$$

For every selfmapping S of M, we denote F_S the mapping defined for all $x \in M$, by $F_S(x) := d(x,Sx)$. For all positive number c, we denote $L_{c,S} := \{x \in M : F_S(x) \leq c\}$.

The first result of this paper is the following

Theorem 1. *Let (M,d) be a complete metric space. Let $\alpha \in (0,1[$ and let S,T be two self-mappings of M such that $S \in A(T,\alpha)$. Then the following four assertions are true.*

(1) There exists a unique point $z \in M$ such that $Fix(S) = Fix(\{S,T\}) = \{z\}$.
(2) S and TS are continuous at the point z.
(3) If $Im(T) \subset Im(S)$ then we have $Fix(S) = Fix(T) = Fix(\{S,T\}) = \{z\}$.
(4) If $\alpha \in [0, \frac{1}{2}[$, then for every $x_0 \in M$ the Picard sequence $\{S^n(x_0)\}$ converges to the unique common fixed point z of S and T.

When $\alpha \in [0, \frac{1}{2}[$, we can find some characterizations of the existence and uniqueness of the common fixed point of S, T for all S in the class $A(T, \alpha)$. These characterizations are stated in the next result.

Theorem 2. *Let (M, d) be a complete metric space. Let $\alpha \in [0, \frac{1}{2}[$ and let S, T be two self-mappings of M such that $S \in A(T, \alpha)$. Then the following four assertions are true and equivalent:*
(1) There exists a unique point $z \in M$ such that $Fix(S) = Fix(\{S, T\}) = \{z\}$.
(2) $\lim_{c \to 0^+} diam(L_{c,S}) = 0$, and the mapping F_S is an r.g.i. on M.
(3) There exists a (unique) point $z \in M$, such that, for each sequence $\{x_n\} \subset M$; $\lim_n d(x_n, Tx_n) = 0$ if and only if $\{x_n\}$ converges to z.
(4) There exists a (unique) point $z \in Im(S)$, (the range of S) such that, for each sequence $\{y_n\} \subset Im(S)$, we have $\lim_{n\to\infty} y_n = z$, if and only if, $\lim_{n\to\infty} F_T(y_n) = 0$.

We recall (see [3] and [6]) that a function $G : M \longrightarrow \mathbb{R}$ is said to be a regular-global-inf (r.g.i.) at $x \in M$ if $G(x) > \inf_M(G)$ implies there exist $\epsilon > 0$ such that $\epsilon < G(x) - \inf_M(G)$ and a neighborhood N_x of x such that $G(y) > G(x) - \epsilon$ for each $y \in N_x$. If this condition is satisfied for each $x \in M$, then G is said to be a r.g.i. on M. As we see, the r.g.i. condition may be considered as a weak type of regularity. In the paper [6] this condition has been extensively used in many problems dealing with metric fixed points. Therefore, in Theorem 1.2 we see, when $\alpha \in [0, \frac{1}{2}[$, that not only all the conclusions of Theorem 4.3 (p. 149) of [6] are still valid for all selfmappings in the class $S \in A(T, \alpha)$ but that, in addition, they are equivalent.

Remark. Let α, β, γ be three nonnegative numbers such that $\alpha + 2\beta + 2\gamma < 1$. Let S, T be two selmappings of M and consider the following contractive condition:

$$d(Sx, TSy) \le \alpha d(x, Sy) + \beta[d(x, Sx) + d(Sy, TSy)] + \gamma[d(x, TSy) + d(Sx, Sy)]. \tag{F}$$

B. Fisher proved in his paper [4] that if S is continuous and S, T verify (F) then S and T have a unique common fixed point. It is clear that if S, T satisfy the condition (F) then $S \in A(T, q)$, where $q := \alpha + 2\beta + 2\gamma$. Therefore, Theorem 1.1 improves the result obtained by B. Fisher in [4]. We point out that L. Nova tried, in his paper [7], to improve Fisher's result but the assumptions used in [7] are still much stronger. So our paper solves the problem posed in [7].

In section 2 we prove Theorem 1.1. The proof of Theorem 1.2 will be given in section 3.

2 Proof of Theorem 1.1.

2.1 First, we begin by proving that (1) is true.

(a) Let x_0 be some point in M, and set

$$\begin{array}{rcl} x_{2n} & = & Sx_{2n-1}, \quad n = 1, 2, \dots \\ x_{2n+1} & = & Tx_{2n}, \quad n = 0, 1, 2, \dots \end{array}$$

We put $t_n := d(x_n, x_{n+1})$ for all integers n. Suppose that $n = 2m$ for some integer m. Then

$$t_n = d(x_{2m}, x_{2m+1}) = d(Sx_{2m-1}, Tx_{2m}) = d(Sx_{2m-1}, TSx_{2m-1})$$
$$\leq \alpha \max\Big\{d(x_{2m-1}, x_{2m}), d(x_{2m-1}, x_{2m}), d(x_{2m}, x_{2m+1}), \frac{1}{2}d(x_{2m-1}, x_{2m+1}))\Big\}$$
$$\leq \alpha \max\Big\{t_{n-1}, t_n, \frac{1}{2}d(x_{2m-1}, x_{2m})\Big\}. \tag{2.1}$$

From (2.1) we deduce that $t_n \leq \max\{t_{n-1}, \frac{1}{2}d(x_{2m-1}, x_{2m})\}$. Indeed, if it is not the case, we will get $t_n > \max\{t_{n-1}, \frac{1}{2}d(x_{2m-1}, x_{2m})\} > 0$, and $t_n \leq \alpha t_n$, since $\alpha \in [0, 1[$, this inequality is impossible. Also, we must have $\frac{1}{2}d(x_{2m-1}, x_{2m}) \leq t_{n-1}$. Otherwise, by using the triangular inequality, we would have

$$t_{n-1} + t_n < d(x_{2m-1}, x_{2m}) \leq t_{n-1} + t_n,$$

which is a contradiction. We conclude that for every even integer greater than two, we have

$$0 \leq t_n \leq \alpha t_{n-1}. \tag{2.2}$$

By similar arguments, it is easy to see that the inequality (2.2) remains valid for odd integers. Since $0 \leq \alpha < 1$, the sequence $\{t_n\}$ must converge to zero.

(b) Now, we show that $\{x_n\}$ is a Cauchy sequence. Since $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ we need only to prove that $\{x_{2n}\}$ is a Cauchy sequence. To obtain a contradiction, let us suppose that there exists a number $\epsilon > 0$ and two sequences of integers $\{2n(k)\}$, $\{2m(k)\}$ with $2k \leq 2m(k) < 2n(k)$, such that

$$d(x_{2n(k)}, x_{2m(k)}) > \epsilon. \tag{2.3}$$

For each integer k, we shall denote $2n(k)$ the least even integer exceeding $2m(k)$ for which (2.3) holds. Then

$$d(x_{2m(k)}, x_{2n(k)-2}) \leq \epsilon \quad \text{and} \quad d(x_{2m(k)}, x_{2n(k)}) > \epsilon.$$

For each integer k, we set

$$p_k := d(x_{2m(k)}, x_{2n(k)}), \qquad s_k := d(x_{2m(k)}, x_{2n(k)+1}),$$
$$q_k := d(x_{2m(k)+1}, x_{2n(k)+1}), \quad \text{and} \quad r_k := d(x_{2m(k)+1}, x_{2n(k)+2}),$$

then by using triangular inequalities, we obtain

$$\begin{aligned} \epsilon < p_k &\le \epsilon + t_{2n(k)-2} + t_{2n(k)-1} \\ \mid s_k - p_k \mid &\le t_{2n(k)}, \\ \mid q_k - s_k \mid &\le t_{2m(k)}, \\ \mid r_k - s_k \mid &\le t_{2n(k)+1}. \end{aligned} \tag{2.4}$$

Since the sequence $\{t_n\}$ converges to 0, we deduce from (2.4) that the sequences: $\{p_k\}$, $\{s_k\}$, $\{q_k\}$ and $\{r_k\}$ have ϵ as a common limit. For all integers k, we have

$$\begin{aligned} r_k &= d(x_{2n(k)+2}, x_{2m(k)+1}) = d(Sx_{2n(k)+1}, TSx_{2m(k)-1}) \\ &\le \alpha \max\Big\{ d(x_{2n(k)+1}, x_{2m(k)}), d(x_{2n(k)+1}, x_{2n(k)+2}), d(x_{2m(k)}, x_{2m(k)+1}), \\ &\qquad \frac{1}{2}\left[d(x_{2n(k)+1}, x_{2m(k)+1}) + d(x_{2n(k)+2}, x_{2m(k)})\right]\Big\} \\ &\le \alpha \max\Big\{ s_k, t_k, t_k, \frac{1}{2}\left[q_k + d(x_{2n(k)+2}, x_{2m(k)})\right]\Big\} \\ &\le \alpha \max\left\{ s_k, t_k, \frac{1}{2}\left[q_k + r_k + t_k\right]\right\}. \end{aligned} \tag{2.5}$$

By letting $k \to \infty$ in (2.5), we get

$$\epsilon \le \alpha \max\{\epsilon, 0, \epsilon\} = \alpha\epsilon < \epsilon,$$

which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since (M, d) is complete, this sequence must have a limit, say z, in M. Next, we shall prove that z is a common fixed point for S and T.

(c) For all positive integers n, we have

$$\begin{aligned} &d(Sz, x_{2n+1}) = d(Sz, Tx_{2n}) = d(Sz, TSx_{2n-1}) \\ &\le \alpha \max\Big\{ d(z, x_{2n}), d(z, Sz), d(x_{2n}, x_{2n+1}), \frac{1}{2}\left[d(z, x_{2n+1}) + d(Sz, x_{2n})\right]\Big\} \end{aligned} \tag{2.6}$$

By taking the limits in both sides of (2.6), we obtain

$$d(Sz,z) \le \alpha d(Sz,z) < d(Sz,z),$$

which is a contradiction. Thus z is fixed by S. Let us show that $Tz = z$. By use of the property (A), we have

$$\begin{aligned} d(z,Tz) &= d(Sz,TSz) \\ &\le \alpha \max\big\{d(z,z), d(z,z), d(z,Tz), \frac{1}{2}\,[d(z,Tz)+d(z,z)]\big\}. \qquad (2.7) \end{aligned}$$

(2.7) implies that $(1-\alpha)d(z,Tz) = 0$. Since $\alpha < 1$, we conclude that $d(z,Tz) = 0$ and then $z \in Fix(\{S,T\})$. We deduce also that $Fix(S) \subset Fix(T)$.

(d) Suppose that there exists another point w fixed by S. Then by using the property (F), we have

$$\begin{aligned} d(w,z)) &= d(Sw,TSz)) \\ &\le \alpha \max\big\{d(w,z), d(w,w), d(z,z), d(w,z)\big\} \\ &\le \alpha d(w,z) \qquad (2.8) \end{aligned}$$

(2.8) implies that $w = z$. We conclude that $Fix(S) = Fix(\{S,T\}) = \{z\}$. This completes the proof of (1).

2.2 Let z be the unique common fixed point of S and T, and let $x \in M$. Then by using the property (A) and the triangular property, we have

$$\begin{aligned} d(Sx,z) &= d(Sx,TSz) \\ &\le \alpha \max\big\{d(x,z), d(x,z)+d(Sx,z), \frac{1}{2}\,[d(x,z)+d(Sx,z)]\big\} \\ &= \alpha[d(x,z)+d(Sx,z)]. \end{aligned}$$

We deduce that

$$d(Sx,z) \le \frac{\alpha}{1-\alpha}\, d(x,z). \qquad (2.9)$$

Therefore, S is continuous at z. Again, by using the property (A) and the triangular property, for every point x in M, we have

$$\begin{aligned} d(z,TSx) &= d(Sz,TSx) \\ &\le \alpha \max\big\{d(z,Sx), d(Sx,z)+d(z,TSx), \frac{1}{2}\,[d(z,TSx)+d(Sx,z)]\big\} \\ &= \alpha[d(Sx,z)+d(z,TSx)] \qquad (2.10) \end{aligned}$$

(2.10) implies that

$$d(z, TSx) \leq \frac{\alpha}{1-\alpha}\, d(Sx, z). \tag{2.11}$$

According to (2.9), the last inequality reduces to $d(z, TSx) \leq \frac{\alpha^2}{(1-\alpha)^2}\, d(x, z)$. Therefore, TS is continuous at z.

2.3 Suppose that $Im(T) \subset Im(S)$. Then, from the subsection (c) in 2.1, we already know that $Fix(S) \subset Fix(T)$. It remains to prove the inverse inclusion. Let $w \in Fix(T)$. Then $w \in Im(S)$ and we can find an $u \in M$, such that $w = Tw = Su$. By using the property (A), we obtain

$$\begin{aligned} d(Sw, w) &= d(Sw, Tw) = d(Sw, TSu) \\ &\leq \alpha \max\big\{d(w,w), d(w,Sw), d(w,w), \frac{1}{2}[d(w,w) + d(Sw,w)]\big\}. \end{aligned} \tag{2.12}$$

(2.12) implies that $[1-\alpha]d(Sw, w) = 0$, which implies that $Sw = w$.

2.4 Suppose that $\alpha \in [0, \frac{1}{2}[$. Let z be the unique common fixed point of S and T. Let y_0 be some point in M. We consider the Picard sequence defined for every integer n, by $y_n := S^n y_0$, where S^n is the $n-$th iterate of S. For each integer n, we put $u_n := d(y_n, z)$. Then by using the property (A), we have

$$\begin{aligned} u_{n+1} &= d(y_{n+1}, z) = d(Sy_n, TSz)) \\ &\leq \alpha \max\big\{u_n, d(y_n, y_{n+1}), 0, \frac{1}{2}[u_n + u_{n+1}]\big\} \\ &\leq \alpha \max\big\{u_n, d(y_n, y_{n+1}) \frac{1}{2}[u_n + u_{n+1}]\big\}. \end{aligned} \tag{2.13}$$

From (2.13) we deduce that $u_{n+1} \leq u_n$ for each integer n. Let l be the limit of u_n. By (17) we get $l \leq 2\alpha l$. Suppose that $l > 0$. Then we must have $\alpha \geq \frac{1}{2}$, which is a contradiction. Therefore, $\lim_{n\to\infty} S^n y_0 = z$, for every $y_0 \in M$. This completes the proof of Theorem 1.1. □

3 Proof of Thorem 1.2

3.1 Let us show that (1) implies (2). Suppose that (1) is satisfied, and let z be the unique common fixed point of S and T. Let x be some point in M. we shall prove that the following inequality is satisfaied

$$d(x, z) \leq \frac{1-\alpha}{1-2\alpha} d(x, Sx). \tag{3.1}$$

Indeed, by using the triangular inequality and (2.9), we have

$$d(x,z) = d(x,Sx) + d(Sx,z) \leq d(x,Sx) + \frac{\alpha}{1-\alpha} d(x,z). \tag{3.2}$$

(3.2) reduces to (3.1). For each positive number, we deduce from (3.2) that $L_{c,S}$ is bounded. It is nonvoid since it contains z. Now, let $x, y \in L_{c,S}$, then we have

$$d(x,y) \leq d(x,z) + d(y,z) \leq \frac{2(1-\alpha)c}{1-2\alpha}. \tag{3.3}$$

(3.3) shows that $diam(L_{c,S})$ tends to zero when c tends to zero. In order to show that F_S is r.g.i., we use Proposition 1.2 of [K-S] and the inequality (3.1).

3.2 Suppose that (2) is satisfied. Let x_0 be some point in M, and consider the associated sequence $\{x_n\}$ given by

$$x_{2n} = Sx_{2n-1}, \quad n = 1, 2, \dots$$
$$x_{2n+1} = Tx_{2n}, \quad n = 0, 1, 2, \dots$$

We recall from a) of 2.1, that the sequence $\{t_n := d(x_n, x_{n+1})\}$ verifies $t_n \leq \alpha t_{n-1}$ for all integers $n \geq 2$. Therefore $\lim_{n\to\infty} F_S(x_n) = 0$. This shows that every $L_{c,S}$ is nonempty and that $\inf_M F_S = 0$. Consider $\{c_n, S\}$ a decreasing sequence of positive numbers converging to zero, and set $A := \cap_n \overline{L_{c_n,S}}$, (where $\overline{L_{c_n,S}}$ designates the closure of $L_{c_n,S}$). By applying Cantor's intersection theorem we ensure the existence of a unique element $z \in A$. For every nonzero integer n, since $z \in \overline{L_{c_n,S}}$, we can find $y_n \in L_{c_n,S}$ such that $d(y_n, z) \leq \frac{1}{n}$. Therefore $\{y_n\}$ converges to z. For each integer n, we have $0 \leq F(y_n) \leq c_n$. Hence $\lim_n F_S(y_n) = 0$. Since F_S is supposed to be regular, then $F_S(z) = \inf_M F_S = 0$. Thus z is a fixed point of T. Since $S \in A(T,\alpha)$, z must be the unique common fixed point of S and T.

Now, let $\{x_n\}$ be a sequence in M such that $\lim_n F_S(x_n) = 0$. Then by using the inequality (3.1), we deduce that $\lim_n x_n = z$. Conversely, according to (2.9), for every $x \in M$, we have

$$d(x,Sx) \leq d(x,z) + d(z,Sx) \leq \frac{1}{1-\alpha} d(x,z).$$

Thus, if $\lim_{n\to\infty} x_n = z$ then $\lim_{n\to\infty} F_S(x_n) = 0$. Thus, (2) implies (3).

3.3 Suppose that (3) is satisfied. Let $w = Sx$ be an element of the range $Im(S)$. Then according to the triangular inequality and (2.11), we have

$$F_T(w) = d(Sx,TSx) \leq d(Sx,z) + d(z,TSx) \leq \frac{1}{1-\alpha} d(Sx,z) = \frac{1}{1-\alpha} d(w,z). \tag{3.4}$$

From (3.4) we obtain the first implication in (4). To prove the converse, let again $w = Sx$ be an element of $Im(S)$. According to (2.11), we have

$$\begin{aligned} d(w, z) &= d(Sx, z) \leq d(Sx, TSx) + d(TSx, z) \\ &\leq d(Sx, TSx) + \frac{\alpha}{1-\alpha} d(Sx, z) = F_T(w) + \frac{\alpha}{1-\alpha} d(w, z). \end{aligned} \tag{3.5}$$

From (3.5), we obtain

$$d(w, z) \leq \frac{1-\alpha}{1-2\alpha} F_T(w).$$

Thus, for every sequence $\{w_n\}$ of points in $Im(S)$, if $\lim_{n\to\infty} F_T(w_n) = 0$, then we must have $\lim_{n\to\infty} w_n = z$. Thus (3) implies (4).

3.4 We observe that if (4) is satisfied then the point z involved in (4) must be fixed by T. It remains to show that z is fixed by S. Let $w \in M$ such that $z = Sw$. According to Property (A), we have

$$\begin{aligned} d(Sz, z) &= d(Sz, TSw) \\ &\leq \alpha \max\left\{d(z, z), d(z, Sz) + d(z, z), \frac{1}{2}\left[d(z, z) + d(Sz, z)\right]\right\} \\ &= \alpha\, d(Sz, z) \end{aligned} \tag{3.6}$$

(3.6) shows that necessarily $Sz = z$. Thus, (4) implies (1), and this completes the proof of Theorem 1.2. □

References

[1] Akkouchi, M. *On a result of W. A. Kirk and L. M. Saliga*, J. Comput. Appl. Math. (To appear).

[2] Akkouchi, M. *Common fixed point theorems by altering the distances between the points in bounded complete metric spaces*, Demonstratio. Math. **4** (2000) 843–850.

[3] Angrisani, M., Clavelli M. *Synthetic approaches to problems of fixed points in metric spaces*, Ann. Mat. Pura Appl., vol. (IV) CLXX (1996), 1–12.

[4] Fisher, B. *Results on common fixed points*, Math. Japonica **22** (1977) 335–338.

[5] Fisher, B. *Theorems on fixed points*, Riv. Mat. Univ. Parma, **4** (1978) 109–114.

[6] Kirk, W. A., Saliga, L. M. *Some results on existence and approximation in metric fixed point theory*, J. Comput. Appl. Math. **113** (2000) 141–152.

[7] Nova, L. *Puntos fijos comunes*, Boletín de Matemáticas, vol. IV (1997) 43–47.

[8] Rashwan, R. A., Sadeek, A. M. *A common fixed point theorem in complete metric spaces*, Electronic Journal : Southwest. Jour. of Pure and Appl. Math., **1** (1996) 6–10.

[9] Walter, W. *Remarks on a paper by F. Browder about contractions*, Nonlinear Anal., vol. TMA 5 (1981) 21–25.