Selmer Groups and Torsion Zero Cycles on the Selfproduct of a Semistable Elliptic Curve

Andreas Langer

Received: February 25, 1997

Communicated by Ulf Rehmann

Abstract. In this paper we extend the finiteness result on the p-primary torsion subgroup in the Chow group of zero cycles on the selfproduct of a semistable elliptic curve obtained in joint work with S. Saito to primes p dividing the conductor. On the way we show the finiteness of the Selmer group associated to the symmetric square of the elliptic curve for those primes. The proof uses p-adic techniques, in particular the Fontaine-Jannsen conjecture proven by Kato and Tsuji.

1991 Mathematics Subject Classification: Primary 14H52; Secondary 19E15, 14F30.
Key words and phrases: torsion zero cycles, semistable elliptic curve, multiplicative reduction primes, Selmer group of the symmetric square, Hyodo-Kato cohomology.

Introduction.

In this note we extend the main finiteness result on p-primary torsion zero-cycles on the selfproduct of a semistable elliptic curve in [L-S] to primes $p \geq 3$ where E has (bad) multiplicative reduction, at least under a certain standard assumption. In the course of the proof we will also derive the finiteness of the Selmer group of the symmetric square $\mathrm{Sym}^2 H^1(E)(1)$ for these primes. However, this latter result has already been proven, under the additional condition that the Galois representation

$$\varrho_p : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{Aut}(E_p)$$

is absolutely irreducible (here $E_p = E_p(\overline{\mathbb{Q}})$ is the subgroup of p-torsion elements of E), in a much more general context by Wiles in his main paper ([W] Theorem 3.1) for Selmer groups associated to deformation theories.

To state the Theorems, let E be a semistable elliptic curve over $\mathbb{Q}$ with conductor N and let $X = E \underset{\mathbb{Q}}{\times} E$ be its self-product. Consider the Chow group $CH_0(X)$ of zero-cycles on X modulo rational equivalence and let $CH_0(X)\{p\}$ be — for a fixed prime p — its p-primary torsion subgroup. For a prime p dividing N consider the following hypothesis:

H 1) The Gersten-Conjecture holds for the Quillen-(Milnor)-sheaf $\mathcal{K}_2$ on a regular model $\mathcal{X}$ of X over $\mathbb{Z}_p$.

Then we have

THEOREM A: *Let E be a semistable elliptic curve and $p \geq 3$ a prime such that $p \mid N$, i.e., E has (bad) multiplicative reduction at p. Assume that the condition H 1) is satisfied. Then $CH_0(X)\{p\}$ is a finite group.*

Let $A = H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$ be the $\mathbb{Q}_p/\mathbb{Z}_p$-realization of the motive $H^2(X)(2)$ with its $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-action. Then we have

THEOREM B: *Let E be a semistable elliptic curve over $\mathbb{Q}$ and $p \geq 3$ a prime such that $p \mid N$. Then the Selmer group $S(\mathbb{Q}, A)$ is finite.*

REMARKS:

— In [L-S] we showed the finiteness of $CH_0(X)\{p\}$ for primes p such that $p \not| 6$ and E has good reduction at p. We also proved that $CH_0(X)\{p\}$ is zero for almost all p. Therefore Theorem A extends this result to bad primes and provides a further step towards a proof that the full torsion subgroup $CH_0(X)_{tors}$ is finite. In order to find a first example where this is true it remains to consider the 2- and 3-primary torsion in $CH_0(X)$.

— The Selmer group $S(\mathbb{Q}, A)$ coincides with $S(\mathbb{Q}, \mathrm{Sym}^2 H^1(\overline{E}, \mathbb{Q}_p/\mathbb{Z}_p(1)))$ that was studied by [Fl], because $S(\mathbb{Q}, \mathbb{Q}_p/\mathbb{Z}_p(1))$ is zero. In [Fl] Flach proved the finiteness of $S(\mathbb{Q}, A)$ for primes $p \geq 5$ such that E has good reduction at p and the representation ϱ_p is surjective. We were able to remove the latter hypothesis by using a rank-argument of Bloch-Kato and reproved Flach's finiteness result for primes p such that $p \not| 6N$ (compare [L-S]). In the proof of Theorem B we combine the criterium of Bloch-Kato with Kolyvagin's argument that was used in Flach's paper. Flach's additional condition on the surjectivity of ϱ_p can be avoided by applying a certain lemma, due to J. Nekovář, that bounds the order of $H^1(\mathrm{Gal}(\mathbb{Q}(E_{p^n})/\mathbb{Q}), (\mathrm{Sym}^2 H^1(\overline{E}, \mathbb{Z}/p^n(1)))(-1))$ independently of n.

The paper is organized as follows:
In the first paragraph we reduce the proof of Theorem A to two Lemmas I and II. Lemma I was already proven in ([L-S], Lemma A). Lemma II is similar to ([L-S], Lemma B), but the statement is different. The difference is caused by the particular semistable situation. In the second paragraph we derive Lemma II and Theorem B from a key proposition that bounds the possible corank (at most 1 !) of the cokernel of the map defining the Selmer group. Finally this proposition is proven in the last paragraph. The methods of the proof are similar to those developed in [L-S]. At the

point where the crystalline conjecture was used in the good reduction case, we now use the Fontaine-Jannsen conjecture (proven by Kato/Tsuji for $p \geq 3$) that relates the log-crystalline cohomology to the p-adic étale cohomology. The role of the syntomic cohomology in the context of Schneider's p-adic points conjecture is now replaced by a semistable analog relating log-syntomic cohomology to $H^1_g(\mathbb{Q}_p, H^2(\overline{X}, \mathbb{Q}_p(2)))$ (compare [L]). When we apply this argument we will also need the computation, due to Hyodo and used by Tsuji, on a filtration on the sheaf of p-adic vanishing cycles in terms of modified logarithmic Hodge-Witt sheaves.

This paper was written during a visit at the University of Cambridge. I want to thank J. Coates and J. Nekovář for their invitation and J. Nekovář for many discussions and the permission to include his proof of Lemma (2.5) in this paper. Finally I thank S. Saito for encouraging me to look at the remaining semistable reduction case of our main finiteness result in [L-S] and I consider this work as having been done very much in the spirit of our joint paper and a continuation of it.

§1

We first fix some notations.

For an Abelian group M let M_{div} be the maximal divisible subgroup of M and $M\{p\}$ its p-primary torsion subgroup. For a scheme Z over a field k let $\overline{Z} = Z \underset{k}{\times} \overline{k}$ where k is an algebraic closure of k. Denote by $G_k = \mathrm{Gal}(\overline{k}/k)$ the absolute Galois group of k. We will consider the Zariski sheaf $\mathcal{K}_2$ associated to the presheaf $U \to \mathcal{K}_2(U)$ of Quillen (-Milnor) K-groups on Z and let $H^j_{Zar}(Z, \mathcal{K}_2)$ be its Zariski cohomology. Let E be a semistable elliptic curve over $\mathbb{Q}$ with conductor N, $\phi : X_0(N) \to E$ a modular parametrization of E, $X = E \underset{\mathbb{Q}}{\times} E$. Let T, A, V be the following $G = G_{\mathbb{Q}}$-modules:

$$T = H^2(\overline{X}, \mathbb{Z}_p(2)) \quad , \quad A = H^2(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \quad , \quad V = H^2(\overline{X}, \mathbb{Q}_p(2)) \quad .$$

Note that as Abelian groups $T \cong \mathbb{Z}_p^6$, $A \cong \mathbb{Q}_p/\mathbb{Z}_p^6$, because the integral cohomology of an Abelian variety is torsion-free and the second Betti number of X b_2 is 6.

Let K be the function field of X. For a prime p let

$$NH^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) := \ker(H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \to H^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2)))$$

and

$$K_N H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) := \ker(NH^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \to H^3(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)))$$

By results of Bloch and Merkurjev-Suslin ([Bl], §5 and [M-S] we have the following exact sequence

$$(1-1) \qquad 0 \to H^1(X, \mathcal{K}_2) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to NH^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \to CH_0(X)\{p\} \to 0$$

Since $H^1(\overline{X}, \mathcal{K}_2) \otimes \mathbb{Q}_p/\mathbb{Z}_p = 0$ we get an exact sequence

$$(1-2) \qquad 0 \longrightarrow H^1(X, \mathcal{K}_2) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow K_N H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow \ker(CH_0(X)\{p\} \longrightarrow CH_0(\overline{X})\{p\}^G) \longrightarrow 0$$

Since X is identified with its Albanese variety, the map $CH_0(X)_{tors} \longrightarrow CH_0(\overline{X})^G_{tors}$ is the Albanese map and therefore $(CH_0(\overline{X})\{p\})^G \cong X(\mathbb{Q})\{p\}$ is finite. Consider the Hochschild-Serre spectral sequence

$$E_2^{a,b} = H^a(\mathbb{Q}, H^b(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \Longrightarrow H^{a+b}(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \quad .$$

Then we have

LEMMA I: *Let the assumptions be as above. Then the composite map*

$$E_2^{2,1} \longrightarrow H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$$

is injective.

This is shown in ([L-S], Lemma (A)) without any assumption on the prime p.

COROLLARY (1.3) *The composite map*

$$\varphi : K_N H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow H^1(G_{\mathbb{Q}}, A)$$

that is obtained by the Hochschild-Serre spectral sequence is injective.

The Corollary will play an important role in the proof of

LEMMA II: *Under the above assumptions let $p \geq 3$ be a prime such that $p \mid N$ and assume that the condition H 1) in the introduction is satisfied. Then we have*

$$H^1(X, \mathcal{K}_2) \otimes \mathbb{Q}_p/\mathbb{Z}_p = K_N H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\text{div}} \quad .$$

REMARK:

Lemma II was proven for primes $p \not| 6N$ in ([L-S, Lemma (B)) because in this case $K_N H^3(X, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\text{div}}$ coincides with $H^1(\mathbb{Q}, A)_{\text{div}}$. This is not stated there explicitly but follows from the proof of Lemma (B) in [L-S].

Now we deduce Theorem A from Lemma II.

The exact sequence (1-1) also holds for a smooth proper model $\mathcal{X}$ of X over $\mathbb{Z}\left[\frac{1}{Np}\right]$. So $CH_0(\mathcal{X})\{p\}$ is a subquotient of $H^3(\mathcal{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))$ and one knows that the latter group is co-finitely generated. Therefore $CH_0(\mathcal{X})\{p\}$ is co-finitely generated as $\mathbb{Z}_p$-module. Since the kernel of the canonical map

$$CH_0(\mathcal{X})\{p\} \longrightarrow CH_0(X)\{p\}$$

is a torsion group by the main result in [Mi], the localization sequence in the Zariski K-cohomology over $\mathcal{X}$ yields a surjection

$$CH_0(\mathcal{X})\{p\} \twoheadrightarrow CH_0(X)\{p\} \quad .$$

So we also know that $CH_0(X)\{p\}$ is co-finitely generated.

On the other hand, by (1-2), the finiteness of $CH_0(\overline{X})\{p\}^G$ and Lemma II we conclude that the maximal divisible subgroup of $CH_0(X)\{p\}$ is zero. Therefore $CH_0(X)\{p\}$ is a finite group.

To complete the proof of Theorem A it remains to show Lemma II.

§2

For each prime ℓ let

$$H^1_e(\mathbb{Q}_\ell, V) \subset H^1_f(\mathbb{Q}_\ell, V) \subset H^1_g(\mathbb{Q}_\ell, V) \subset H^1(\mathbb{Q}_\ell, V)$$

be defined as in ([BK], 3.7)). Let

$$H^1_f(\mathbb{Q}_\ell, T) \subset H^1_g(\mathbb{Q}_\ell, T) \subset H^1(\mathbb{Q}_\ell, T)$$

be the inverse image of $H^1_f(\mathbb{Q}_\ell, V)$ and $H^1_g(\mathbb{Q}_\ell, V)$. Put

$$H^1_f(\mathbb{Q}_\ell, A) := H^1_f(\mathbb{Q}_\ell, T) \otimes \mathbb{Q}_p/\mathbb{Z}_p \subset H^1(\mathbb{Q}_\ell, A)$$

and

$$H^1_g(\mathbb{Q}_\ell, A) := H^1_g(\mathbb{Q}_\ell, T) \otimes \mathbb{Q}_p/\mathbb{Z}_p \subset H^1(\mathbb{Q}_\ell, A)$$

Write $\wedge_\ell = H^1(\mathbb{Q}_\ell, T)/H^1_f(\mathbb{Q}_\ell, T)$. Then we have

$$\wedge_\ell \otimes \mathbb{Q}_p/\mathbb{Z}_p = H^1(\mathbb{Q}_\ell, A)_{\mathrm{div}}/H^1_f(\mathbb{Q}_\ell, A)$$

Consider as in ([L-S], §3) the composite map

$$\psi : H^1(X, \mathcal{K}_2) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow K_N H^3(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(2))_{\mathrm{div}} \xrightarrow{\alpha'} \bigoplus_{\ell} \wedge_\ell \otimes \mathbb{Q}_p/\mathbb{Z}_p$$

where α' is the restriction of the map

$$\alpha : H^1(\mathbb{Q}, A) \longrightarrow \bigoplus_{all\ \ell} \frac{H^1(\mathbb{Q}_\ell, A)}{H^1_f(\mathbb{Q}_\ell, A)}$$

the kernel of which defines the Selmer group $S(\mathbb{Q}, A)$.

In analogy to ([L-S], Lemma 3.1) we will prove the following

PROPOSITION (2.1): *Let the notations be as in §1. Let $p \geq 3$ a prime, such that E has multiplicative reduction at p. Assume that condition (H 1) holds. Then we have*

a) $$\operatorname{coker}\psi = H^1(\mathbb{Q}_p, A)_{\mathrm{div}}/H^1_g(\mathbb{Q}_p, A)$$

b) $$\operatorname{Im}\psi = \operatorname{Im}\alpha'$$

We will give the proof of Proposition 2.1 in the next section.

In the following we will compute the coranks of $H^1(\mathbb{Q}_p, A)_{\mathrm{div}}/H^1_g(\mathbb{Q}_p, A)$ and $H^1_g(\mathbb{Q}_p, A)/H^1_f(\mathbb{Q}_p, A)$. Let

$$\Omega_p = H^1_g(\mathbb{Q}_p, V)/H^1_f(\mathbb{Q}_p, V) \quad \text{and} \quad \theta_p = H^1(\mathbb{Q}_p, V)/H^1_g(\mathbb{Q}_p, V)$$

as in ([L-S], §4). It is well known that $X_{\mathbb{Q}_p} = E \times E_{\mathbb{Q}_p}$ has a regular proper model $\mathcal{X}$ over $\mathbb{Z}_p$ with semistable reduction. Let X_p be its closed fiber. By local Tate-Duality ([B-K], §3.8), Ω_p is the $\mathbb{Q}_p$-dual of $H^1_f(\mathbb{Q}_p, V(-1))/H^1_e(\mathbb{Q}_p, V(-1))$ and this quotient is — by the computations in [B-K], 3.8 — isomorphic to $(B_{\mathrm{crys}} \otimes V(-1))^{G_{\mathbb{Q}_p}}/1 - f$, which is by Kato's and Tsuji's proof of the Fontaine-Jannsen-Conjecture ([Ka], §6), ([Tsu]) isomorphic to $(D_2)^{N=0}/1 - f)$, where

$$D_2 = H^2_{\log\ \mathrm{crys}}((X_p, M_1)/W(\mathbb{F}_p), W(L), O^{\mathrm{crys}}) \otimes \mathbb{Q}_p$$

denotes the log-crystalline cohomology introduced by Hyodo-Kato [H-K], $N = 0$ denotes the kernel under the action of the monodromy operator N, and f acts as $p^{-1}\varphi$, where φ is the Frobenius acting on D_2. Therefore we have by Poincaré duality for Hyodo-Kato cohomology that Ω_p is isomorphic to $(\operatorname{coker} N : D_2 \to D_2)^{\varphi=p}$. Since the functor $D_{st}(\cdot) = (B_{st} \otimes \cdot)^{G_{\mathbb{Q}_p}}$ commutes with tensor products and a Tate-elliptic curve has ordinary semistable reduction in the sense of ([Il], Definition 1.4) we have a Hodge-Witt-decomposition ([Il], Proposition 1.5)

$$D_2 = \bigoplus_{i+j=2} H^i(X_p, Ww^j) \otimes \mathbb{Q}_p \quad .$$

Here $H^i(X_p, Ww^j)$ is the cohomology of the modified Hodge-Witt-sheaves. From the action of the Frobenius φ on D_2 it is clear that $(D_2)_{\varphi=p}$ is contained in $H^1(X_p, Ww^1)_{\mathbb{Q}_p}$. By ([Mo], §6) we know that the monodromy filtration and the weight filtration on D_2 coincide. Using the formula $N\varphi = p\varphi N$ we have that

$$N(H^0(X_p, Ww^2)) \subset H^1(X_p, Ww^1)$$

and the map

$$N^2 : H^0(X_p, Ww^2) \longrightarrow H^2(X_p, Ww^0)$$

is an isomorphism. Since $\dim H^i(X_p, Ww^j)_{\mathbb{Q}_p} = \dim H^i(X_{\mathbb{Q}_p}\Omega^j)$ by ([Il], Corollaire 2.6), we see that

$$\dim(\operatorname{coker} N : D_2 \to D_2)^{\varphi=p} = \dim(D_2)^{N=0}_{\varphi=p} \leq 3 \quad .$$

On the other hand the B_{St}-comparison-isomorphism provides an injection

$$\operatorname{Pic}(X) \otimes \mathbb{Q}_p \hookrightarrow H^2(\overline{X}, \mathbb{Q}_p(1))^{G_{\mathbb{Q}_p}} \hookrightarrow (D_2)^{N=0}_{\varphi=p} \quad .$$

Since $\operatorname{Pic}(X)$ has rank 3 we have

LEMMA (2.2):

$$\dim \Omega_p = \dim(D_2)^{N=0}_{\varphi=p} = 3 \quad .$$

By the same methods and the proof of ([L-S], Lemma 4.4) we get

LEMMA (2.3):

$$\dim \theta_p = 1 \quad .$$

From Lemma (2.2) and ([L-S], Lemma 4.1) we get

LEMMA (2.4): *The image of the composite map*

$$(\mathrm{Pic}(X) \otimes \mathbb{Q}^*) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow H^1(X, \mathcal{K}_2) \otimes \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\psi_p} \wedge_p \otimes \mathbb{Q}_p/\mathbb{Z}_p$$

is

$$H^1_g(\mathbb{Q}_p, A)/H^1_f(\mathbb{Q}_p, A) \quad .$$

Now we will give the proof of Theorem B and we distinguish between two cases.

CASE I:

The map α'_p, i.e. the p-component of α' is surjective.

This case is actually obstructed by the Gersten-conjecture as we will see in the proof of Proposition (2.1). Since we do not assume (H 1) in Theorem B we also consider this case. Using the surjectivity-property of ψ_ℓ, i.e. the ℓ-component of ψ, for $\ell \neq p$ that follows from Prop. 2.1, and where the condition (H 1) is not needed, we see that $\operatorname{coker} \alpha$ has $\mathbb{Z}_p$-corank 0. Now apply the modified version of ([B-K], Lemma 5.16) that is given in ([L-S], Lemma (3.3)): All the assumptions there are also satisfied for our choice of p:

— V is a de Rham representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ by Falting's proof of the de Rham conjecture.

— For the characteristic polynomial $P_\ell(V,t)$ we have $P_\ell(V,1) \neq 0$. For $\ell \neq p$ the proof is the same as in ([L-S], §3). For $\ell = p$, we have $\mathrm{Crys}(V)^{f=1} = (D_2)^{N=0}_{\varphi=p^2}$. By the same methods as in the proof of Lemma (2.2) we have $(D_2)^{N=0}_{\varphi=p^2} = 0$. By the same arguments as in the proof of ([L-S], Theorem 3.2) we get the formula $\operatorname{corank}(\ker \alpha) = \operatorname{corank}(\operatorname{coker} \alpha) = 0$. Therefore $S(\mathbb{Q}, A) = \ker \alpha$ is finite.

CASE II:

$\operatorname{Im} \alpha'_p = H^1_g(\mathbb{Q}_p, A)/H^1_f(\mathbb{Q}_p, A)$

By Lemmas (2.3) and (2.4) this is the only remaining case to consider.

Let $T' = \mathrm{Sym}^2 H^1(\overline{E}, \mathbb{Z}_p(1))$. By Lemma (2.2) and Lemma (2.4) we have $H^1_g(\mathbb{Q}_p, T')/H^1_f(\mathbb{Q}_p, T') = 0$. Let $c(\ell)$ for $\ell \not| N$ be the elements in $H^1(X, \mathcal{K}_2)$ that were constructed by Mildenhall and Flach. In the notation of ([Fl], Prop. (1.1)) we therefore have $\mathrm{res}_{r=p} c(\ell) \in H^1_f(\mathbb{Q}_p, T')$. We get this property with little effort whereas in [Fl] this was one of the harder parts in the whole paper. It is now easy to check that all the other required properties on the elements $c(\ell)$ in ([Fl], Prop. (1.1)) are also satisfied for our choice of p. Thus we apply Kolyvagin's argument in ([Fl], Prop. (1.1)). At the point where Flach needs the surjectivity of the Galois representation ϱ_p in order to derive the finiteness of $S(\mathbb{Q}, A(-1))$, we use the following Lemma, due to Nekovář, that finishes, after applying Poitou-Tate Duality, the proof of Theorem B.

LEMMA (2.5): *Let $\mathbb{Q}(E_{p^n})/\mathbb{Q}$ be the Galois extension obtained by adjoining the coordinates of all p^n-torsion points on E and let T' be as above. Then there exists a $c > 0$, such that the exponent of $H^1(\mathrm{Gal}(\mathbb{Q}(E_{p^n})/\mathbb{Q}), T'(-1)/p^n)$ divides p^c for all $n \geq 0$.*

REMARK: Flach uses the vanishing of this cohomology group that follows from his additional assumption on the surjectivity of ϱ_p.

PROOF: Put $G := \mathrm{Im}(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}_{\mathbb{Z}_p}(T_p(E)))$. Since E is without complex multiplication over $\overline{\mathbb{Q}}$, G is of finite index in $\mathrm{Aut}_{\mathbb{Z}_p}(T_p(E)) = GL_2(\mathbb{Z}_p)$. Put $G_n := \ker(G \to GL_2(\mathbb{Z}/p^n), T' := \mathrm{Sym}^2(T_p(E)), \tilde{G} := \mathrm{Im}(G \to \mathrm{Aut}_{\mathbb{Z}_p}(T')) = G/Z \cap G$, where $Z = $ center of $GL_2(\mathbb{Z}_p) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \lambda \in \mathbb{Z}_p^* \right\}$.
Consider the following diagram with horizontal and vertical exact sequences:
(note that $G/G_n \cong \mathrm{Gal}(\mathbb{Q}(E_{p^n})/\mathbb{Q})$.

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
 & & H^1(G, T'(-1)) \otimes \mathbb{Z}/p^n & & \\
 & & \downarrow & & \\
0 \to H^1(G/G_n, T'(-1)/p^n) & \xrightarrow{\mathrm{inf}} & H^1(G, T'(-1)/p^n) & \xrightarrow{\mathrm{res}} & H^1(G_n, T'(-1)/p^n)^{G/G_n} \\
 & & \downarrow & & \\
 & & H^2(G, T'(-1))_{p^n} & &
\end{array}$$

It is clear that $H^i(G, T'(-1)) = H^i_{\mathrm{cont}}(G, T'(-1)) = H^i_{\mathrm{naive}}(G, T'(-1))$ are $\mathbb{Z}_p$-modules of finite type. Therefore $H^2(G, T'(-1))_{p^\infty}$ is finite. We have an exact sequence

$$0 \to H^1(\tilde{G}, T'(-1)) \xrightarrow{\mathrm{inf}} H^1(G, T'(-1)) \xrightarrow{\mathrm{res}} H^1(Z \cap G, T'(-1))^{G/Z\cap G}$$
$$= \mathrm{Hom}_{\mathrm{cont}}(Z \cap G, (T'(-1))^{G/Z\cap G})$$

But $(T'(-1))^{G/Z\cap G}$ is zero (E has no CM). Thus $H^1(\tilde{G},T'(-1)) = H^1(G,T'(-1))$. By result of Lazard there is an injection

$$\begin{aligned} H^1(\tilde{G},T'(-1))\otimes\mathbb{Q} \quad &\hookrightarrow \quad H^1(Lie(\tilde{G}),T'(-1)\otimes\mathbb{Q}) \\ &\quad\;\; = H^1(sl(2),T'(-1)\otimes\mathbb{Q}) \end{aligned}$$

and H^1 vanishes for semisimple Lie-algebras (and every representation). So $H^1(G,T'(-1))$ is finite and Lemma 2.5 follows.

Finally it is easy to see that Corollary (1.3), Proposition (2.1) b) and Theorem B imply Lemma II and as a consequence also Theorem A. It remains to show Proposition (2.1). This will be accomplished in the next paragraph.

§3

The surjectivity of the map

$$\psi' = \underset{\ell\neq p}{\oplus}\psi_\ell : H^1(X,\mathcal{K}_2)\otimes\mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \underset{\ell\neq p}{\oplus} H^1(\mathbb{Q}_\ell,A)_{\rm div}/H^1_f(\mathbb{Q}_\ell,A)$$

follows from ([L-S], Lemmas (4.1), (4.3), (4.4) and (4.5)). On the other hand the composite map

$$\mathrm{Pic}(X)\otimes p^{\mathbb{Z}})\otimes\mathbb{Q}_p/\mathbb{Z}_p \to H^1(X,\mathcal{K}_2)\otimes\mathbb{Q}_p/\mathbb{Z}_p \overset{\psi_p}{\to} H^1_g(\mathbb{Q}_p,A)/H^1_f(\mathbb{Q}_p,A)$$

is surjective by Lemma (2.2), whereas the image of $(\mathrm{Pic}(X)\otimes p^{\mathbb{Z}})\otimes\mathbb{Q}_p/\mathbb{Z}_p$ under the map ψ' is zero. To finish the proof of Proposition (2.1) we therefore have to show that the image of α'_p, the p-component of α' is contained in $H^1_g(\mathbb{Q}_p,A)/H^1_f(\mathbb{Q}_p,A)$.

By the theory of Bloch-Ogus and the work of Merkurjev-Suslin [M-S] we have an isomorphism

$$H^1(X,\mathcal{K}_2/p^n)\cong NH^3_{et}(X,\mathbb{Z}/p^n(2)) \quad .$$

Let $\mathcal{X}$ be a proper regular semistable model of $X_{\mathbb{Q}_p}$ over $\mathbb{Z}_p$, $i : X_p \to \mathcal{X}$ and $j : X_{\mathbb{Q}_p}\hookrightarrow\mathcal{X}$ the inclusions of the closed and generic fiber.

Let $H^3_{et}(\mathcal{X},\tau_{\leq 2}Rj_*\mathbb{Z}/p^n(2))$ be the cohomology of the truncated complex of p-adic vanishing cycles. Then we have

Lemma (3.1): *Assume that the Gersten-Conjecture holds for the Zariski sheaf $\mathcal{K}_2$ on the regular scheme $\mathcal{X}$. Then we have the inclusion*

$$H^1(X_{\mathbb{Q}_p},\mathcal{K}_2/p^n)\subset H^3_{et}(\mathcal{X},\tau_{\leq 2}Rj_*\mathbb{Z}/p^n(2)) \quad .$$

Proof:

This follows from the proof of ([L-S], Lemma (5.4)).

LEMMA (3.2): $H^3(\overline{X}_{\mathbb{Q}_p}, \mathbb{Q}_p(2))^{G_{\mathbb{Q}_p}} = 0.$

PROOF:

Using the Künneth formula and the fact that $H^2(\overline{E}, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p$ (the Brauer group of a curve over an algebraically closed field is zero), it suffices to show that $H^1(\overline{E}, \mathbb{Q}_p(1))^{G_{\mathbb{Q}_p}} = 0$. This follows from ([J], Theorem 5a).

Using Lemma (3.2) and the Hochschild-Serre spectral sequence we get a canonical map

$$\sigma : \varprojlim_n H^1(X_{\mathbb{Q}_p}, \mathcal{K}_2/p^n) \otimes \mathbb{Q}_p \longrightarrow H^1(\mathbb{Q}_p, V) \quad .$$

When we deal with a variety over a local field, all cohomology groups under consideration are (co-)finitely generated. The map α'_p certainly factors through $\varinjlim_n H^1(X_{\mathbb{Q}_p}, \mathcal{K}_2/p^n)_{\mathrm{div}}$. The assertion that $\varinjlim_n H^1(X_{\mathbb{Q}_p}, \mathcal{K}_2/p^n)_{\mathrm{div}}$ is contained in $H^1_g(\mathbb{Q}_p, A)$ is therefore equivalent to the assertion that the image of σ is contained in $H^1_g(\mathbb{Q}_p, V)$. In view of Lemma (3.1) we see that Proposition (2.1) follows from the following

LEMMA (3.3): *Under the condition H1) we have:* $\mathrm{Im}\,\sigma \subset H^1_g(\mathbb{Q}_p, V)$.

To prove Lemma (3.3) it suffices to show that the image of the map

$$H^3(\mathcal{X}, \tau_{\leq 2} Rj_* \mathbb{Q}_p(2)) \longrightarrow H^1(\mathbb{Q}_p, V)$$

is contained in $H^1_g(\mathbb{Q}_p, V)$.

Let $s_n^{\log}(2)$ be the log-syntomic complex in $D_{et}(\mathcal{X})$ constructed by Kato ([Ka], §6) and Tsuji [Tsu] together with a canonical map

$$s_n^{\log}(2) \longrightarrow \tau_{\leq 2} i_* i^* Rj_* \mathbb{Z}/p^n(2) \quad .$$

This gives rise to a composite map

$$\eta : H^3_{et}(\mathcal{X}, s^{\log}_{\mathbb{Q}_p}(2)) \longrightarrow H^1(\mathbb{Q}_p, V) \quad .$$

Since $(D_2)^{N=0}_{\varphi=p^2} = (D_3)^{N=0}_{\varphi=p^2} = 0$ (D_i denotes the i-th log-crystalline cohomology of X_p) we may apply the main result in [L] on a semistable analogue of Schneider's p-adic points conjecture to get

Lemma (3.4) $\operatorname{Im}\eta = H^1_g(\mathbb{Q}_p, V)$.

Tsuji has proven that there is a canonical isomorphism between the cohomology $\mathcal{H}^2(i^* s_n^{\log}(2))$ and the sheaf $M_n^2 = i^* R^2 j_* \mathbb{Z}/p^n(2)$ of p-adic vanishing cycles ([Tsu], Theorem 3.2). His proof relies on a filtration $\mathrm{Fil}^{\cdot}$ on M_n^2 that was defined by Hyodo ([H], (1.4)) and is induced by a symbol map on Milnor K-Theory. Hyodo has shown ([H], Theorem (1.6)) that the highest graded quotient $gr^0 M_n^2$ sits in an extension (change of notation: $Y := X_p$, the closed fiber of $\mathcal{X}$)

$$0 \longrightarrow W_n w^1_{Y,log} \longrightarrow gr^0 M_n^2 \longrightarrow W_n w^2_{Y,log} \longrightarrow 0$$

where $W_n w^i_{Y,log}$ are the modified logarithmic Hodge-Witt-sheaves ([H] (1.5)). On the other hand Hyodo and Kato ([H-K] Prop. 1.5) constructed an exact sequence of Hodge-Witt-sheaves

$$0 \longrightarrow W_n w^1_Y \longrightarrow W_n \tilde{w}^2_Y \longrightarrow W_n w^2_Y \longrightarrow 0$$

and used the connecting homomorphism on the level of cohomology to define the monodromy operator on log-crystalline cohomology. It follows from the work of Tsuji ([Tsu], §2.4) that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & W_n w^1_{Y,log} & \to & gr^0 M_n^2 & \to & W_n w^2_{Y,log} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & W_n w^1_Y & \to & W_n \tilde{w}^2_Y & \to & W_n w^2_Y & \to & 0 \end{array}$$

such that the upper exact sequence is obtained by taking the kernel of $1 - F$ acting on the lower exact sequence, where F is the Frobenius. From the Hodge-Witt-decomposition of $H^r(Y, Ww^{\cdot})$ ([Il], Proposition (1.5)) it is easy to derive a Hodge-Witt-decomposition for $H^r(Y, W\tilde{w}^{\cdot}_Y)$

$$H^r(Y, W\tilde{w}^{\cdot}_Y) = \bigoplus_{i+j=r} H^i(Y, W\tilde{w}^j_Y) \quad .$$

From the action of the Frobenius φ on $H^r(Y, W\tilde{w}^{\cdot}_Y)$ we get

$$H^3(Y, W\tilde{w}^{\cdot}_Y)_{\varphi=p^2} = H^1(Y, W\tilde{w}^2_Y)^{F=1} \quad .$$

On the other hand it is shown in the proof of the semistable analogue of the p-adic points conjecture on log-syntomic cohomology [L], (2.6), Prop. (2.9), Prop. (2.13) that there is a surjection

$$H^3_{et}(\mathcal{X}, s^{\log}_{\mathbb{Q}_p}(2)) \twoheadrightarrow (H^3(Y, W\tilde{w}^{\cdot}_Y)_{\mathbb{Q}_p})_{\varphi=p^2}$$

and the above arguments yield a commutative diagram

$$\begin{array}{ccccc}
H^3(\mathcal{X}, s^{\log}_{\mathbb{Q}_p}(2)) & & & & \\
\downarrow & \searrow & & & \\
H^3_{et}(\mathcal{X}, \tau_{\leq 2} Rj_* \mathbb{Q}_p(2)) & \to & H^1(Y, gr^0 M^2_{\mathbb{Q}_p}) & \to & (H^3(Y, W\tilde{\omega}^{\cdot}_Y)_{\mathbb{Q}_p})_{\varphi = p^2} \\
\downarrow & & & & \downarrow \\
H^1(\mathbb{Q}_p, V) & & \longrightarrow & & H^1(\mathbb{Q}_p, B_{\text{crys}} \otimes V)
\end{array}$$

It follows from ([L], (2.10)) that the composite

$$(H^3(Y, W\tilde{\omega}^{\cdot}_Y)_{\mathbb{Q}_p})_{\varphi = p^2} \longrightarrow H^1(\mathbb{Q}_p, B_{\text{crys}} \otimes V) \longrightarrow H^1(\mathbb{Q}_p, B_{st} \otimes V)$$

is the zero map. Using the fact that $H^1_{st} = H^1_g$ (unpublished result of Hyodo, see also Nekovář ([Ne](1.24)) we conclude that the image of the map

$$H^3_{et}(\mathcal{X}, \tau_{\leq 2} Rj_* \mathbb{Q}_p(2)) \longrightarrow H^1(\mathbb{Q}_p, V)$$

is $H^1_g(\mathbb{Q}_p, V)$ in view of Lemma (3.4). This finishes the proof of Lemma (3.3) and Proposition (2.1).

References

[Bl] S. Bloch, *Lectures on Algebraic Cycles.* Duke Univ. Math. Series, Durham, 1980.

[B-K] S. Bloch, K. Kato, *L-functions and Tamagawa numbers of motives.* In: The Grothendieck Festschrift 1990.

[Fl] M. Flach, *A finiteness theorem of the symmetric square of an elliptic curve.* Invent. Math. 109 (1992), 307–327.

[H] O. Hyodo, *A note on p-adic étale cohomology in the semistable reduction case.* Invent. Math. 91 (1988), 543–557.

[H-K] O. Hyodo, K. Kato, *Semistable reduction and crystalline cohomology with logarithmic poles.* In: Périodes p-adiques, Astérisque 223 (1994), 221–268.

[Il] L. Illusie, *Réduction semistable ordinaire, cohomologie étale p-adique et cohomologie de de Rham d'après Bloch-Kato et Hyodo.* In: Périodes p-adiques, Astérisque 223 (1994), 209–220.

[J] U. Jannsen, *On the ℓ-adic cohomology of varieties over number fields and its Galois-cohomology.* In: Galois groups over $\mathbb{Q}$, MSRI Publications (1989), 315–360.

[K] K. Kato, *Semistable reduction and p-adic etal cohomology.* In: Périodes p-adiques, Astérisque 223 (1994), 269–293.

[L] A. Langer, *Local points of motives in semistable reduction.* To appear in Compositio Mathematica.

[L-S] A. Langer, S. Saito, *Torsion zero-cycles on the selfproduct of a modular elliptic curve.* In: Duke Math. Journal, November 1996.

[Mi] S. Mildenhall, *Cycles in a product of elliptic curves and a group analogous to the class group.* Duke Math. J. 67 (1992), 387–406.

[Mo] A. Mokrane, *La suite spectrale des poids en cohomologie de Hyodo-Kato.* Duke Math. J. 72 (1993), 301–337.

[M-S] A. S. Merkurjev, A. A. Suslin, *K-cohomology of Severi-Brauer. Varieties and the norm residue homomorphism.* Math. USSR Izvestiya 21, No. 2 (1983), 307–340.

[Ne] J. Nekovář, *On p-adic height-pairings.* In: Seminaire de Théorie de Nombres 1990–1991 (1993), 127–202.

[Tsu] T. Tsuji, *p-adic etale cohomology and crystalline cohomology in the semi-stable reduction case.* Preprint 1996.

[W] A. Wiles, *Modular elliptic curves and Fermat's last Theorem.* Annals of Mathematics 142 (1995), 443–551.

Andreas Langer
Mathematisches Institut
der Universität Münster
Einsteinstr. 62
D - 48149 Münster
Germany