On the Construction of the Kan Loop Group

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Abstract. A re-make, not of the construction, but of its description.

By an *ordered graph* will be meant a triple of sets (P, N, E) together with a pair of structure maps, $N \leftarrow E \rightarrow P$. These data are to be thought of as 'positive vertices', 'negative vertices', 'edges', and 'incidence relations', respectively.

An ordered graph is a sort of ordered simplicial complex. It can be made into a simplicial set by adding degenerate simplices. The details of this step can be neatly described by means of an auxiliary category C_Γ associated to the ordered graph Γ. The set of objects of C_Γ is the disjoint union $N \dot\cup P$; the set of non-identity morphisms is the set E, and the source and target functions on E are given by the two maps $E \rightarrow N$ and $E \rightarrow P$, respectively. The category is a little unusual insofar as no two morphisms in it can be composable unless at least one of them is an identity morphism.

The *nerve* construction produces a simplicial set $N(C_\Gamma)$ now: an m-simplex is a functor $[m] \rightarrow C_\Gamma$ (where $[m]$ denotes the ordered set $(0 < 1 < \cdots < m)$, regarded as a category). The set of m-simplices is thus a disjoint union $N \dot\cup E \dot\cup \ldots \dot\cup E \dot\cup P$, with one entry "$E$" for each surjective monotone map $[m] \rightarrow [1]$. The simplicial set $N(C_\Gamma)$ is 1-*dimensional* in the sense that every non-degenerate simplex has dimension ≤ 1. Instead of $N(C_\Gamma)$ we will henceforth write $N(\Gamma)$ for this simplicial set.

The geometric realization $|N(\Gamma)|$ is a CW complex of dimension ≤ 1. The 0-cells of $|N(\Gamma)|$ are indexed by the set $N \dot\cup P$ (disjoint union), and the 1-cells are indexed by the set E.

We will suppose now that the ordered graph Γ is *connected* (equivalently, that the CW-complex $|N(\Gamma)|$ is) and *pointed* (i.e., equipped with the choice of an element $x \in P$). We may then speak of the *fundamental group* $\pi_1(\Gamma, x)$. It can be described as the fundamental group of the CW-complex $|N(\Gamma)|$ based at $|x|$ or else, in somewhat more combinatorial terms, as the *edge path group* of Γ based at x.

We may also speak of the *universal covering* of Γ (with respect to the chosen basepoint x). This is an ordered graph $\widetilde\Gamma$. It comes equipped with an action of $\pi_1(\Gamma, x)$, and with a map $\widetilde\Gamma \rightarrow \Gamma$; and these two pieces of data are such that they make $\widetilde\Gamma$ into a principal $\pi_1(\Gamma, x)$-bundle over Γ (by definition, this means that the action is free, and that the quotient by the action is identified to Γ by the given map). The construction of all this is as follows, by covering space theory. An element of $\widetilde N$ (a 'negative vertex' of $\widetilde\Gamma$) consists of a pair of data in Γ, namely (i) a 'negative vertex' v of Γ and (ii) a homotopy class of paths connecting v to the chosen basepoint x. The map $\widetilde N \rightarrow N$ is defined as the forgetful map which forgets the path; and the action of $\pi_1(\Gamma, x)$ on $\widetilde N$ is given by composing the path with the loop in question. The other data are given similarly.

As we have implicitly used before, the ordered graphs are the objects of a category in an evident way: a map in this category is a triple of maps of sets, $P \to P'$, $N \to N'$, $E \to E'$, so that these maps are compatible to the structure maps of the two ordered graphs in question. It makes sense, consequently, to speak of a *simplicial ordered graph*, a simplicial object in the category of ordered graphs. We note that a simplicial ordered graph will give rise to a bisimplicial set, by *nerve*, and hence to a CW-complex, by geometric realization (this particular geometric realization uses 'prisms'; an equivalent construction, up to canonical isomorphism, would be to pass to the diagonal simplicial set first and then take the geometric realization of that diagonal simplicial set). We are in a position now to describe our basic construction. The construction is implicit in Kan's paper [1], but it was not made explicit there.

CONSTRUCTION. Let $X\colon \Delta^{\mathrm{op}} \to (\mathrm{sets})$, $[n] \mapsto X_n$, be a simplicial set. There is an associated simplicial ordered graph. It has $P_n = X_n$, $N_n = X_0$, $E_n = X_{n+1}$, and the maps $E_n \to P_n$ and $E_n \to N_n$ are given by the 'last face' map and 'last vertex' map, respectively. This simplicial ordered graph will be denoted ΓX.

(Here are some more details. The simplicial set $P.$ is defined to be isomorphic to X itself, while $N.$ is defined as the set X_0 considered as a simplicial set in a trivial way. Concerning $E.$, if $\alpha\colon [n] \to [n']$ is a monotone map then $\alpha^* : E_{n'} \to E_n$ is defined to be the map $X_{n'+1} \to X_{n+1}$ induced from $\alpha \cup \{\infty\} : [n] \cup \{\infty\} \to [n'] \cup \{\infty\}$. The map $E_n \to P_n$ is defined to be the map $X_{n+1} \to X_n$ induced from the injective map $[n] \to [n{+}1]$ which misses $n{+}1$, and the map $E_n \to N_n$ is defined to be the map $X_{n+1} \to X_0$ induced from the map $[0] \to [n{+}1]$ taking 0 to $n{+}1$.)

Considering X as a simplicial ordered graph in a trivial way (no edges, no negative vertices) we have a natural inclusion $X \to \Gamma X$. The following will be shown later.

LEMMA. *The map $X \to \Gamma X$ is a weak homotopy equivalence.*

We will suppose now that the simplicial set X is connected and that it is equipped with a basepoint (that is, the choice of an element in X_0). Then $\Gamma_n X$, the ordered graph in degree n of the simplicial ordered graph ΓX, will also be connected (a proof of this fact will be given below) and it will be equipped with a basepoint x_n (namely the degenerate in degree n of the chosen element in X_0). ΓX can thus be considered as a simplicial object of *pointed* ordered graphs, and we can therefore define a simplicial group $G = G(X)$,

$$[n] \ \mapsto \ G_n := \pi_1(\Gamma_n X, x_n) \ .$$

THEOREM. *The simplicial group G is a loop group for X.*

PROOF. In view of the lemma it will suffice to show that there is a principal G-bundle over ΓX with weakly contractible total space ('weakly contractible' means that the map to the one-point-space is a weak homotopy equivalence). For by pulling back such a bundle along the map $X \to \Gamma X$ we can obtain a principal G-bundle over X, and the total space of the latter bundle will again be weakly contractible. (This is so since, for example, a map of principal bundles of simplicial sets is also a map of Kan fibrations [2, Satz 9.5] and the geometric realization of a Kan fibration is a Serre fibration [3]. So the Whitehead theorem applies.)

The universal covering of a pointed ordered graph, as described above, is *functorial*. Hence we have a simplicial ordered graph $[n] \mapsto \widetilde{\Gamma}_n X$, it is obtained from $[n] \mapsto \Gamma_n X$ (that is, from ΓX) by taking the universal covering degreewise. This simplicial ordered graph is weakly contractible in every degree; hence (by [4, Appendix A] for example) it is also weakly contractible globally.

The desired principal bundle is now obtained by observing that the simplicial group G acts on $\widetilde{\Gamma} X$, that the action is free, and that the quotient of $\widetilde{\Gamma} X$ by the action is just ΓX again. □

Proof of Lemma. We give two proofs, both fairly self-contained. The short proof is in an appendix; here is the pedestrian one.

The category of ordered graphs, as well as the category of simplicial sets, is a *functor category*, namely the category of $\nwarrow\!\nearrow$-shaped, respectively of $\Delta^{\rm op}$-shaped, diagrams in the category of sets; and colimits in such a functor category are computed 'pointwise'. It results that the functor $X \mapsto \Gamma X$ commutes with colimits and, what is more to the point here, that the functor $X \mapsto N(\Gamma X)$ (and therefore also $X \mapsto |N(\Gamma X)|$) does so, too. We can apply this fact in two ways. First, by direct limit, we can reduce to proving the lemma for those simplicial sets which are *finite*; that is, there are only finitely many non-degenerate simplices. Next, a finite simplicial set can be obtained from a 'smaller' one by the attaching of a simplicial set *standard k-simplex*, for some k; by induction and the gluing lemma we can therefore reduce to proving the lemma for just the latter kind of simplicial set. In other words, we are reduced now to showing that $|N(\Gamma\Delta^k)|$ is contractible.

To show this, we will work out the cell structure of the CW-complex $|N(\Gamma\Delta^k)|$ explicitly. The cells in this complex are of three kinds. First, there are the cells coming from the *positive vertices*; these contribute the copy of $|\Delta^k|$ coming from the inclusion $\Delta^k \to N(\Gamma\Delta^k)$. Next, there are the cells coming from the *negative vertices*; these cells are all 0-dimensional, and there is one such for every vertex of Δ^k.

And, finally, there are the cells coming from the *non-degenerate edges*; of these there is a 'basic' edge for every negative vertex. Namely suppose that the negative vertex corresponds to the l-th vertex of Δ^k. Let $\mathrm{front}_l(\Delta^k)$ denote the copy of Δ^l inside Δ^k whose vertices are the vertex l and its predecessors. Then the *last degenerate* of the generating simplex of $\mathrm{front}_l(\Delta^k)$ gives an l-dimensional edge of the simplicial ordered graph, and this edge is non-degenerate. Conversely, every non-degenerate edge is either of this kind or is a face of one such. Indeed, suppose the edge corresponds to a simplex y of Δ^k and suppose that l is the highest vertex of Δ^k occurring in y. If any vertex $< l$ occurs twice in y, or if the vertex l occurs more than twice, then the edge associated to y is degenerate—contrary to assumption. If, on the other hand, some vertex $< l$ does not occur at all, or if the vertex l occurs only once rather than twice, then the edge associated to y is a proper face of one of higher dimension.

Returning to the 'basic' edge, we note that the associated cell has dimension $l{+}1$. Its closure is the image of a copy of $|\Delta^l| \times |\Delta^1|$ which is mapped in such a way that all of $|\Delta^l| \times 0$ is identified to a point (corresponding to the negative vertex in question), while $|\Delta^l| \times 1$ is identified to the geometric realization of $\mathrm{front}_l(\Delta^k)$. By induction, there are no identifications over faces of Δ^k which are not of this kind. It results that $|N(\Gamma\Delta^k)|$ is the union of the cones on $|\Delta^0|, |\Delta^1|, \ldots, |\Delta^k|$, each glued along its base to the appropriate subsimplex in $|\Delta^k|$. This complex is indeed contractible. □

APPENDIX (*on generators and relations*).

If the groups $G_n = \pi_1(\Gamma_n X, x_n)$ are expressed as *edge path groups*, one obtains a sort of description of the simplicial group G in terms of the structure of X. This description occurs as a definition of the loop group in [1, section 12]. Another definition of the loop group is given in [1, sections 7 and 9] in terms of *generators and relations*. The equivalence of the two definitions can be explained by combinatorial group theory. Namely, in a connected graph one can choose a *maximal tree*. The fundamental group of the graph can then be identified to the free group freely generated by the edges of the graph *not* in that maximal tree; equivalently, the fundamental group can be identified to the group generated by *all* the edges of the graph, where, however, the edges of the chosen maximal tree are also introduced as relations.

To make this description effective, one needs to know what a maximal tree in the ordered graph $\Gamma_n X$ will look like. The answer is as follows. If the simplicial set X is *reduced* (that is, if X_0, the set of 0-simplices, has only one element) then there is a maximal tree in $\Gamma_n X$ which is such that it contains exactly those edges where the corresponding simplex of X is a *last degenerate*. In the general case of a connected, but not necessarily reduced X, one has to choose a *maximal tree* in X first (a sub-simplicial-set which contains all of X_0 and whose geometric realization is a simply-connected CW-complex of dimension ≤ 1); the pieces in $\Gamma_n X$ coming from this sub-simplicial-set are then, additionally, in the maximal tree in $\Gamma_n X$.

We will justify this description of the maximal tree now (for much of the following, cf. [1, Lemma 9.1] and [1, section 14] in particular). We begin by explaining why, for connected X and for every n, the graph $\Gamma_n X$ is connected. First, every positive vertex of $\Gamma_n X$ can be connected to some negative vertex. Indeed, if the positive vertex corresponds to $x \in X_n$ then the last degenerate of x gives an edge in $\Gamma_n X$ which will connect this positive vertex to a negative vertex (namely the one associated with the 'last vertex' of that last degenerate or, what amounts to the same thing, the 'last vertex' of x itself). Next, all the negative vertices of $\Gamma_n X$ come from $\Gamma_0 X$, by degeneracy, hence it will suffice to show that they can be connected to each other inside $\Gamma_0 X$. It will, in fact, suffice to show this in the special case of two negative vertices where the associated 0-simplices of X are *adjacent* (in making this reduction we are using the assumed fact that X is connected). We are thus in the special case now where the two 0-simplices of X are the faces of some $y \in X_1$. We see that in this case the two negative vertices can be connected to each other by an edge path of length 2 in $\Gamma_0 X$; the two edges in the path are provided by the simplex y on the one hand and by the 1-dimensional degenerate of the last face of y on the other.

Next, suppose that the simplicial set X is a *tree*. We want to show that, in this case, the ordered graph $\Gamma_n X$ is a tree, too, for every n. Now the nerve $N(\Gamma_n X)$ is 1-dimensional, and connected; so it will be a tree if (and only if) it is *acyclic*. To prove the latter, since the functor $X \mapsto N(\Gamma_n X)$ commutes with colimits, we can further reduce, by direct limit and (inductively) the gluing lemma, to dealing with just the two cases where $X = \Delta^0$ or $X = \Delta^1$. We will write P_n, E_n, N_n, respectively, for the sets of positive vertices, edges, and negative vertices of $\Gamma_n X$. In the case $X = \Delta^0$, each of these sets has exactly one element, so $N(\Gamma_n \Delta^0)$ is isomorphic to Δ^1. In the case $X = \Delta^1$, the set P_n has $n{+}2$ elements which we denote $p_0, p_1, \ldots, p_{n+1}$ (where p_{n+1} stands for the map $[n] \to [1]$ with image consisting of only $0 \in [1]$ and where, otherwise, p_i stands for the monotone map $[n] \to [1]$ having the property that

$i \in [n]$ is the smallest element whose image is $1 \in [1]$); the set E_n has $n{+}3$ elements, $e_0, e_1, \ldots, e_{n+2}$, and the set N_n has two elements, n_0 and n_1. The map $E_n \to P_n$ takes e_i to p_i for all $i \le n{+}1$, and, in addition, it takes e_{n+2} to p_{n+1}. The map $E_n \to N_n$ takes the element e_{n+2} into n_0 and it takes all other elements of E_n into n_1. We see that $N(\Gamma_n \Delta^1)$ is a one-point-union of $n{+}1$ copies of Δ^1, together with one extra copy of Δ^1 hanging on to one of the whiskers. It is a tree indeed.

Let X be a connected simplicial set now. Choose a maximal tree T in X. Let P', E', N' denote, respectively, the sets of positive vertices, edges, and negative vertices of $\Gamma_n T$. Let P'' denote the subset of X_n which is complementary to the subset T_n. Let E'' be defined as the subset of X_{n+1} given by the image of P'' under the 'last degeneracy' map. One of the structure maps of $\Gamma_n X$ restricts to a map $E'' \to N'$ (all the negative vertices of $\Gamma_n X$ are contained in N' since T contains all the 0-simplices of X), and the other structure map restricts to a map $E'' \to P''$. The latter map is given by the 'last face' map, and is actually inverse to the above map $P'' \to E''$; in particular it is an isomorphism. In view of this fact, and using the fact established above, that the ordered graph

$$N' \,,\; E' \,,\; P' \,, \quad N' \leftarrow E' \to P'$$

is indeed a tree, we can now conclude that the sets, and maps,

$$N' \,,\; E' \cup E'' \,,\; P' \cup P'' \;, \quad N' \leftarrow E' \cup E'' \to P' \cup P''$$

do form a tree, too. The isomorphisms $N' \approx X_0$ and $P' \cup P'' \approx X_n$ show that this tree contains all the vertices of $\Gamma_n X$. It is therefore a maximal tree. □

Appendix (*another view at the lemma*).

The geometric realization $|\Gamma X|$ may be identified to the double mapping cylinder of the following diagram (the terms involved have been defined in connection with the definition of ΓX),

$$|P_{\bullet}| \longleftarrow |E_{\bullet}| \longrightarrow |N_{\bullet}| \,.$$

As a consequence, the assertion of the lemma, that the inclusion

$$|X| \approx |P_{\bullet}| \longrightarrow |\Gamma X|$$

is a homotopy equivalence, will therefore result once one knows that the map

$$E_{\bullet} \longrightarrow N_{\bullet}$$

is a (weak) homotopy equivalence. But this is a well known fact: $E_{\bullet}$ is obtained from the simplicial set X by *shifting*, it is a sort of path space on X, and it is homotopy equivalent to the *subspace of constant paths*; that is, the set X_0 regarded as a simplicial set in a trivial way. The latter statement is in fact true with the strongest possible interpretation of homotopy equivalence, namely *simplicial homotopy equivalence*. An account can be found in [4, proposition 1.5]; another in [5, lemma 1.5.1].

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