Which Moments of a Logarithmic Derivative Imply Quasiinvariance?

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Abstract. In many special contexts quasiinvariance of a measure under a one-parameter group of transformations has been established. A remarkable classical general result of A.V. Skorokhod [6] states that a measure μ on a Hilbert space is quasiinvariant in a given direction if it has a logarithmic derivative β in this direction such that $e^{a|\beta|}$ is μ-integrable for some $a > 0$. In this note we use the techniques of [7] to extend this result to general one-parameter families of measures and moreover we give a complete characterization of all functions $\psi : [0, \infty) \to [0, \infty)$ for which the integrability of $\psi(|\beta|)$ implies quasiinvariance of μ. If ψ is convex then a necessary and sufficient condition is that $\log \psi(x)/x^2$ is not integrable at ∞.

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1 Overview

The paper is divided into two parts. The first part does not mention quasiinvariance at all. It treats only one-dimensional functions and, implicitly, one-dimensional measures. The reason is as follows: A measure μ on $\mathbb{R}$ has a logarithmic derivative ρ if and only if μ has an absolutely continuous Lebesgue density f, and ρ is given by $\rho(x) = \frac{f'}{f}(x)$ μ-a.e.. Then the μ-integrability of $\psi(|\rho|)$ is equivalent to the Lebesgue-integrability of $\psi(|\frac{f'}{f}|)f$. The quasiinvariance of μ is equivalent to the statement that $f(x) \neq 0$ Lebesgue-a.e.. Therefore in the case of one-dimensional measures, a function ψ allows a quasiinvariance criterion, as indicated in the abstract, iff for all absolutely continuous functions $f \geq 0$, the integrability of $\psi(|\frac{f'}{f}|)f$ implies that f is strictly positive. The main result of the first part, Theorem 1, gives necessary and sufficient reformulations of this property which are easier to check. The most simple of these reformulations is the divergence of the integrals $\int_c^\infty \log \psi_*(x)/x^2 dx$ where ψ_* is the lower nondecreasing convex envelope of ψ. Moreover we give, for every ψ with this property, explicit lower estimates for the values of f on an interval I, in terms

of the length of this interval and of the integral $\int_I \psi_*(|\frac{f'}{f}|) f dx$. Finally we give an example showing that the introduction of the lower convex hull in these results really is necessary.
The second part of the paper then proves that the one-dimensional situation is typical. The quasiinvariance criterion works on the real axis if and only if it works for the transport of a measure under an arbitrary measurable flow, or even more generally for general one parameter families of measures which are differentiable in the sense of [7]. If this criterion applies then one gets even the typical Cameron-Martin type formula for the Radon-Nikodym-densities between the members of such a family (cf.e.g. [3], [1], [5], [7]). In the situation of Skorokhod's result mentioned in the summary, we see that the exponential functions $\psi(x) = e^{ax}, a > 0$ can be replaced by $\exp(\frac{x}{\log x})$ but not by $\exp(\frac{x}{(\log x)^2})$. This shows that Skorokhod's exponential criterion is not strictly optimal but it gives the optimal power of $\log \psi$.

2 A class of one-dimensional functions

Theorem 1: *For a measurable function $\psi : [0,\infty) \to [0,\infty)$ the following six conditions are equivalent:*

(A) *Let $f : \mathbb{R} \to [0,\infty)$ be absolutely continuous such that*

$$\int_{-\infty}^{\infty} \psi(|\frac{f'}{f}(x)|)\ f(x)dx < \infty \tag{1}$$

and $f \neq 0$. Then $f(x) > 0$ for Lebesgue-all $x \in \mathbb{R}$.

(A') *Let $f : \mathbb{R} \to [0,\infty)$ be absolutely continuous such that $x \mapsto \psi(|\frac{f'}{f}(x)|)\ f(x)$ is locally Lebesgue integrable and $f \neq 0$. Then $f(x) > 0$ for all $x \in \mathbb{R}$.*

(B) *For some $a > 0$ the following implication holds*

$$\sum_{i=1}^{\infty} z_i < \infty,\ z_i > 0 \Longrightarrow \sum_{i=1}^{\infty} z_i\ \psi(\frac{1}{z_i})e^{-ai} = \infty. \tag{2}$$

.

(B') *The implication (2) holds for all $a > 0$.*

(C) *Let ψ_* be the largest nondecreasing convex function $\leq \psi$ und suppose $\psi_*(c) > 0$. Then*

$$\int_c^{\infty} \frac{\log \psi_*(x)}{x^2} dx = \infty. \tag{3}$$

(C') *Similarly, $\lim_{x\to\infty} \psi_*(x) = \infty$, and for d in the range of $\log \psi_*$,*

$$\int_d^{\infty} \frac{1}{(\log \psi_*)^{-1}(x)} dx = \infty. \tag{4}$$

In particular the conditions $(A)-(B)$ hold for ψ if and only if they hold for ψ_*. If ψ is convex and nondecreasing and some power ψ^p with $p>0$ satisfies one of the conditions then the same is true for ψ.

Proof: Clearly $(A') \Longrightarrow (A)$.
$(A) \Longrightarrow (B)$: Let $z_i>0$ and $b:=\sum_{i=1}^\infty z_i<\infty$. Define $f:\mathbb{R}\longrightarrow[0,\infty)$ by

$$f(s)=\exp(-\frac{s-(z_1+\ldots+z_{i-1})}{z_i}-(i-1))$$

for $s\in[z_1+\ldots+z_{i-1},z_1+\ldots+z_i],\ i=1,2,\ldots.$ Note that $e^{-i}\le f(s)\le e^{-(i-1)}$ and $\frac{f'}{f}(x)=(\log f)'(x)=-\frac{1}{z_i}$ in this interval. Moreover, set $f(s)=0$ for $s\ge b$ and $f(-s):=f(s)$ for $s\ge 0$. Then f is absolutely continuous but not strictly positive. Therefore by assumption (A) the integral in (1) diverges. Hence

$$\begin{aligned}\sum_{i=1}^\infty \psi(\frac{1}{z_i})e^{-(i-1)}z_i &\ge \sum_{i=1}^\infty \psi(\frac{1}{z_i})\int_{z_1+\ldots+z_{i-1}}^{z_1+\ldots+z_i} f(x)dx\\ = \int_0^b \psi(|\frac{f'}{f}(x)|)\ f(x)dx=\infty\end{aligned}$$

which proves (B) with $a=1$.
$(B)\Longleftrightarrow(B')$: Denote by (B_a) the statement that (B) holds with the constant a. Clearly, $(B_b)\Longrightarrow(B_c)$ if $c\le b$. We prove $B_a\Longrightarrow B_{2a}$: Suppose $\sum_{i=1}^\infty z_i<\infty,\ z_i>0$ and let $y_{2j}=y_{2j-1}=z_j$ for $j\in\mathbb{N}$. Then $\sum_{j=1}^\infty y_j<\infty$ and hence

$$2\sum_{i=1}^\infty z_i e^{-2ai}\psi(\frac{1}{z_i})\ge e^{-a}\sum_{j=1}^\infty y_j e^{-aj}\psi(\frac{1}{y_j})=\infty.$$

$(B')\Longrightarrow(C')$: Let $h(t)=(\log\psi_*)^{-1}(t)$. Define the number z_i^* by $z_i^*=\frac{1}{h(i)}$. From (B) it follows easily that $\frac{\psi(x)}{x}\to\infty$ as $x\to\infty$. Thus the same holds for ψ_*. Since ψ_* is convex and increasing the function $1/h$ is continuous and decreasing. Therefore for the proof of (4) it is sufficient to prove that the sum $\sum_{i=1}^\infty z_i^*$ diverges.
Suppose, on the contrary, that $\sum_{i=1}^\infty z_i^*<\infty$. Choose $y_i\le 2z_i^*$ such that $y_i\psi(\frac{1}{y_i})\le c_i+1$ where

$$c_i=\inf_{x\ge\frac{1}{2z_i^*}}\frac{\psi(x)}{x}.$$

This is possible by definition of this infimum c_i. The affine function $l_i:x\mapsto c_ix-\frac{c_i}{2z_i^*}$ is $\le\psi$ since it is negative on $[0,\frac{1}{2z_i^*})$, and on $[\frac{1}{2z_i^*},\infty)$ even the larger function $x\mapsto c_ix$ is bounded by ψ. Therefore, from the definition of ψ_*, we get

$$\psi_*(\frac{1}{z_i^*})\ge l_i(\frac{1}{z_i^*})=\frac{1}{2}\frac{c_i}{z_i^*}. \tag{5}$$

We apply (B') with $a=1$ and use the summability of the z_i^* and hence of the y_i to get

$$\sum_{i=1}^\infty c_ie^{-i}\ge\sum_{i=1}^\infty y_i\psi(\frac{1}{y_i})e^{-i}-\sum_{i=1}^\infty e^{-i}=\infty.$$

Now $\psi_*(\frac{1}{z_i^*}) = e^i$ by construction of the z_i^* , thus (5) gives

$$\sum_{i=1}^{\infty} z_i^* = \sum_{i=1}^{\infty} z_i^* \psi_*(\frac{1}{z_i^*})e^{-i} \geq \frac{1}{2}\sum_{i=1}^{\infty} c_i e^{-i} = \infty$$

which is a contradiction.
$(C) \iff (C')$: Both, (3) and (4), imply that ψ_* is continuous, nondecreasing and unbounded at infinity. Therefore there is some c such that ψ_* is even strictly increasing on $[c,\infty)$, and the assertion follows from lemma 1 below, applied to $\varphi = \log \psi_*$.
$(C') \implies (A')$. Presumably, this is the most useful implication. We formulate the main part of the proof as the separate Theorem 2 since it involves only integrals over finite intervals and can be applied also to functions which do not satisfy the conditions of the theorem. In order to deduce our implication from Theorem 2 assume (4) and let $\Psi(x) = \psi_*(x) - \psi_*(0)$. Then $\lim_{x\to\infty} \log \Psi(x) - \log \psi_*(x) = 0$ and hence using the equivalence of (3) and (4) we get

$$\int_0^{\infty} \frac{1}{(\log \Psi)^{-1}(y)}\; dy = \infty.$$

Now if f is absolutely continuous and $x \mapsto \psi(|\frac{f'}{f}(x)|)f(x)$ is locally integrable then also the function $x \mapsto \Psi(|\frac{f'}{f}(x)|)f(x)$ is locally integrable and hence (9) below gives a lower bound for the values of f on any interval $[s,t]$ such that $f(s) > 0$. The case $f(t) > 0$ follows by reflection. In particular f is strictly positive which is the assertion of (A').
Finally we prove the last statement. Let ψ be convex and nondecreasing and suppose that ψ^p satisfies one of the conditions. If $p < 1$ then $\psi \geq \max(1,\psi^p)$ and using criterion (B) it follows that ψ satisfies the same condition. If $p > 1$ then ψ^p, by Jensen's inequality, is also convex nondecreasing and hence $\psi = \psi_*$ and $\psi^p = (\psi^p)_*$. Since $\log \psi^p = p \log \psi$, the criterion (C) carries over from ψ^p to ψ. □

In the proof we have used the following elementary fact.

Lemma 1: *Let $c > 0$ and let $\varphi : [c,\infty) \to [d,\infty)$ be a homeomorphism. Then*

$$\int_d^{\infty} \frac{1}{\varphi^{-1}(y)}\; dy = \int_c^{\infty} \frac{\varphi(x)}{x^2}\; dx - \frac{d}{c}, \tag{6}$$

i.e. both integrals converge at the same time and if they do (6) holds.

Proof: The change of variables $y = \varphi(x)$ gives

$$\int_d^{\varphi(T)} \frac{1}{\varphi^{-1}(y)} dy = \int_c^T \frac{1}{x} d\varphi(x)(x) = \frac{\varphi(x)}{x}\Big|_c^T + \int_c^T \frac{\varphi(x)}{x^2} dx. \tag{7}$$

Since $\frac{\varphi(T)}{T} > 0$ for large T the left-hand side of (6) dominates the right-hand side. For the converse inequality assume that the integral on the right-hand side of (6) is finite. The indefinite integral on the left-hand side of (7) is monotone in T, so it has a finite or infinite limit. Therefore by (7) the limit $\lim_{T\to\infty} \frac{\varphi(T)}{T}$ exists and it must be 0 because otherwise the integral on the right-hand side of (6) would be infinite. This implies (6). □

The following result gives a quantitative version of the implication $(C') \Longrightarrow (A')$ in Theorem 1.

Theorem 2: *Let $\Psi : \mathbb{R} \longrightarrow [0,\infty)$ be a convex even function with $\Psi(0) = 0$. Let $f : [s,t] \longrightarrow [0,\infty)$ be absolutely continuous such that $f(s) > 0$. Then*

$$\frac{1}{f(s)} \int_s^t \Psi(\frac{f'(x)}{f(x)}) f(x) dx \geq \int_0^{-\min\limits_{s\le x\le t} \log(f(x)/f(s))} \frac{1}{(\log \Psi)^{-1}(x)} dx - (t-s). \qquad (8)$$

Remark: Let $I = \int_s^t \Psi(\frac{f'(x)}{f(x)}) f(x)\, dx$ be finite. Define $F(y) := \int_0^y \frac{1}{(\log \psi)^{-1}(x)}\, dx$ for $y \geq 0$. If the range of F contains the number $\frac{I}{f(s)} + t - s$ (which certainly is true if $F(y) \to \infty$ for $y \to \infty$) then (8) can be rewritten as

$$f(t) \geq f(s) \exp\Big(- F^{-1}(\frac{I}{f(s)} + t - s)\Big). \qquad (9)$$

This gives a lower estimate of the fluctuation of the function f in terms of the integral I and the length of the interval $[s,t]$.

Remark: In the special case $\psi(x) = e^{ax}$ there is an elegant more abstract proof of property (A) of Theorem 1, see [4], prop. 2.18. That proof does not give a lower bound for the values of f in terms of I but on the other hand it works also in higher dimensions whereas our method is strictly one-dimensional.

Proof: Both sides of (8) remain unchanged if f is multiplied by some positive constant. Therefore we may and shall, for notational convenience, assume $f(s) = 1$. For $a > 0,\ i \in \mathbb{N}_0$ let $x_i^{(a)} := \inf\{y \geq s : f(y) = e^{-ai}\}$, We also introduce the numbers $z_i^{(a)} := x_i^{(a)} - x_{i-1}^{(a)}$, $\overline{z}_i^{(a)} := \frac{a}{(\log \psi)^{-1}(ai)}$ for $i \in \mathbb{N}$ and finally $N_a := \sup\{n \in \mathbb{N} : x_n^{(a)} \leq t\}$. We apply Jensen's inequality in the second step of the following estimates

$$\begin{aligned}
\int_s^t \Psi(\frac{f'(x)}{f(x)})f(x)\;dx &\geq \sum_{i=1}^{N_a} e^{-ai}\int_{x_{i-1}^{(a)}}^{x_i^{(a)}} \Psi(\frac{f'(x)}{f(x)})\;dx \\
&\geq \sum_{i=1}^{N_a} e^{-ai} z_i^{(a)}\Psi\Big(\frac{1}{z_i^{(a)}}\int_{x_{i-1}^{(a)}}^{x_i^{(a)}}\frac{-f'(x)}{f(x)}dx\Big) \\
&= \sum_{i=1}^{N_a} e^{-ai} z_i^{(a)}\Psi\Big(\frac{1}{z_i^{(a)}}(-\ln f(x_i^{(a)}) + \ln f(x_{i-1}^{(a)}))\Big) \\
&= \sum_{i=1}^{N_a} e^{-ai} z_i^{(a)}\Psi(\frac{a}{z_i^{(a)}}) \\
&\geq \sum_{\substack{i=1 \\ z_i^{(a)}\leq \overline{z}_i^{(a)}}}^{N_a} z_i^{(a)}\Psi(\frac{a}{z_i^{(a)}})e^{-ai}.
\end{aligned}$$

Since Ψ is convex and $\Psi(0) = 0$ the function $y \mapsto \frac{\Psi(y)}{y}$ is increasing on $[0,\infty)$. Moreover $\Psi(\frac{a}{\overline{z}_i^{(a)}}) = e^{ai}$ and hence the last sum can be further estimated from below by

$$\begin{aligned}
\sum_{\substack{i=1 \\ z_i^{(a)}\leq \overline{z}_i^{(a)}}}^{N_a} \overline{z}_i^{(a)}\Psi(\frac{a}{\overline{z}_i^{(a)}})e^{-ai} &= \sum_{\substack{i=1 \\ z_i^{(a)}\leq \overline{z}_i^{(a)}}}^{N_a} \overline{z}_i^{(a)} \\
\geq \sum_{i=1}^{N_a}\overline{z}_i^{(a)} - \sum_{i=1}^{N_a} z_i^{(a)} &\geq \sum_{i=1}^{N_a}\frac{a}{(\log\Psi)^{-1}(ai)} - (t-s).
\end{aligned}$$

Because of

$$\sum_{i=1}^{N_a}\frac{a}{(\log\Psi)^{-1}(ai)} \underset{a\downarrow 0}{\longrightarrow} \int_0^b \frac{1}{(\log\Psi)^{-1}(y)}\;dy$$

where

$$b = \lim_{a\downarrow 0} N_a a = -\lim_{a\downarrow 0}\log f\big(x_{N_a}^{(a)}\big) = -\min_{s\leq x\leq t}\log f(x),$$

the proof is complete. □

Example 1 *For every $0 < p < 1$ there is a convex increasing function $\psi : [0,\infty) \longrightarrow [0,\infty)$ which satisfies the conditions of Theorem 1, but ψ^p does not.*

This function then satisfies

$$\int_c^\infty \frac{\log\psi^p(x)}{x^2}\;dx = \int_c^\infty p\frac{\log\psi(x)}{x^2}\;dx = p\int_c^\infty \frac{\log\psi_*(x)}{x^2}\;dx = \infty$$

but (C) does not hold for ψ^p. This shows that in (C) (and in (C')) the convex lower envelope cannot be replaced by the function itself. Switching roles of ψ and ψ^p, the example also shows that in the last statement of the theorem the convexity of ψ cannot be replaced by the convexity of ψ^p for $p > 1$. With some additional effort one could modify the example in such a way that for no $p < 1$ the function ψ^p satisfies the conditions of Theorem 1.

Construction: We write q instead of $\frac{1}{p}$. We start by setting $b_0 = 0$, $\gamma_0 = 0$, $\beta_0 = 1$. We shall choose recursively points $a_1 < b_1 < a_2 < b_2 < \dots$ and real numbers $\alpha_k, \beta_k, \gamma_k,\ \ k \in \mathbb{N}$ and set

$$\psi(x) = \begin{cases} a_k^q & \text{for} \quad x = a_k \\ \alpha_k e^{qx} & \text{for} \quad a_k < x < b_k \\ \beta_k x + \gamma_k & \text{for} \quad b_k \leq x < a_{k+1} \end{cases} \tag{10}$$

So the function alternates between affine and exponential type. The constants are chosen in such a way that at the points a_k the graph of ψ is bent upwards, while at the points b_k the two one-sided derivatives agree.
Assume that all numbers $a_i, b_i, \alpha_i, \beta_i, \gamma_i$ with $i < k$ are already chosen such that (10) gives a continuous convex increasing function on some interval $[0, b_{k-1} + \varepsilon]$ which is differentiable with the possible exception of the points a_i for $i < k$. In the case $k = 1$ let $a_1 = 1$. For $k > 1$ we then know that $a_{k-1} \geq 1$ and, comparing logarithmic derivatives of ψ and of x^q, respectively, we see that $\psi(x) > x^q$ on the interval $(a_{k-1}, b_{k-1}]$, in particular $\beta_{k-1} b_{k-1} + \gamma_{k-1} > b_{k-1}^q$. Since $q > 1$ this implies that there is a solution $> b_{k-1}$ of the equation

$$\beta_{k-1} x + \gamma_{k-1} = x^q \tag{11}$$

which we choose as a_k. Then ψ is defined on $[b_{k-1}, a_k]$ by the third part of (10). Choose α_k such that $\alpha_k e^{q a_k} = a_k^q$, i.e. $\alpha_k = a_k^q e^{-q a_k}$. Let $b_k = \frac{3}{2} a_k$ and define ψ on $[a_k, b_k]$ according to the second part of (10). The numbers β_k and γ_k are determined by the equation of the (left) tangent of ψ at b_k.
Verification: By construction,

$$\psi'(a_k-) = \beta_{k-1} = \psi'(b_{k-1}-) = q\psi(b_{k-1}) < q\psi(a_k) = \psi'(a_k+).$$

i.e. this extension of ψ continues to be convex and continuous. Moreover,

$$\psi(b_k) = \alpha_k e^{q b_k} = a_k^q e^{-q a_k} e^{q \frac{3}{2} a_k} = a_k^q e^{\frac{q}{2} a_k}. \tag{12}$$

The sequence (a_k) is certainly unbounded by the choice of the b_k. By construction of a_k, at this point the slope of $y = x^q$ is bigger than β_{k-1}. Thus,

$$q a_k^{q-1} > \beta_{k-1} = \psi'(b_{k-1}) > q\psi(a_{k-1}) = q a_{k-1}^q$$

and hence $\frac{a_k}{a_{k-1}} > (a_{k-1})^{\frac{1}{q-1}}$. This implies

$$\frac{a_k}{a_{k-1}} \longrightarrow \infty. \tag{13}$$

Because of (12) and (13)

$$\begin{aligned}\int_{a_k}^{a_{k+1}} \frac{\log\psi(x)}{x^2}\,dx &\geq \log\psi(b_k)\int_{b_k}^{a_{k+1}}\frac{1}{x^2}\,dx\\ &= (q\log a_k + \frac{q}{2}a_k)\Big(\frac{1}{\frac{3}{2}a_k} - \frac{1}{a_{k+1}}\Big)\\ &\geq \frac{q}{4}\end{aligned}$$

for eventually all k. Together with the convexity this shows that ψ satisfies condition (C).

On the other hand, for $z_k = \frac{1}{a_k}$, (13) implies $\sum_{k=1}^{\infty} z_k < \infty$. But

$$z_k\psi^p(\frac{1}{z_k}) = \frac{1}{a_k}(a_k^q)^{\frac{1}{q}} = 1$$

and, therefore, $\sum_{k=1}^{\infty} z_k\psi^p(\frac{1}{z_k})e^{-i} < \infty$. So ψ^p does not have property (B). This concludes the discussion of the example.

3 Logarithmic derivatives and quasiinvariance

Let $\mathcal{M}(E)$ be the linear space of finite signed measures on a measurable space $(E,\mathcal{B})$, equipped with the total variation norm $\|\cdot\|$. Let C be a linear space of bounded test functions on E which is normdefining for $\mathcal{M}(E)$, $\|\mu\| = \sup\{\int\varphi\,d\mu : \varphi\in C, \|\varphi\|_\infty \leq 1\}$ for all $\mu\in\mathcal{M}(E)$. Typical examples of spaces C with this property are the space of bounded continuous functions for a topology for which $\mathcal{B}$ is the class of Baire sets, i.e. the σ-field generated by C, or the space of smooth cylindrical functions on a Hilbert space. Let I be a real interval and let $(\mu_t)_{t\in I}$ be a family of elements of $\mathcal{M}(E)$. We call this map τ_C-differentiable at $t\in I$ with *logarithmic derivative* $\rho_t\in L^1(\mu_t)$ if for every $\varphi\in C$ the function $s\mapsto\int\varphi\,d\mu_s$ is differentiable at t with derivative

$$\frac{d}{ds}_{|s=t}\int\varphi\,d\mu_s = \int\varphi\rho_t\,d\mu_t. \tag{14}$$

The measure $\rho_t\mu_t$ is the derivative of the $\mathcal{M}(E)$-valued curve (μ_t) with respect to the topology $\tau_C = \sigma(\mathcal{M}(E),C)$ and is denoted by μ_t'. An important special class of examples are families $(\mu_t)_{t\in\mathbb{R}}$ which are induced by a measurable flow: If $\mathcal{T} = (T_t)_{t\in\mathbb{R}}$ is a one-parameter group of bimeasurable bijections of E, and $\mu\in\mathcal{M}(E)$ is a fixed measure, one considers the family of measures $\mu_t = \mu\circ T_t^{-1}$. If (μ_t) satisfies the above differentiability condition at one (and then at all) t we call μ differentiable along $\mathcal{T}$ with logarithmic derivative $\rho = \rho_0$. In this case the logarithmic derivative for general t is given by

$$\rho_t(x) = \rho(T_{-t}x). \tag{15}$$

This extends the concept of the differentiability of a measure on a linear space in a certain direction which was the main subject of [2] and the relevant parts of [6]. The more general aspects have been studied, starting with [3], in [5] and [7], for a

comparison with concepts of the Gross-Malliavin calculus see e.g. [8].
We need two results from [7]: (a) Suppose that ρ_t exists for all $t \in I$, and that

$$\int_I \|\rho_t\|_{1,\mu_t}\ dt = \int_I \|\mu_t'\|\ dt < \infty. \tag{16}$$

Then there are a probability measure ν on $\mathcal{B}$ and $\mathcal{B}\times\mathcal{B}(I)$-measurable functions g, g' on $E \times I$ such that $\mu_t(dx) = g(t,x)\nu(dx), \mu_t'(dx) = g'(t,x)\nu(dx)$ and thus $\rho_t(x) = \frac{g'(t,x)}{g(t,x)}$ ν-a.e. for Lebesgue almost all $t \in I$ and finally

$$g(b,x) - g(a,x) = \int_a^b g'(s,x)ds \quad \textit{for all } x \in E \textit{ and } a, b \in I. \tag{17}$$

(b) If, moreover, the pointwise integrability condition

$$\int_a^b |\rho_s(x)|\ ds < \infty \quad |\mu_a| + |\mu_b| \ - a.e. \tag{18}$$

holds then all measures $\mu_t, a \leq t \leq b$ are equivalent and we have the 'abstract Cameron-Martin' formula

$$\frac{d\mu_b}{d\mu_a}(x) = \exp \int_a^b \rho_s(x)\ ds. \tag{19}$$

The condition (18) clearly is necessary for (19) to make sense. But how can one verify it ? The interaction of the Radon-Nikodym derivatives ρ_t for varying t may be complicated. Therefore, it seems desirable to have sufficient conditions for the equivalence of the μ_t in terms of the onedimensional laws of the ρ_t with respect to the measures μ_t. The following continuation of the main result of the first section provides an answer of this type.

THEOREM 3: *A function $\psi : [0,\infty) \to [0,\infty)$ satisfies the conditions $(A) - (C')$ of Theorem 1 if and only if the following holds: For every τ_C-differentiable and $\|\cdot\|$-bounded family $(\mu_t)_{t\in I}, I \subset \mathbb{R}$ of signed measures on a measurable space and $a, b \in I$ with*

$$\int_a^b \|\psi(|\rho_t|)\|_{1,\mu_t}\ dt < \infty \tag{20}$$

the measures $\mu_t, a \leq t \leq b$ are equivalent to each other. Moreover, for such functions ψ the condition (20) implies the abstract Cameron-Martin formula formula (19).

PROOF: 1. Suppose that ψ has the indicated property. We want to show that condition (A') of Theorem 1 is fulfilled. Let f be an absolutely continuous nonnegative function on the real axis for which $x \mapsto \psi(|\frac{f'}{f}(x)|)\ f(x)$ is locally integrable and such that f does not vanish everywhere. We have to show that f is strictly positive. Otherwise there are two points a, b with $f(a) > 0$ and $f(b) = 0$. Without loss of generality $a < b$. Let $\mu_t(dx) = f(x+t)dx$. In order to apply our condition, we have to make sure that these measures are finite. For this, redefine f on $[b,\infty)$ by $f(x)=0$ and on $[-\infty,a)$ by $f(x)=f(a)\exp(x-a)$ Then

$$\int_{-\infty}^{a}\psi(|\frac{f'}{f}(x)|)f(x)\ dx=\int_{-\infty}^{a}\psi(1)f(a)\exp(x-a)\ dx<\infty$$

and, similarly, $\int_b^\infty \psi(|\frac{f'}{f}(x)|)f(x)\ dx=0<\infty$. The modified f still satisfies (1) and it is certainly Lebesgue integrable. Thus, we have the flow situation mentioned above with $T_t x = x-t$. The family μ_t is differentiable (even for the topology induced by the total variation norm) with $\rho_t(x)=\frac{f'}{f}(x+t)$. Then the local integrability assumption and the two tail estimates imply $\int\psi(|\rho_t(x)|)\ \mu_t(dx)=\int\psi(|\rho_0(x)|)\ \mu(dx)<\infty$ for all t. Therefore, the condition (20) is satisfied. By our assumption on ψ this implies that the measures μ_t are all equivalent, i.e. the function f cannot vanish on a half-line as our f does. This contradiction shows that f must be strictly positive. Hence ψ has property (A').

2. Suppose, conversely, that ψ is a function of the type considered in Theorem 1. Let μ_t be a τ_C-differentiable and $\|\cdot\|$-bounded family $(\mu_t)_{t\in I}, I\subset\mathbb{R}$ of signed measures on a measurable space and let $a,b\in I$ with (20) be given. First we claim that (16) holds. In fact from condition (C) in Theorem 1 we find positive constants u,v such that $v\psi(y)\geq y$ for all $y>u$. Then

$$\begin{aligned}\|\rho_t\|_{1,\mu_t}=\int_E|\rho_t(x)|\ d\mu_t &\leq v\int_{|\rho_t|>u}\psi(|\rho_t|)\ d|\mu_t|+\int_E u\ d|\mu_t|\\ &\leq v\|\psi(|\rho_t|)\|_{1,\mu_t}+u\|\mu_t\|.\end{aligned}$$

Since the measures are $\|\cdot\|$-bounded (20) implies (16). Therefore we can choose g,g' and ν with the properties listed after (16). Then (20) can be rewritten as

$$\int_a^b\int_E\psi(|\frac{g'(t,x)}{g(t,x)}|)g(t,x)\ \nu(dx)\ dt<\infty.$$

By Fubini, there is a ν-nullset N such that $\int_a^b\psi(|\frac{g_t'}{g_t}(x)|)g_t(x)\ dt<\infty$ for every $x\in E\setminus N$. Then extending $t\mapsto g(t,x)$ outside of the interval $[a,b]$ by exponential tails (or zero) as in the first part of this proof we can apply condition (A) in Theorem 1 and conclude that for each $x\in E\setminus N$ either $g(t,x)>0$ for all $t\in[a,b]$ or $g(t,x)=0$ for all $t\in[a,b]$. This implies that the measures $\mu_t, t\in[a,b]$ are all equivalent.

3. Moreover the function $g(\cdot,x)$ is continuous by (17) and therefore it is bounded away from 0 by some constant $\delta(a,b,x)$ on the interval $[a,b]$ for μ_a- (and μ_b-) almost all $x\in E$. Then (17) and the representation $\rho_t(x)=\frac{g'(t,x)}{g(t,x)}$ show that (18) and hence also (19) hold. □

In particular, we get the following version of Skorokhod's theorem for the function given by $\psi(y)=\exp(\frac{y}{|\log(y)|})$ for $y>0$.

COROLLARY 4 *Let μ be a probability measure on a measurable space E and let $\mathcal{T}=(T_t)_{t\in\mathbb{R}}$ be a measurable flow on E. Suppose μ is τ_C-differentiable along $\mathcal{T}$ with logarithmic derivative ρ. If ρ satisfies the following integrability condition*

$$\int_E\exp\Big(\frac{|\rho(x)|}{|\log(|\rho(x)|)|}\Big)\mu(dx)<\infty$$

then μ is quasiinvariant under the flow $\mathcal{T}$ and the corresponding Radon-Nikodym derivatives are given by (19). But, even for translation families on the real axis, the quasiinvariance is not implied by the weaker condition

$$\int_E \exp\Big(\frac{|\rho(x)|}{\log(|\rho(x)|)^2}\Big)\mu(dx) < \infty.$$

Proof: We consider the function $\psi(y) = \exp(\frac{y}{|\log(y)|})$ for $y > 0$. Then it is easily verified that ψ is convex and increasing for sufficiently large y and, thus, it satisfies the criterion (C). Because of (15) we have

$$\|\psi(|\rho_t|)\|_{1,\mu_t} = \int \psi(|\rho|) \circ T_t^{-1}\ d\mu_t = \int \psi(|\rho|)\ d\mu$$

for all t, and hence our integrability assumption implies (20).
On the other hand $\psi(y) = \exp(\frac{y}{\log(y)^2})$ for $y > 0$ defines a function which does not satisfy the condition (C). The function f used in the proof of $(A) \Longrightarrow (B)$ in Theorem 1 then satisfies (1) for this function ψ but f has compact support. Therefore the logarithmic derivative $\rho = \frac{f'}{f}$ of the measure $\mu \in \mathcal{M}(\mathbb{R})$ whose density is f, satisfies the weakened integrability condition of our Corollary, but this measure is not quasiinvariant under the flow of translations. □

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