De Rham-Witt Cohomology and Displays

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Abstract. Displays were introduced to classify formal p-divisible groups over an arbitrary ring R where p is nilpotent. We define a more general notion of display and obtain an exact tensor category. In many examples the crystalline cohomology of a smooth and proper scheme X over R carries a natural display structure. It is constructed from the relative de Rham-Witt complex. For this we refine the comparison between crystalline cohomology and de Rham-Witt cohomology of [LZ]. In the case where R is reduced the display structure is related to the strong divisibility condition of Fontaine [Fo].

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1 Introduction

Displays of formal p-divisible groups were introduced in [Z2]. They are one possible extension of classical Dieudonné theory to more general ground rings. In [LZ] we gave a direct construction of a display for an abelian scheme by the relative de Rham-Witt complex. In the case where the p-divisible group of the abelian scheme is local the construction leads to the display of [Z2].
We define here a more general notion of display over a ring R, where a given prime number p is nilpotent. If R is a perfect field a display is just a finitely generated free $W(R)$-module M endowed with an injective Frobenius linear map $F : M \to M$, while a display of [Z2] is a Dieudonné module, where V acts topologically nilpotent. Our category of displays is an exact tensor category which contains the displays of [Z2] as a full subcategory. There is also a good notion of base change for displays with respect to arbitrary ring morphisms $R \to R'$. Neither the construction of the tensor product nor the construction of base change is straightforward. Special types of tensor products are related

in [Z2] to biextensions of formal groups. Other operations of linear algebra as exterior products and duals up to Tate twist may be performed but we don't discuss them here, since we don't use them essentially and their construction requires just the same ideas. We add that the exact category of displays is Karoubian [T] and has a derived category.

In many examples we have a display structure on the cohomology of a projective and smooth scheme which arises as follows: Let p be a fixed prime number and let R be a ring such that p is nilpotent in R. We denote by $W(R)$ the ring of Witt vectors and we set $I_R = VW(R)$. Let X be a projective and smooth scheme over R. Let $W\Omega^{\cdot}_{X/R}$ be the de Rham-Witt complex. We define for $m \geq 0$ the Nygaard complex $\mathcal{N}^m W\Omega^{\cdot}_{X/R}$ of sheaves of $W(R)$-modules:

$$(W\Omega^0_{X/R})_{[F]} \xrightarrow{d} \ldots \xrightarrow{d} (W\Omega^{m-1}_{X/R})_{[F]} \xrightarrow{dV} W\Omega^m_{X/R} \xrightarrow{d} W\Omega^{m+1}_{X/R} \xrightarrow{d} \ldots.$$

Here F indicates restriction of scalars with respect to the Frobenius $F : W(R) \to W(R)$. We remark that $\mathcal{N}^0 W\Omega^{\cdot}_{X/R} = W\Omega^{\cdot}_{X/R}$. These complexes were considered by Nygaard, Illusie and Raynaud [I-R], and Kato [K] if R is a perfect field.

Let m be a nonnegative integer and consider the hypercohomology groups

$$P_i = \mathbb{H}^m(X, \mathcal{N}^i W\Omega_{X/R})$$

for $i \geq 0$. The structure of the de Rham-Witt complex gives naturally three sets of maps (compare: Definition 2.2):

1) A chain of morphisms of $W(R)$-modules

$$\ldots \to P_{i+1} \xrightarrow{\iota_i} P_i \to \ldots \to P_1 \xrightarrow{\iota_0} P_0.$$

2) For each $i \geq 0$ a $W(R)$-linear map

$$\alpha_i : I_R \otimes_{W(R)} P_i \to P_{i+1}.$$

3) For each $i \geq 0$ a Frobenius linear map

$$F_i : P_i \to P_0.$$

The composition of ι and α is the multiplication $I_R \otimes P_i \to P_i$. Moreover we have the equation:

$$F_{i+1}(\alpha_i({}^V\eta \otimes x)) = \eta F_i x, \quad \eta \in I_R,\ x \in P_i \tag{1}$$

We will call a set of data $\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$ with the properties above a predisplay. The predisplays form an abelian category. The equation (1) implies:

$$F_i(\iota_i(y)) = pF_{i+1}(y)$$

i.e. the Frobenius F_0 becomes more and more divisible by p if it is restricted to the Nygaard complexes.

We are interested in predisplays, which are obtained by the following construction. We start with a set of data which are called standard:

A sequence $L_0, \ldots, L_d$ of finitely generated projective $W(R)$-modules.

A sequence of Frobenius linear maps for $i = 0, \ldots d$:

$$\Phi_i : L_i \to L_0 \oplus \ldots \oplus L_d$$

We require that the map $\oplus_i \Phi_i$ is a Frobenius linear automorphism of $L_0 \oplus \ldots \oplus L_d$.
From these data one defines a predisplay $\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$, with

$$P_i = (I_R \otimes L_0) \oplus \ldots \oplus (I_R \otimes L_{i-1}) \oplus L_i \oplus \ldots \oplus L_d$$

for $i \in \mathbb{Z}, i \geq 0$. The definition of the maps ι_i, α_i, F_i (compare Definition 2.2) is not obvious, but we skip it for the moment. We should warn the reader that the P_i for $i > d$ are obviously isomorphic, but these isomorphisms are not canonical, i.e. they depend on our construction and not only on the predisplay $\mathcal{P}$.

Definition: *A predisplay is called a display if it is isomorphic to a predisplay associated to standard data.*

A decomposition $P_0 = L_0 \oplus L_1 \oplus \ldots \oplus L_d$ which is given by standard data is called a *normal decomposition*.
If we start with standard data for $d = 1$ we obtain exactly the $3n$-displays of [Z2], which are called displays in [Me]. In this work we call them 1-displays.
If we assume that the L_i are free the map $\oplus \Phi_i$ is represented by a block matrix (A_{ij}), where A_{ij} is the matrix of the Frobenius linear map $L_j \to L_i$ induced by $\oplus \Phi_i$, where $0 \leq i, j \leq d$. Conversely any block matrix (A_{ij}) from $\mathrm{GL}(W(R))$ defines standard data for a display. Over a local ring R it would be possible to define the category of displays in terms of matrices.
We note that the maps ι_i for a display $\mathcal{P}$ are generally not injective unless the ring R is reduced. In this case the whole display is uniquely determined by the Frobenius module (P_0, F_0). Indeed the display property implies that:

$$P_i = \{x \in P_0 \mid F_0(x) \in p^i P_0\} \tag{2}$$

One has $F_i = (1/p^i) F_0$. This makes sense because p is not a zero divisor in $W(R)$ if R is reduced. Therefore over a reduced ring a display is a special kind of Frobenius module.
If $R = k$ is a perfect field a display is just the same as a Frobenius module (P_0, F_0). Indeed, consider the map $F_0 : P_0 \otimes \mathbb{Q} \to P_0 \otimes \mathbb{Q}$. We obtain inclusions of $W(k)$-modules:

$$P_0 \subset F_0^{-1} P_0 \subset P_0 \otimes \mathbb{Q}.$$

By the theory of elementary divisors we find a decomposition by $W(R)$-modules $P_0 = L_0 \oplus L_1 \oplus \ldots \oplus L_d$, such that

$$F_0^{-1} P_0 = L_0 \oplus p^{-1} L_1 \oplus \ldots \oplus p^{-d} L_d.$$

Therefore the restriction of $p^{-i}F_0$ to L_i defines a map $\Phi_i : L_i \to P_0$, for $i = 0, \dots, d$. These are the standard data for the display associated to the Frobenius module (P_0, F_0).
If $pR = 0$ Moonen and Wedhorn [MW] introduced the structure of an F-zip. It is defined in terms of the de Rham cohomology of the scheme X/R. As one should expect any display gives rise to an F-zip (compare the remark after Definition 2.6.).
For an arbitrary projective and smooth variety X/R we can't expect that the crystalline cohomology $H^m_{crys}(X/W(R))$ has a display structure. Therefore we consider the following assumptions: There is a compatible system of smooth liftings $\tilde{X}_n/W_n(R)$ for $n \in \mathbb{N}$ of X/R such that the following properties hold:

(*) The cohomology groups $H^j(\tilde{X}_n, \Omega^i_{\tilde{X}_n/W_n(R)})$ are for each n, i and j locally free $W_n(R)$-modules of finite type.

(**) The de Rham spectral sequence degenerates at E_1

$$E_1^{ij} = H^j(\tilde{X}_n, \Omega^i_{\tilde{X}_n/W_n(R)}) \Rightarrow \mathbb{H}^{i+j}(\tilde{X}_n, \Omega^\cdot_{\tilde{X}_n/W_n(R)}).$$

Theorem: *Let X be smooth and projective over a reduced ring R, such that the assumptions (*) and (**) are satisfied. Let d be an integer $0 \leq m < p$. Consider the Frobenius module $P_0 = H^m_{crys}(X/W(R))$ and define P_i by the formula (2).*
Then the P_i form a display and P_i coincides with the hypercohomology of the Nygaard complex $\mathcal{N}^i W\Omega^\cdot_{X/R}$.
It would follow from the general conjecture made below that this theorem holds without the restriction $m < p$.
Finally we indicate how to proceed if the ring R is not reduced. In order to overcome the problem with the p-torsion in $W(R)$ we use frames [Z1]. A frame for R is a triple (A, σ, α), where A is a p-adic ring without p-torsion, $\sigma : A \to A$ is an endomorphism which lifts the Frobenius on A/pA, and $\alpha : A \to R$ is a surjective ring homomorphism whose kernel has divided powers. Let us assume that X admits a lifting to a smooth formal scheme $\mathcal{Y}$ over $\operatorname{Spf} A$, which satisfies assumptions analogous to (*) and (**). We define "displays" relative to A which we call windows (see [Z1]). Theorem 5.5 says that under the conditions made $H^m_{crys}(X/A, \mathcal{O}_{X/A})$ has a window structure for $m < p$. There is a morphism $A \to W(A) \to W(R)$ which allows to pass from windows to displays. We remark that because of this morphism the assumptions (*) and (**) for A are stronger than the original assumption for $W(R)$. In equal characteristic we obtain e.g. the following:

Theorem *Let X be smooth and projective over a ring R, such that $pR = 0$. Let us assume that there is a frame $A \to R$ and a smooth p-adic lifting $\mathcal{Y}/\operatorname{Spf} A$ of X, which satisfies the conditions analogous to (*) and (**).*
Then there is a canonical display structure on $H^m_{crys}(X/W(R))$ for $m < p$, which does not depend on the lifting $\mathcal{Y}$ nor on the frame A.

We discuss three examples where the assumptions $(*)$ and $(**)$ hold. In these examples the assumptions made on X in the two preceding theorems are fullfilled.

Let X be a $K3$-surface over R. We assume without restriction of generality that R is noetherian. We denote by $\mathcal{T}_{X/R}$ the tangent bundle of X. The cohomology group $H^2(X, \mathcal{T}_{X/R})$ commutes with base change by [M] §5 Cor.3. From the case where R is an algebraically closed field, we deduce that this cohomology group vanishes. It follows that X has a formal lifting over $\operatorname{Spf} W(R)$ resp. $\operatorname{Spf} A$. From the Hodge numbers of a K3-surface over an algebraically closed field [De1] one deduces that $H^1(X, \mathcal{O}_X) = 0$, $H^0(X, \Omega^1_{X/R}) = 0$, $H^2(X, \Omega^1_{X/R}) = 0$, $H^1(X, \Omega^2_{X/R}) = 0$. It follows that the cohomology of X commutes with arbitrary base change and is therefore locally free [M] loc.cit.. The degeneration of the de Rham spectral sequence follows now because the Hodge numbers above are zero, because there is no room for non-zero differentials.

Let X be an abelian variety over R. In this case the assumptions $(*)$ and $(**)$ are fullfilled by [BBM] 2.5.2.

Finally let X be a smooth relative complete intersection in a projective space over R. Then the conditions $(*)$ and $(**)$ are fullfilled by [De2] Thm.1.5.

Let p be a prime number. Let R be a ring such that p is nilpotent in R. In [LZ] Thm. 3.5 we proved a comparison between the crystalline cohomology and the hypercohomology of the de Rham-Witt complex extending a result of Illusie [I] if R is a perfect field. We show here a filtered version of this comparison, which is the key to the display structure. We conjecture a more precise comparison, which would lead to a wide generalization of the theorems above.

Let $W_n(R)$ be the truncated Witt vectors. We set $I_{R,n} = VW_{n-1}(R)$. This ideal is 0 for $n = 1$.

Let X/R be a smooth and projective scheme. We consider the crystalline site $Crys(X/W_n(R))$ with its structure sheaf $\mathcal{O}_{X/W_n(R)}$. Let us denote by $\mathcal{J}_{X/W_n(R)} \subset \mathcal{O}_{X/W_n(R)}$ the sheaf of pd-ideals. We denote by $\mathcal{J}^{[m]}_{X/W_n(R)}$ its m-th divided power. Let

$$u_n : Crys(X/W_n(R))^{\sim} \longrightarrow X^{\sim}_{zar}$$

be the canonical morphism of topoi.

The comparison isomorphism [LZ] is an isomorphism in the derived category $D(X_{zar})$ of sheaves of $W_n(R)$-modules on X_{zar}:

$$Ru_{n*}\mathcal{O}_{X/W_n(R)} \longrightarrow W_n\Omega^{\cdot}_{X/R}$$

We will prove a filtered version of this. Let m be a natural number. Let $\mathcal{I}^m W_n\Omega^{\cdot}_{X/R}$ be the following subcomplex of the de Rham-Witt complex:

$$p^{m-1}VW_{n-1}\Omega^0_{X/R} \xrightarrow{d} p^{m-2}VW_{n-1}\Omega^1_{X/R} \ldots \xrightarrow{d} VW_{n-1}\Omega^{m-1}_{X/R} \xrightarrow{d} W_n\Omega^m_{X/R} \ldots$$

The filtered comparison Theorem 4.6 says that for $m < p$ we have an isomorphism in the derived category

$$Ru_{n*}\mathcal{J}^{[m]}_{X/W_n(R)} \longrightarrow \mathcal{I}^m W_n\Omega^{\cdot}_{X/R} \tag{3}$$

We would like to have a similar comparison theorem for the truncated Nygaard complex $\mathcal{N}^m W_n\Omega^{\cdot}_{X/R}$ instead of $\mathcal{I}^m W_n\Omega^{\cdot}_{X/R}$:

$$(W_{n-1}\Omega^0_{X/R})_{[F]} \xrightarrow{d} \ldots \xrightarrow{d} (W_{n-1}\Omega^{m-1}_{X/R})_{[F]} \xrightarrow{dV} W_n\Omega^m_{X/R} \xrightarrow{d} W_n\Omega^{m+1}_{X/R} \xrightarrow{d} \ldots$$

The advantage of the Nygaard complex is that the restriction of the Frobenius from $W\Omega^{\cdot}_{X/R}$ to $\mathcal{N}^m W\Omega^{\cdot}_{X/R}$ is in a natural way divisible by p^m even if p is a zero divisor. For a reduced ring R the Nygaard complex $\mathcal{N}^m W\Omega^{\cdot}_{X/R}$ is quasi-isomorphic to $\mathcal{I}^m W\Omega^{\cdot}_{X/R}$. Unfortunately in general we don't know a definition for the Nygaard complex in terms of crystalline cohomology. Nevertheless we make the conjecture 4.1:

Conjecture: *Assume that $\tilde{X}/W_n(R)$ is a smooth lifting of X. Then the Nygaard complex is in the derived category canonically isomorphic to the following complex $\mathcal{F}^m\Omega^{\cdot}_{\tilde{X}/W_n(R)}$:*

$$I_{R,n} \otimes_{W_n(R)} \Omega^0_{\tilde{X}/W_n(R)} \xrightarrow{pd} \ldots \xrightarrow{pd} I_{R,n} \otimes_{W_n(R)} \Omega^{m-1}_{\tilde{X}/W_n(R)} \xrightarrow{d} \Omega^m_{\tilde{X}/W_n(R)} \xrightarrow{d} \ldots.$$

Assume that we have for varying n a compatible system of smooth liftings $\tilde{X}_n/W_n(R)$. We obtain a formal scheme $\mathcal{X} = \varinjlim \tilde{X}_n$. We set:

$$\mathcal{F}^m\Omega^{\cdot}_{\mathcal{X}/W(R)} = \varprojlim_n \mathcal{F}^m\Omega^{\cdot}_{\tilde{X}_n/W_n(R)} \qquad \mathcal{N}^m W\Omega^{\cdot}_{X/R} = \varprojlim_n \mathcal{N}^m W_n\Omega^{\cdot}_{X/R}$$

We show the following weak form of the conjecture (Corollary 4.7):

Theorem: *Assume that R is reduced and that $m < p$. Then there is a natural isomorphism in the derived category of $W(R)$-modules on X_{zar}:*

$$\mathcal{N}^m W\Omega^{\cdot}_{X/R} \cong \mathcal{F}^m\Omega^{\cdot}_{\mathcal{X}/W(R)}$$

Moreover we can show in support of our conjecture, that the complexes $\mathcal{N}^m W_n\Omega^{\cdot}_{X/R}$ and $\mathcal{F}^m\Omega^{\cdot}_{\tilde{X}_n/W_n(R)}$ are always locally quasi-isomorphic on X_{zar}.

The last theorem is closely related to strong divisibility in the sense of [Fo] 1.3: Assume the assumptions $(*)$ and $(**)$ are satisfied. By the last theorem the splitting of the Hodge filtration of the formal scheme $\mathcal{X}$ defines a normal decomposition:

$$\mathbb{H}^m(X, \mathcal{F}^j\Omega^{\cdot}_{\mathcal{X}/W(R)}) = I_R L_0 \oplus \ldots \oplus I_R L_{j-1} \oplus L_j \oplus \ldots \oplus L_d$$

It is obvious from Definition 2.2 that the Frobenius $F_j : H^m(X, \mathcal{N}^j W\Omega^{\cdot}_{X/R}) \to \mathbb{H}^m(X, W\Omega^{\cdot}_{X/R})$ is bijective if j is bigger than the dimension. Therefore $F_0 \oplus F_1 \oplus \ldots \oplus F_d$: induces a bijection:

$$I_R L_0 \oplus \ldots \oplus I_R L_d \to L_0 \oplus \ldots \oplus L_d$$

This is what strong divisibility asserts.

2 The Category of Displays

Let R be a ring, and let $W(R)$ be the ring of Witt vectors. We set $I_R = VW(R)$. If no confusion is possible we sometimes use the abbreviation $I = I_R$. Let $\Phi : M \to N$ a Frobenius-linear homomorphism of $W(R)$-modules. We define a Frobenius-linear homomorphism $\tilde{\Phi}$:

$$\begin{array}{rcl} \tilde{\Phi} : I_R \otimes_{W(R)} M & \to & N \\ {}^V\xi \otimes m & \mapsto & \xi\Phi(m) \end{array} \tag{4}$$

Definition 2.1 *A predisplay over R consists of the following data:*

1) *A chain of morphisms of $W(R)$-modules*

$$\ldots \to P_{i+1} \xrightarrow{\iota_i} P_i \to \ldots \to P_1 \xrightarrow{\iota_0} P_0.$$

2) *For each $i \geq 0$ a $W(R)$-linear map*

$$\alpha_i : I_R \otimes_{W(R)} P_i \to P_{i+1}.$$

3) *For each $i \geq 0$ a Frobenius linear map*

$$F_i : P_i \to P_0.$$

The following axioms should be fulfilled

(D1) *For $i \geq 1$ the diagram below is commutative and its diagonal $I_R \otimes P_i \to P_i$ is the multiplication.*

$$\begin{array}{ccc} I_R \otimes P_i & \xrightarrow{\alpha_i} & P_{i+1} \\ {\scriptstyle I_R \otimes \iota_{i-1}}\downarrow & & \downarrow{\scriptstyle \iota_i} \\ I_R \otimes P_{i-1} & \xrightarrow{\alpha_{i-1}} & P_i \end{array}$$

For $i = 0$ the following map is the multiplication:

$$I_R \otimes P_0 \xrightarrow{\alpha_0} P_1 \xrightarrow{\iota_0} P_0$$

(D2) $F_{i+1}\alpha_i = \tilde{F}_i : I_R \otimes P_i. \to P_0$

We will denote a predisplay as follows:

$$\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i), \quad i \in \mathbb{Z}_{\geq 0}.$$

Let X be a smooth and proper scheme over a scheme S. Then we obtain a predisplay structure on the crystalline cohomology through the Nygaard complexes $\mathcal{N}^m W_n\Omega_{X/S}$ which are built from the de Rham-Witt complex as follows:

$$(W_{n-1}\Omega^0_{X/S})_{[F]} \xrightarrow{d} \ldots \xrightarrow{d} (W_{n-1}\Omega^{m-1}_{X/S})_{[F]} \xrightarrow{dV} W_n\Omega^m_{X/S} \xrightarrow{d} W_n\Omega^{m+1}_{X/S} \ldots$$

This is considered as a complex of $W_n(\mathcal{O}_S)$-modules. The index $[F]$ means that we consider this term as a $W_n(\mathcal{O}_S)$-module via restriction of scalars $F: W_n(\mathcal{O}_S) \to W_{n-1}(\mathcal{O}_S)$.
Let $I_{S,n} = VW_{n-1}(\mathcal{O}_S) \subset W_n(\mathcal{O}_S)$ be the sheaf of ideals. We define three sets of maps:

$$
\begin{array}{rccc}
\hat{\alpha}_m : & I_{S,n} \otimes_{W_n(\mathcal{O}_S)} \mathcal{N}^m W_n\Omega^{\cdot}_{X/S} & \to & \mathcal{N}^{m+1} W_n\Omega^{\cdot}_{X/S} \\
\hat{\iota}_m : & \mathcal{N}^{m+1} W_n\Omega^{\cdot}_{X/S} & \to & \mathcal{N}^m W_n\Omega^{\cdot}_{X/S} \\
\hat{F}_m : & \mathcal{N}^m W_n\Omega^{\cdot}_{X/S} & \to & W_{n-1}\Omega^{\cdot}_{X/S}
\end{array}
\tag{5}
$$

These maps are given in this order by the maps between the following vertically written procomplexes (the index n is omitted):

$$
\begin{array}{ccccccc}
I_S \otimes (W\Omega^0_{X/S})_{[F]} & \longrightarrow & (W\Omega^0_{X/S})_{[F]} & \xrightarrow{p} & (W\Omega^0_{X/S})_{[F]} & \xrightarrow{id} & W\Omega^0_{X/S} \\
{\scriptstyle I_S \otimes d}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow \\
\cdots & & \cdots & & \cdots & & \cdots \\
I_S \otimes (W\Omega^{m-1}_{X/S})_{[F]} & \longrightarrow & (W\Omega^{m-1}_{X/S})_{[F]} & \xrightarrow{p} & (W\Omega^{m-1}_{X/S})_{[F]} & \xrightarrow{id} & W\Omega^{m-1}_{X/S} \\
{\scriptstyle \mathrm{id} \otimes dV}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle dV}\downarrow & & {\scriptstyle d}\downarrow \\
I_S \otimes (W\Omega^{m}_{X/S}) & \xrightarrow{\tilde{F}} & (W\Omega^{m}_{X/S})_{[F]} & \xrightarrow{V} & W\Omega^{m}_{X/S} & \xrightarrow{F} & W\Omega^{m}_{X/S} \\
{\scriptstyle \mathrm{id} \otimes d}\downarrow & & {\scriptstyle dV}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow \\
I_S \otimes W\Omega^{m+1}_{X/S} & \xrightarrow{mult} & W\Omega^{m+1}_{X/S} & \xrightarrow{id} & W\Omega^{m+1}_{X/S} & \xrightarrow{pF} & W\Omega^{m+1}_{X/S} \\
{\scriptstyle \mathrm{id} \otimes d}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow & & {\scriptstyle d}\downarrow \\
I_S \otimes W\Omega^{m+2}_{X/S} & \xrightarrow{mult} & W\Omega^{m+2}_{X/S} & \xrightarrow{id} & W\Omega^{m+2}_{X/S} & \xrightarrow{p^2F} & W\Omega^{m+2}_{X/S} \\
\cdots & & \cdots & & \cdots & & \cdots
\end{array}
$$

The first unlabeled arrows on the left hand side denote the maps ${}^V\xi \otimes \omega \mapsto \xi\omega$, where the product is taken in $W\Omega^i_{X/S}$ (without restriction of scalars).

Definition 2.2 *Let $S = \operatorname{Spec} R$ be an affine scheme. Let X/S be a smooth and proper scheme. Then we associate a predisplay. We set:*

$$P_i = \mathbb{H}^d(X, \mathcal{N}^i W\Omega_{X/S})$$

The predisplay structure on the P_i is easily obtained by taking the cohomology of the maps (5).

Here we write $\mathcal{N}^m W\Omega^{\cdot}_{X/R} = \varprojlim_n \mathcal{N}^m W_n\Omega^{\cdot}_{X/R}$. The P_i coincide with the cohomology of $R\varprojlim_n R\Gamma(X, \mathcal{N}^i W_n\Omega_{X/S})$ by the proof of [LZ] Prop. 1.13 (compare [BO] Appendix).

Remark: Let $S = \operatorname{Spec} k$ be the spectrum of a perfect field. Then $I(k)$ is isomorphic to $W(k)$ as $W(k)$-module. The maps of complexes which define $\hat{\alpha}_i$ and $\hat{\iota}_i$ are in this case the maps $\tilde{F}$ and $\tilde{V}$ used by Kato in his definition of the F-gauges $GH^d(X/S)$.

Let A/S be an abelian scheme. Then the predisplay structure on the crystalline cohomology $H^1(A/W(R), \mathcal{O}_{A/W(R)})$ is in fact a 3n-display structure in the sense of [Z2]. We will introduce additional properties of predisplay structures which arise in the crystalline cohomology of smooth and proper varieties.
Let $\mathcal{P}$ be a predisplay. Then we have a commutative diagram:

$$\begin{array}{ccc} P_i & \xrightarrow{F_i} & P_0 \\ {\scriptstyle\iota_i}\uparrow & & \uparrow{\scriptstyle p} \\ P_{i+1} & \xrightarrow{F_{i+1}} & P_0 \end{array} \tag{6}$$

Indeed, let $y \in P_{i+1}$. Then we obtain from (D1) that

$$\alpha_i({}^V 1 \otimes \iota_i(y)) = {}^V 1 y$$

If we apply F_{i+1} to the last equation and use (D2), we obtain:

$$F_i(\iota_i(y)) = pF_{i+1}(y)$$

Definition 2.3 *A predisplay $\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$ is called separated if the map of P_{i+1} to the fibre product induced by the commutative diagram (6) is injective.*

Remark: Predisplays form obviously an abelian category. To each predisplay $\mathcal{P}$ we can associate a separated predisplay $\mathcal{P}^{sep}$ and a canonical surjection $\mathcal{P} \to \mathcal{P}^{sep}$. This is defined inductively: $P_0^{sep} = P_0$ and P_{i+1}^{sep} is the image of P_{i+1} in the fibre product of:

$$P_i^{sep} \xrightarrow{F_i^{sep}} P^0 \xleftarrow{p} P^0$$

The functor $\mathcal{P} \mapsto \mathcal{P}^{sep}$ to the category of separated displays is left adjoint to the forgetful functor, but it is not exact.

It is not difficult to prove that a separated predisplay has the following property: Consider the iteration of the maps α:

$$I^{\otimes k} \otimes P_i \xrightarrow{\alpha_i} I^{\otimes k-1} \otimes P_{i+1} \xrightarrow{\alpha_{i+1}} \ldots \xrightarrow{\alpha_{i+k-1}} P_{i+k} \tag{7}$$

Here the maps α pick up the last factor of $I^{\otimes}$. The following map is called the "Verjüngung":

$$\begin{array}{cccc} \nu^{(k)}: & I^{\otimes k} & \to & I \\ & {}^V\xi_1 \otimes \ldots \otimes {}^V\xi_k & \mapsto & {}^V(\xi_1 \cdot \ldots \cdot \xi_k) \end{array} \tag{8}$$

For a separated display the iteration (7) factors through the Verjüngung:

$$I^{\otimes k} \otimes P_i \xrightarrow{\nu^{(k)}} I \otimes P_i \longrightarrow P_{i+k}$$

The last arrow will be called $\alpha_i^{(k)}$. In particular this shows that the iteration (7) is independent of the factors we picked up, when forming α_j.
For a separated display the data α_i, $i \geq 0$ are uniquely determined by the remaining data. This is seen by the following commutative diagram:

$$\begin{tikzcd}
& P_i \arrow[r, "F_i"] & P_0 & \\
& P_{i+1} \arrow[u, "\iota_i"] \arrow[r, "F_{i+1}"] & P_0 \arrow[u, "p"] & (*) \\
I \otimes P_i \arrow[uur] \arrow[ur, "\alpha_i"'] \arrow[urr, "\tilde{F}_i"'] & & &
\end{tikzcd}$$

For a predisplay $\mathcal{P}$ the cokernel $E_{i+1} := Coker\ \alpha_i$ is annihilated by I. It is therefore an R-module.

Definition 2.4 *We say that a predisplay is of degree d (or a d-predisplay), if the maps α_i are surjective for $i \geq d$.*

A separated predisplay of degree d is already uniquely determined by the data:

$$P_0, \ldots P_d, \iota_0, \ldots \iota_{d-1}, F_0, \ldots, F_d, \alpha_0, \ldots, \alpha_{d-1} \tag{9}$$

For this consider the diagram $(*)$ above for $i = d$. The data already given determine a map of $I \otimes P_d$ to the fibre product. This map is α_d and the image is P_{d+1}. Thus inductively all data of the display are uniquely determined.
Conversely assume that we have data (9) which satisfy all predisplay axioms reasonable for these data. Then we define P_{d+1} by the diagram $(*)$ above. We obtain also the maps α_d, ι_d, and F_{d+1}. The axioms for the extended data are trivially satisfied, except for the requirement that

$$I \otimes P_{d+1} \rightarrow I \otimes P_d \rightarrow P_{d+1}$$

is the multiplication. But this follows easily by composing the diagram $(*)$ for $i = d$, with the arrow $\mathrm{id} \otimes \iota_d : I \otimes P_{i+1} \to I \otimes P_i$. Inductively we see that a set of data (9) satisfying the predisplay axioms may be extended uniquely to a predisplay of degree d.

We may define the twist of a predisplay. Let

$$\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$$

be a predisplay. Then we define its *Tate-twist*

$$\mathcal{P}(1) = (P'_i, \iota'_i, \alpha'_i, F'_i) \tag{10}$$

as follows: For $i \geq 1$ we set $P'_i = P_{i-1}, \iota'_i = \iota_{i-1}, \alpha'_i = \alpha_{i-1}, F'_i = F_{i-1}$. We set $P'_0 = P_0 = P'_1, F'_0 = pF_0, \iota'_0 = \mathrm{id}_{P_0}$. Finally $\alpha'_0 : I \otimes P_0 \to P_0$ is defined to be the multiplication. If we repeat this n-times we write $\mathcal{P}(n)$.

We define a predisplay $\mathcal{U} = (P_i, \iota_i, \alpha_i, F_i)$ called the *unit display* as follows: $P_0 = W(R)$, $P_i = I$ for $i \geq 1$. The chain of the maps ι is as follows:

$$\ldots I \xrightarrow{p} I \ldots \xrightarrow{p} I \to W(R), \tag{11}$$

where the last map ι_0 is the natural inclusion.
The maps $F_i : I = P_i \to W(R)$ for $i \geq 1$ coincide with the map

$$V^{-1} : I \to W(R), \qquad {}^{V}\xi \mapsto \xi.$$

The map F_0 is the Frobenius on $W(R)$. The map $\alpha_0 : I \otimes W(R) \to I$ is the multiplication. The maps $\alpha_i : I \otimes I \to I$ are the Verjüngung $\nu^{(2)}$. Since the "Verjüngung" is surjective the unit display has degree zero.

A 3n-display (P, Q, F, V^{-1}) as defined in [Z2] gives naturally rise to data of type (9) with $P_0 = P$, $P_1 = Q$, $F_0 = F$, $F_1 = V^{-1}$ and therefore extends naturally to a predisplay of degree 1 as we explained above. We will make this explicit later on.

Let R be a reduced ring. Then the multiplication by p is injective on $W(R)$. Let M be a projective $W(R)$-module, and $F : M \to M$ be a Frobenius linear map. Then we set:

$$P_i = \{x \in M \mid F(x) \in p^i M\}$$

We obtain maps

$$F_i = (1/p^i)F : P_i \to P_0 = M$$

For ι_i we take the natural inclusion $P_{i+1} \to P_i$. For α_i we take the maps $I \otimes P_i \to IP_i \subset P_{i+1}$ induced by multiplication. The data $(P_i, \iota_i, \alpha_i, F_i)$ constructed in this way are a separated predisplay.

The predisplays we are interested in arise from a construction which we explain now.

Definition 2.5 *The following set of data we will call standard data for a display of degree d.*

A sequence $L_0, \ldots, L_d$ of finitely generated projective $W(R)$-modules.

A sequence of Frobenius linear maps for $i = 0, \ldots d$:

$$\Phi_i : L_i \to L_0 \oplus \ldots \oplus L_d$$

We require that the map $\oplus_i \Phi_i$ is a Frobenius linear automorphism of $L_0 \oplus \ldots \oplus L_d$

From these data we obtain a predisplay in the following manner: We set:

$$P_i = (I \otimes L_0) \oplus \ldots \oplus (I \otimes L_{i-1}) \oplus L_i \oplus \ldots \oplus L_d$$

for $i \in \mathbb{Z}, i \geq 0$.

We note that $P_i = P_{d+1}$ for $i > d$. But these identifications are not part of the predisplay structure we are going to define. They depend on the standard data!

We define "divided" Frobenius maps:

$$F_i : P_i \to P_0$$

The restriction of F_i to $I \otimes L_k$ for $k < i$ is $\tilde{\Phi}_k$, and to L_{i+j} for $j \geq 0$ is $p^j \Phi_{i+j}$.

The map $\iota_i : P_{i+1} \to P_i$ is given by the following diagram:

$$\begin{array}{ccccccccc} (I \otimes L_0) \oplus & \ldots & \oplus (I \otimes L_{i-1}) \oplus & (I \otimes L_i) \oplus & L_{i+1} \oplus & \ldots & \oplus L_d \\ \downarrow p & & \downarrow p & \downarrow mult & \downarrow \mathrm{id} & & \downarrow \mathrm{id} \\ (I \otimes L_0) \oplus & \ldots & \oplus (I \otimes L_{i-1}) \oplus & L_i & \oplus L_{i+1} \oplus & \ldots & \oplus L_d \end{array} \tag{12}$$

The map $\alpha_i : I \otimes P_i \to P_{i+1}$ is given by the following diagram:

$$\begin{array}{ccccccc} I \otimes (I \otimes L_0) \oplus & \ldots & \oplus I \otimes (I \otimes L_{i-1}) \oplus & I \otimes L_i & \oplus I \otimes L_{i+1} \oplus & \ldots & \oplus I \otimes L_d \\ \downarrow \nu & & \downarrow \nu & \downarrow \mathrm{id} & \downarrow mult & & \downarrow mult \\ (I \otimes L_0) & \oplus \ldots \oplus & (I \otimes L_{i-1}) & \oplus (I \otimes L_i) \oplus & L_{i+1} & \oplus \ldots \oplus & L_d \end{array} \tag{13}$$

Here $\nu = \nu^{(2)}$ is the Verjüngung. We leave the verification that $\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$ is a separated predisplay to the reader.

DEFINITION 2.6 *A predisplay is called a display if it is isomorphic to a predisplay associated to standard data.*

Remark: Let us assume that $pR = 0$. There is the notion of an F-zip by Moonen and Wedhorn. The relation to displays is as follows. Let $\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$ be a display over R. We define an F-zip structure on $M = P_0/I_R P_0$ by the following two filtrations. Let C^i as the image of P_i in $P_0/I_R P_0$ given by the composite of the maps ι_k. This gives the decreasing "Hodge filtration":

$$\ldots \subset C^d \subset C^{d-1} \subset \ldots \subset C^1 \subset C^0 = M.$$

We set $D_i = W(R)F_i P_i + I_R P_0/I_R P_0$ and obtain an increasing filtration, called the "conjugate filtration":

$$0 = D_{-1} \subset D_0 \subset D_1 \subset D_2 \subset \ldots \subset D_d \subset \ldots \subset M.$$

The morphisms F_i for $i \geq 0$ induce Frobenius linear morphisms:

$$F_i : C_i/C_{i+1} \to D_i/D_{i-1} \tag{14}$$

These are Frobenius linear isomorphisms of R-modules. Indeed, if we choose a normal decomposition $\{L_i\}$ we obtain identification:

$$C^i/C^{i+1} \cong L_i/I_R L_i \quad \text{and} \quad D_i/D_{i-1} \cong W(R)F_i L_i/I_R W(R) F_i L_i$$

The two filtrations $C^{\cdot}$ and $D_{\cdot}$ together with the operators (14) form an F-zip [MW] Def. 1.5.

Let $\mathcal{P}$ be the display associated to the standard data (L_i, Φ_i) as above. Let $\mathcal{Q} = (Q_i, \iota_i, \alpha_i, F_i)$ be a predisplay. Assume we are given homomorphisms $\rho_i : L_i \to Q_i$. Then we define maps τ_i:

$$P_i = (I \otimes L_0) \oplus \ldots \oplus (I \otimes L_{i-1}) \oplus L_i \oplus \ldots \oplus L_d \longrightarrow Q_i$$

On the summand $(I \otimes L_{i-k})$ the map τ_i is the composite:

$$I \otimes L_{i-k} \xrightarrow{\mathrm{id} \otimes \rho_{i-k}} I \otimes Q_{i-k} \xrightarrow{\alpha^{(k)}} Q_k$$

On the summand L_{i+j} the map τ_i is the composite:

$$L_{i+j} \xrightarrow{\rho_{i+j}} Q_{i+j} \xrightarrow{\iota^{(j)}} Q_i,$$

where the last arrow is a compositions of $\iota' s$.

Proposition 2.7 *The maps τ_i define a homomorphism of predisplays $\mathcal{P} \to \mathcal{Q}$ if and only if the following diagrams are commutative:*

$$\begin{tikzcd} L_i \arrow[r, "\rho_i"] \arrow[d, "\Phi_i"'] & Q_i \arrow[d, "F_i"'] \\ P_0 \arrow[r, "\tau_0"] & Q_0 \end{tikzcd}$$

We omit the verification.

If $\mathcal{P} = (P, Q, F, V^{-1})$ is 3n-display in the sense of [Z2], then any normal decomposition $P = L_0 \oplus L_1$, $Q = IL_0 \oplus L_1$ defines standard data, which determine this display.

We will now define the tensor product of displays: Assume that $\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$ and $\mathcal{P}' = (P'_i, \iota'_i, \alpha'_i, F'_i)$ are displays over R.

A tensor product $\mathcal{T} = (T_i, \overset{\circ}{\iota}_i, \overset{\circ}{\alpha}_i, \overset{\circ}{F}_i)$ may be constructed as follows. We choose normal decompositions

$$P_0 = \underset{n\geq 0}{\oplus} L_n, \quad P'_0 = \underset{n\geq 0}{\oplus} L'_n.$$

More precisely this means, that we fix isomorphisms of $\mathcal{P}$ resp. $\mathcal{P}'$ with standard displays. We obtain:

$$P_i = I \otimes L_0 \oplus \cdots \oplus I \otimes L_{i-1} \oplus L_i \oplus \ldots$$

We denote the restriction of $F_i : P_i \longrightarrow P_0$ to the direct summand L_i by Φ_i. We obtain data for a standard display $K_l, \overset{\circ}{\Phi}_l, l \geq 0$, if we set

$$K_l = \underset{n+m=l}{\oplus} (L_n \otimes L'_m).$$

Then $\oplus_l K_l = P_0 \otimes P'_0$, and we define Frobenius linear maps

$$\overset{\circ}{\Phi}_l : K_l \longrightarrow P_0 \otimes P'_0,$$

by

$$\overset{\circ}{\Phi}_l = \sum_{n+m=l} \Phi_n \otimes \Phi'_m$$

From the standard data $K_l, \overset{\circ}{\Phi}_l$ we obtain a display

$$\mathcal{T} = (T_i, \overset{\circ}{\iota}_i, \overset{\circ}{\alpha}_i, \overset{\circ}{F}_i) \tag{15}$$

We will show that $\mathcal{T}$ is up to canonical isomorphism independent of the normal decompositions of $\mathcal{P}$ resp. $\mathcal{P}'$.

In order to do this we define bilinear forms of displays. Let $\mathcal{T}$ be an arbitrary predisplay. A bilinear form

$$\lambda : \mathcal{P} \times \mathcal{P}' \longrightarrow \mathcal{T}.$$

consists of the following data.
λ is a sequence of maps of $W(R)$-modules

$$\lambda_{ij} : P_i \otimes P'_j \longrightarrow T_{i+j}.$$

We require that the following diagrams are commutative:

$$\begin{tikzcd}
P_i \otimes P'_j \arrow[r] \arrow[d, "F_i \otimes F'_j"'] & T_{i+j} \arrow[d, "\overset{\circ}{F}_{i+j}"] \\
P_0 \otimes P'_0 \arrow[r, "\mathrm{id}"'] & T_0
\end{tikzcd}$$

$$\begin{tikzcd}
P_i \otimes P'_j \arrow[r] & T_{i+j} & P_i \otimes P'_j \arrow[r] & T_{i+j} \\
P_{i+1} \otimes P'_j \arrow[r] \arrow[u, "\iota \otimes \mathrm{id}"] & T_{i+j+1} \arrow[u, "\overset{\circ}{\iota}"'] & P_i \otimes P'_{j+1} \arrow[r] \arrow[u, "\mathrm{id} \otimes \iota'"] & T_{i+j+1} \arrow[u, "\overset{\circ}{\iota}"']
\end{tikzcd}$$

$$\begin{tikzcd}
I_R \otimes P_i \otimes P'_j \arrow[r] \arrow[d, "\alpha_i \otimes \mathrm{id}"'] & I_R \otimes T_{i+j} \arrow[d, "\overset{\circ}{\alpha}_{i+j}"] & I_R \otimes P_i \otimes P'_j \arrow[r] \arrow[d, "\mathrm{id} \otimes \alpha'_j"'] & I \otimes T_{i+j} \arrow[d, "\overset{\circ}{\alpha}_{i+j}"] \\
P_{i+1} \otimes P'_j \arrow[r] & T_{i+j+1} & P_i \otimes P'_{j+1} \arrow[r] & T_{i+j+1}.
\end{tikzcd}$$

Remark: We will consider also the maps

$$P_i \otimes P_j \longrightarrow T_k, \text{ for } i + j \geq k,$$

which are the compositions of λ_{ij} and $T_{i+j} \longrightarrow T_k$, the iteration of ι.
If $i + j > k$ we obtain a commutative diagram:

$$\begin{array}{cccc} & P_{i-1} \otimes P_j & \longrightarrow & T_k \\ \iota \otimes \mathrm{id} & \uparrow & & \uparrow \\ & P_i \otimes P_j & \longrightarrow & T_{k+1}. \end{array} \tag{16}$$

We will denote the set of bilinear forms of displays in this sense by

$$Bil(\mathcal{P} \times \mathcal{P}', \mathcal{T}).$$

We return to the display $\mathcal{T}$ given by the standard data $K_l, \overset{\circ}{\Phi}_l$. We will now define maps $\lambda_{ij} : P_i \otimes P'_j \longrightarrow T_{i+j}$. For this we write $P_i \otimes P'_j$ according to the normal decompositions:

$$\begin{aligned}P_i \otimes P'_j = (\bigoplus_{\substack{n<i\\ m<j}} I \otimes I \otimes L_n \otimes L'_m) \oplus (\bigoplus_{\substack{n<i\\ m\geq j\\ m+n<i+j}} I \otimes L_n \otimes L'_m)\\ \oplus(\bigoplus_{\substack{n\geq i\\ m<j\\ n+m<i+j}} (I \otimes L_n \otimes L'_m)) \oplus (\bigoplus_{\substack{n<i\\ m\geq j\\ n+m\geq i+j}} I \otimes L_n \otimes L'_m)\\ \oplus(\bigoplus_{\substack{m\geq i\\ m\leq j\\ n+m\geq i+j}} (I \otimes L_n \otimes L'_m) \oplus (\bigoplus_{\substack{n\geq i\\ m\geq j}} L_n \otimes L'_m).\end{aligned} \tag{17}$$

We have six direct sums in brackets, which we denote by Z_i, $i = 1, \dots, 6$ in the order as above.
By definition T_{i+j} has the decomposition

$$T_{i+j} = (\bigoplus_{n+m<i+j} I \otimes L_n \otimes L'_m) \oplus (\bigoplus_{n+m\geq i+j} L_n \otimes L'_m). \tag{18}$$

We define $\lambda_{ij} : P_i \otimes P'_j \longrightarrow T_{i+j}$ as a bigraded map with respect to $n, m \geq 0$, which is on the homogeneous components as follows.
Case Z_1: $n < i, m < j$

$$\begin{aligned}I \otimes I \otimes L_n \otimes L'_m &\longrightarrow I \otimes L_n \otimes L'_m\\ {}^V\xi \otimes {}^V\eta \otimes l_n \otimes l'_m &\longmapsto {}^V(\xi\eta) \otimes l_n \otimes l'_m\end{aligned}$$

Case Z_2: $n < i,\ m \geq j,\ n + m < i + j$

$$p^{m-j}\,\mathrm{id} : I \otimes L_n \otimes L'_m \longrightarrow I \otimes L_n \otimes L'_m$$

Case Z_3: $n \geq i,\ m < j,\ n + m < i + j$

$$p^{n-i}\,\mathrm{id} : I \otimes L_n \otimes L'_m \longrightarrow I \otimes L_n \otimes L'_m$$

Case Z_4: $n < i,\ m \geq j,\ n + m \geq i + j$

$$p^{i-n-1}\,\mathrm{id} : I \otimes L_n \otimes L'_m \longrightarrow I \otimes L_n \otimes L'_m$$

Case Z_5: $n \geq i,\ m < j,\ n + m \geq i + j$

$$p^{j-m-1}\,\mathrm{id} : I \otimes L_n \otimes L'_m \longrightarrow I \otimes L_n \otimes L'_m$$

Case Z_6: $n \geq i,\ m \geq j$

$$\mathrm{id} : L_n \otimes L'_m \longrightarrow L_n \otimes L'_m.$$

Proposition 2.8 *The homomorphism* $\lambda_{ij} : P_i \otimes P'_j \longrightarrow T_{i+j}$ *defined by* $Z_1 - Z_6$ *above define a bilinear form of displays.*

Proof: We omit the tedious but simple verification.

Lemma 2.9 *The homomorphism*

$$\oplus_{i+j=k} P_i \otimes P'_j \longrightarrow T_k$$

given by the sum of λ_{ij} *is surjective.*

Proof: We have to show that all summand of (18) are in the image. Consider the submodule $L_n \otimes L'_m \subset T_k$ where $n + m \geq k$. We set $i = n$ and $j = k - i = k - n \leq m$. By Z_6 this submodule is in the image of $P_i \otimes P'_j \longrightarrow T_k$. Next we consider a submodule $I \otimes L_n \otimes L'_m \subset T_k$, where $n + m < k$. We set $i = n$ and $j = k - i = k - n > m$. Thus we are in the case Z_3 with factor $p^{n-i} = 1$. Again the submodule is in the image of $P_i \otimes P'_j \longrightarrow T_k$. *Q.E.D.*

Proposition 2.10 *Let* $\mathcal{P}$ *and* $\mathcal{P}'$ *be displays. Let* $\mathcal{T} = (T_i, \mathring{\iota}_i, \mathring{\alpha}_i, \mathring{F}_i)$ *be the display (15). Let* $\mathcal{Q}$ *be a separated predisplay. There is a canonical isomorphism of abelian groups*

$$Bil(\mathcal{P} \times \mathcal{P}', \mathcal{Q}) \cong \mathrm{Hom}(\mathcal{T}, \mathcal{Q}).$$

Proof: Assume that we are given a bilinear form. We set $\mathcal{T} = \mathcal{P} \otimes \mathcal{P}'$. The maps $T_i \longrightarrow Q_i$ are constructed inductively. For $i = 0$ this map is λ_{00}, where λ denotes the bilinear form. For the induction step to $i + 1$ we consider the diagram

$$\begin{tikzcd} T_i \arrow[r] & Q_i \arrow[r, "F_i"] & Q_0 \\ T_{i+1} \arrow[u] \arrow[r, "F_{i+1}"] & T_0 \arrow[r] & Q_0 \arrow[u, "p"'] \end{tikzcd} \tag{19}$$

We claim that (19) is commutative. By Lemma 2.9 it suffices to show the commutativity if we compose the diagram with the maps $P_s \otimes P'_r \longrightarrow T_{i+1}$, for $s + r = i + 1$. This amounts to the commutativity of the following diagram

$$\begin{tikzcd}
 & Q_i \arrow[r, "F_i"] & Q_0 \\
 & Q_{i+1} \arrow[u, "\iota"'] \arrow[r] & Q_0 \arrow[u, "p"'] \\
P_r \otimes P'_s \arrow[uur] \arrow[ur] \arrow[urr, "F_r \otimes F_s"'] & &
\end{tikzcd}$$

But the diagram is commutative by the definition of a bilinear form. Now the commutativity of (19) gives a map: $T_{i+1} \longrightarrow Q_i \times_{F_i, Q_0, p} Q_0$. It is clear from the diagram above and Lemma 2.9 that this map factors through Q_{i+1}. *Q.E.D.*

Corollary 2.11 *The display (15)*

$$\mathcal{T} = (T_i, \mathring{\iota}_i, \mathring{\alpha}_i, \mathring{F}_i)$$

does not depend up to canonical isomorphism on the normal decompositions of $\mathcal{P}$ and $\mathcal{P}'$. We write

$$\mathcal{T} = \mathcal{P} \otimes \mathcal{P}'$$

This is clear because of the universal property of $\mathcal{T}$ proved in the last proposition. *Q.E.D.*

Remark: Assume that $\mathcal{P}$ and $\mathcal{P}'$ are given by standard data (L_i, Φ_i) and (L'_i, Φ'_i). Assume we are given bilinear forms of $W(R)$-modules:

$$\beta_{ij} : L_i \otimes L'_j \rightarrow Q_{i+j}.$$

Composing this with the iteration of ι, $Q_{i+j} \rightarrow Q_0$, we obtain a bilinear form

$$P_0 \otimes P'_0 = (\oplus_i L_i) \otimes (\oplus_j L'_j) \rightarrow Q_0$$

Let us assume that the following diagrams are commutative:

$$\begin{tikzcd}
L_i \otimes L'_j \arrow[r, "\Phi_i \otimes \Phi'_j"] \arrow[d, "\beta_{ij}"'] & P_0 \otimes P'_0 \arrow[d] \\
Q_{i+j} \arrow[r, "F_{i+j}"] & Q_0
\end{tikzcd}$$

Then the β_{ij} extend uniquely to a bilinear form

$$\mathcal{P} \times \mathcal{P}' \rightarrow \mathcal{Q}$$

In [Z2] Definition 18 the notion of a bilinear form of 1-displays was defined. It is obvious from the formulas there, that a bilinear form on two 1-displays in the sense of loc.cit. is the same as a bilinear form

$$\mathcal{P} \times \mathcal{P}' \to \mathcal{U}(1),$$

where the right hand side is the twisted unit display (11).
Starting from the normal decomposition of a display $\mathcal{P}$ it is easy to write down the standard data of a candidate for the exterior power $\bigwedge^k \mathcal{P}$. It comes with an alternating map $\otimes^k \mathcal{P} \to \bigwedge^k \mathcal{P}$. One proves as above that $\bigwedge^k \mathcal{P}$ has the universal property.
We will now define the base change for displays. Let $R \longrightarrow S$ be a homomorphism of rings. Let $\mathcal{P} = (P_i, \iota_i, \alpha_i, F_i)$ be a display over R. We will define a display $\mathcal{P}_S = (Q_i, \iota_i, \alpha_i, F_i)$ over S, with the following properties. There are $W(R)$-linear maps

$$P_i \longrightarrow Q_i,$$

such that the following diagrams are commutative

$$\begin{array}{ccc} P_i & \longrightarrow & Q_i \\ {\scriptstyle \iota_i}\uparrow & & \uparrow{\scriptstyle \iota_i} \\ P_{i+1} & \longrightarrow & Q_{i+1} \end{array} \qquad \begin{array}{ccc} Q_i & \xrightarrow{F_i} & Q_0 \\ \uparrow & & \uparrow \\ P_i & \xrightarrow{F_i} & P_0 \end{array} \qquad \begin{array}{ccc} I_R \otimes Q_i & \xrightarrow{\alpha_i} & Q_{i+1} \\ \uparrow & & \uparrow \\ I_R \otimes P_i & \xrightarrow{\alpha_i} & P_{i+1} \end{array} \tag{20}$$

Proposition 2.12 *There is a unique display $\mathcal{P}_S$ as above which enjoys the following universal property.*
If $\mathcal{T} = (T_i, \iota_i, \alpha_i, F_i)$ is an arbitrary display over S and

$$P_i \longrightarrow T_i$$

is a set of $W(R)$-linear morphisms, such that the diagrams above, with Q_i replaced by T_i are commutative, then there is a unique morphism of displays over S

$$\mathcal{P}_S \longrightarrow \mathcal{T},$$

such that the following diagrams are commutative:

$$\begin{tikzcd} Q_i \arrow[rr] & & T_i \\ & P_i \arrow[ul] \arrow[ur] & \end{tikzcd}$$

The display $\mathcal{P}_S$ may be constructed using a normal decomposition of $\mathcal{P}$. Let $P_0 = \oplus L_i$ be such a decomposition, and let $\Phi_i : L_i \longrightarrow P_0$ be the maps induced by F_i. Then L_i, Φ_i are standard data for a display over R. We can define $\mathcal{P}_S$ to be the display over S associated to the standard data $W(S) \otimes_{W(R)} L_i$, with the Frobenius linear maps $F \otimes_{W(R)} \Phi_i = \Phi'_i$.
We will now see that this definition is up to canonical isomorphism independent of the normal decomposition chosen. It suffices to see that $\mathcal{P}_S$ has the universal property Proposition 2.12.
The obvious maps $P_i \longrightarrow Q_i$ make the diagrams (20) commutative.

Lemma 2.13 *The following $W(S)$-module homomorphism is surjective*

$$W(S) \otimes_{W(R)} P_i \oplus I_S \otimes_{W(S)} Q_{i-1} \longrightarrow Q_i.$$

Proof: This is clear from the definitions.
Assume that $P_i \longrightarrow T_i$ is a set of maps as in Proposition 2.12. We construct inductively maps $Q_i \longrightarrow T_i$, which are compatible with F_i, ι_i, α_i. Therefore we obtain the desired morphism of displays $\mathcal{P}_S \longrightarrow \mathcal{T}$. Since $P_0 \longrightarrow T_0$ is $W(R)$-linear, we obtain a map

$$Q_0 = W(S) \otimes_{W(R)} P_0 \longrightarrow T_0.$$

Assume we have already constructed $W(S)$-module homomorphisms

$$Q_j \longrightarrow T_j,$$

which are compatible with F, ι and α for $j \leq i$.
Consider the diagram

$$\begin{array}{ccc} T_i & \xrightarrow{F_i} & T_0 \\ \uparrow & & \uparrow p \\ Q_{i+1} & \longrightarrow & T_0. \end{array} \tag{21}$$

The arrow $Q_{i+1} \longrightarrow T_i$ is the composition $Q_{i+1} \xrightarrow{\iota} Q_i \longrightarrow T_i$ and the arrow $Q_{i+1} \longrightarrow T_0$ is the composition $Q_{i+1} \xrightarrow{F_{i+1}} Q_0 \longrightarrow T_0$. By Lemma 2.13 we deduce that (21) is commutative. Thus it induces a map

$$Q_{i+1} \longrightarrow T_i \times_{F_i, T_0, p} T_0. \tag{22}$$

It suffices to show that the last map factors through T_{i+1}. This is seen easily by composing (22) with the morphism of the lemma.
The uniqueness of the constructed morphism $\mathcal{P}_S \longrightarrow \mathcal{T}$ is obvious. This proves the proposition. *Q.E.D.*

3 Degeneracy of some Spectral Sequences

Proposition 3.1 *Let $\pi : X \to Y$ be a separated and quasicompact morphism. Let $K^\cdot$ be a complex of of flat $\pi^{-1}\mathcal{O}_Y$-modules on X which is bounded above. We assume that each K^i is a quasicoherent $\mathcal{O}_X$-module. Then for each m the hypercohomology groups $\mathbb{R}^m\pi_*K^\cdot$ are quasicoherent $\mathcal{O}_Y$-modules. If M is a quasicoherent $\mathcal{O}_Y$-module there is a canonical isomorphism*

$$\mathbb{R}\pi_*(K^\cdot \otimes^{\mathbb{L}}_{\pi^{-1}(\mathcal{O}_Y)} \pi^{-1}M) \cong \mathbb{R}\pi_*K^\cdot \otimes^{\mathbb{L}}_{\mathcal{O}_Y} M \tag{23}$$

Proof: We may assume that Y is affine. Let $\mathcal{U} = \{U_i\}$ be a finite affine covering of X. Let $F^\cdot = C^\cdot(\mathcal{U}, K^\cdot)$ be the Czech complex. It is the complex of global sections of a sheafified Czech complex on Y: $\mathcal{F}^\cdot = \mathcal{C}^\cdot(\mathcal{U}, K^\cdot)$. The sheaves in this complex are acyclic with respect to π_* because the cohomology of an affine scheme vanishes. One concludes [EGA III] Prop. 1.4.10 that $\mathbb{R}\pi_*K^m$ are quasicoherent $\mathcal{O}_Y$-modules namely the sheaves associated to the cohomology of $F^\cdot$. Since the modules and sheaves involved are flat with respect to Y the projection formula reduces to the trivial equation:

$$C^\cdot(\mathcal{U}, K^\cdot \otimes_{\mathcal{O}_Y} M) \cong F^\cdot \otimes_{\Gamma(Y,\mathcal{O}_Y)} \Gamma(Y, M)$$

Q.E.D.

Let $\pi : X \to S$ be a proper morphism of schemes, such that S is affine. In this section we consider a bounded complex $K^\cdot$ of flat $\pi^{-1}(\mathcal{O}_S)$-modules. We assume that each K^i is a quasicoherent $\mathcal{O}_X$-module. Moreover we assume that the following conditions are satisfied:

(i) $R^j\pi_*K^i$ is a locally free $\mathcal{O}_S$-module of finite type for any i and j.

(ii) the spectral sequence of hypercohomology degenerates:

$$E_1^{ij} = R^j\pi_*K^i \Rightarrow \mathbb{R}^n\pi_*K^\cdot$$

One can easily see that with these assumptions the simple complex associated to $C^\cdot(\mathcal{U}, K^\cdot)$ as above is quasi-isomorphic to the direct sum of its cohomology groups. It follows that $\mathbb{R}^m\pi_*K^\cdot$ commutes with arbitrary base change for any m. For the same reason the cohomology groups $\mathbb{R}^j\pi_*K^i$ commute with arbitrary base change.

The degeneration of this spectral sequence may be reformulated as follows. Let us denote the by $\sigma^{\geq m}K^\cdot$ and $\sigma^{<m}K^\cdot$ the truncated complexes with respect to the naive truncation. Then the cohomology sequence of

$$0 \to \sigma^{\geq m}K^\cdot \to K^\cdot \to \sigma^{<m}K^\cdot \to 0,$$

splits into short exact sequences:

$$0 \to \mathbb{R}^q\pi_*(\sigma^{\geq m}K^\cdot) \to \mathbb{R}^q\pi_*K^\cdot \to \mathbb{R}^q\pi_*(\sigma^{<m}K^\cdot) \to 0. \tag{24}$$

Indeed, take a Cartan-Eilenberg resolution $K^{\cdot} \to I^{\cdot}$ by injective sheaves of abelian groups. Let $L^{\cdot} = \pi_* I^{\cdot}$. This complex comes with a filtration $Fil^m L^{\cdot}$ which is induced by the naive filtration of K. The spectral sequence in question is the spectral sequence of this filtered complex. The condition (24) is equivalent to the requirement that the maps

$$H^q(Fil^{m+1}L^{\cdot}) \to H^q(Fil^m L^{\cdot})$$

are injective for each q and m, as one may see easily from the exact cohomology sequence. This injectivity may be restated as follows:

$$d(Fil^m L^{q-1}) \cap Fil^{m+1} L^q = d(Fil^{m+1} L^{q-1}).$$

We conclude by [De3] Prop. 1.3.2.
The observation shows that the spectral sequences of hypercohomology of the truncated complexes $\sigma^{\geq m} K^{\cdot}$ and $\sigma^{<m} K^{\cdot}$ degenerate too.

PROPOSITION 3.2 *Let $\pi : X \to S$ and $K^{\cdot}$ be as in Proposition 3.1. Let $\ldots \to M_0 \to M_1 \to M_2 \to \ldots$ be a sequence of $\mathcal{O}_S$-modules (not necessarily a complex). We consider the complex*

$$L^{\cdot} : \ldots \to K^0 \otimes_{\mathcal{O}_S} M_0 \to K^1 \otimes_{\mathcal{O}_S} M_1 \to K^2 \otimes M_2 \to \ldots$$

Then the spectral sequence

$$E_1^{ij} : R^j \pi_* L^i \Rightarrow \mathbb{R}^{p+q} \pi_* L^{\cdot}$$

degenerates.

PROOF: We assume without loss of generality that $K^i = 0$ for $i < 0$. We say that a sequence $M_0 \to M_1 \to \ldots$ is m-stationary if it is isomorphic to a sequence of the form:

$$M_0 \to \ldots \to M_{m-1} \to M_m = M_m = \ldots$$

Because $K^{\cdot}$ is bounded it suffices to show the theorem for m-stationary sequences. We argue by induction. For $m = 0$ this is clear from the projection formula (23). Assume that the proposition holds for r-stationary sequences with $r < m$. For an m-stationary sequence we consider the following morphism of complexes:

$$L^{\cdot} \to I^{\cdot} \tag{25}$$

$$\begin{array}{ccccc}
K^0 \otimes M^0 \ldots K^{m-2} \otimes M^{m-2} & \longrightarrow & K^{m-1} \otimes M^{m-1} & \longrightarrow & K^m \otimes M^m \ldots \\
\mathrm{id}\big\downarrow \qquad \mathrm{id}\big\downarrow & & \big\downarrow & & \mathrm{id}\big\downarrow \\
K^0 \otimes M^0 \ldots K^{m-2} \otimes M^{m-2} & \longrightarrow & K^{m-1} \otimes M^m & \longrightarrow & K^m \otimes M^m \ldots
\end{array}$$

If we apply the induction assumption to $I^{\cdot}$ we obtain an exact sequence for each q and the given m.

$$0 \to \mathbb{R}^q \pi_*(\sigma^{\geq m} I^{\cdot}) \to \mathbb{R}^q \pi_* I^{\cdot} \to \mathbb{R}^q \pi_*(\sigma^{<m} I^{\cdot}) \to 0. \tag{26}$$

The morphism of complexes (25) induces a commutative diagram:

$$\begin{tikzcd} \mathbb{R}^q\pi_*\sigma^{\geq m}L^{\cdot} \arrow[r] \arrow[d, "\mathrm{id}"'] & \mathbb{R}^q\pi_*L^{\cdot} \arrow[d] \\ \mathbb{R}^q\pi_*\sigma^{\geq m}I^{\cdot} \arrow[r] & \mathbb{R}^q\pi_*I^{\cdot} \end{tikzcd}$$

By our induction assumption (26) it follows that the upper horizontal arrow is injective.
We have to prove that the following sequences are exact for arbitrary integers q and n.

$$0 \to \mathbb{R}^q\pi_*(\sigma^{\geq n}L^{\cdot}) \to \mathbb{R}^q\pi_*L^{\cdot} \to \mathbb{R}^q\pi_*(\sigma^{<n}L^{\cdot}) \to 0.$$

We have seen this for $n = m$. For $n > m$ we have to consider the maps.

$$\mathbb{R}^q\pi_*(\sigma^{\geq n}L^{\cdot}) \to \mathbb{R}^q\pi_*(\sigma^{\geq m}L^{\cdot}) \to \mathbb{R}^q\pi_*L^{\cdot}$$

It suffices to show that the first arrow is injective. But this follows from the beginning of our induction.
Finally we consider the case $n < m$. By the cohomology sequence it is sufficient to see that the map

$$\mathbb{R}^q\pi_*L^{\cdot} \to \mathbb{R}^q\pi_*(\sigma^{<n}L^{\cdot})$$

is surjective. But this map factors as:

$$\mathbb{R}^q\pi_*L^{\cdot} \to \mathbb{R}^q\pi_*(\sigma^{<m}L^{\cdot}) \to \mathbb{R}^q\pi_*(\sigma^{<n}L^{\cdot})$$

We need to show that the second map is surjective. But the complex $\sigma^{<m}L^{\cdot}$ is the tensor product of $\sigma^{<m}K^{\cdot}$ with an $(m-1)$-stationary sequence of modules. Therefore the map is surjective by induction assumption and we are done. *Q.E.D.*

Proposition 3.3 *Let $T : \mathcal{C} \to \mathcal{D}$ be a left exact functor of abelian categories such that $\mathcal{C}$ has enough injective objects. Let $K^{\cdot}$ be a complex in $\mathcal{C}$ which is bounded below. We assume that the spectral sequences in hypercohomology*

$$E_1^{ij} = R^jTK^i \Rightarrow \mathbb{R}^{i+j}TK^{\cdot}$$

degenerates. Let $f^{\cdot} : K^{\cdot} \to K^{\cdot}$ be a homomorphism of complexes. Then for each integer m the corresponding spectral sequence of hypercohomology associated to the complex

$$K(m,f): \quad \xrightarrow{d} K^{m-2} \xrightarrow{d} K^{m-1} \xrightarrow{f^m d} K^m \xrightarrow{d} K^{m+1} \to \dots$$

degenerates.

We omit the proof because it uses exactly the same arguments as above.

4 Filtered Comparison Theorems for the de Rham-Witt complex

Let R be a ring such that p is nilpotent in R. We consider a smooth scheme X over R. We will fix a natural number n. Assume we are given a smooth lifting $\tilde{X}/W_n(R)$. If $\tilde{X}$ admits a Witt-lift ([LZ] Def.3.3) $\mathcal{O}_{\tilde{X}} \longrightarrow W_n(\mathcal{O}_X)$ we obtain a morphism of complexes

$$\Omega^{\cdot}_{\tilde{X}/W_n(R)} \longrightarrow \Omega^{\cdot}_{W_n(X)/W_n(R)} \longrightarrow W_n\Omega^{\cdot}_{X/R}. \tag{27}$$

It is shown in [LZ] 3.2 and 3.3, that even if $\tilde{X}$ admits no Witt lift, there is a natural isomorphism in the derived category $D^+(X_{zar}, W_n(R))$ of sheaves of $W_n(R)$-modules on X:

$$\Omega^{\cdot}_{\tilde{X}/W_n(R)} \longrightarrow W_n\Omega^{\cdot}_{X/R}.$$

The aim of this section is to prove a filtered version of this isomorphism.
For typographical reasons we use the abbreviations:

$$\tilde{\Omega}^{\cdot}_n = \Omega^{\cdot}_{\tilde{X}/W_n(R)}, \qquad W_n\Omega^{\cdot} = W_n\Omega^{\cdot}_{X/R}.$$

Let us denote by $\mathcal{F}^m\Omega^{\cdot}_{\tilde{X}/W(R)}$ the complex

$$I_{R,n} \otimes_{W_n(R)} \tilde{\Omega}^0_n \xrightarrow{pd} \ldots \xrightarrow{pd} I_{R,n} \otimes_{W_n(R)} \tilde{\Omega}^{m-1}_n \xrightarrow{d} \tilde{\Omega}^m_n \xrightarrow{d} \tilde{\Omega}^{m+1}_n \rightarrow \ldots. \tag{28}$$

Conjecture 4.1 *There is a canonical isomorphism in the derived category $D^+(X_{zar}, W_n(R))$ between the Nygaard complex and the complex* (28)*:*

$$\mathcal{N}^m W_n\Omega^{\cdot}_{X/R} \cong \mathcal{F}^m\Omega^{\cdot}_{\tilde{X}/W_n(R)}$$

This question is closely related to the work of Deligne and Illusie [DI]. We will now see that the complexes in question are always locally isomorphic.
Let us assume we are given a Witt-lift. It induces a map

$$\kappa : \tilde{\Omega}^{\cdot}_n \longrightarrow W_n\Omega^{\cdot}.$$

By composition with the Frobenius $F : W_n\Omega^{\cdot} \rightarrow W_{n-1}\Omega^{\cdot}_{[F]}$ we obtain a map

$$\begin{aligned} \tilde{F} : & I_{R,n} \otimes_{W_n(R)} \tilde{\Omega}^{\cdot}_n \longrightarrow W_{n-1}\Omega_{[F]}. \\ & {}^V\xi \otimes \omega \longmapsto \xi\ {}^F\kappa(\omega) \end{aligned}$$

Using $\tilde{F}$ we obtain a morphism of complexes of $\mathcal{F}^m\tilde{\Omega}^{\cdot} \longrightarrow \mathcal{N}^m W_n\Omega$:

$$\begin{tikzcd} I_R \otimes \tilde{\Omega}^0_n \arrow[r, "pd"] \arrow[d, "\tilde{F}"'] & \ldots \arrow[r, "pd"] & I_R \otimes \tilde{\Omega}^{m-1}_n \arrow[r, "d"] \arrow[d, "\tilde{F}"'] & \tilde{\Omega}^m_n \arrow[r, "d"] \arrow[d] & \ldots \\ W_{n-1}\Omega^0_{[F]} \arrow[r, "d"] & \ldots \arrow[r, "d"] & W_{n-1}\Omega^{m-1}_{[F]} \arrow[r, "dV"] & W_n\Omega^m \arrow[r, "d"] & \ldots \end{tikzcd} \tag{29}$$

Let us consider the morphism (29) in the following simple situation:
Let $A = R[T_1,\dots,T_d]$ and $X = \operatorname{Spec} A$. We set $\tilde{A} = W_n(R)[T_1,\dots T_d]$ and $\tilde{X} = \operatorname{Spec}\tilde{A}$. We consider the Witt-lift:

$$\begin{aligned} \tilde{A} &\longrightarrow W_n(A) \\ T_i &\longrightarrow [T_i]. \end{aligned} \tag{30}$$

It is the unique map of $W_n(R)$-algebras, which maps T_i to its Teichmüller representative in $W_n(A)$.

Proposition 4.2 *For the Witt-lift* (30) *the induced morphism*

$$\mathcal{F}^m\Omega^{\cdot}_{\tilde{X}/W_n(R)} \longrightarrow \mathcal{N}^m W_n\Omega^{\cdot}_{X/R} \tag{31}$$

is for any $m \geq 0$ a quasi-isomorphism.

Proof: We use the $W_n(R)$-basis of $\Omega^l_{\tilde{A}/W_n(R)}$ given by p-basic differential forms. For each weight function $k : [1,d] \to \mathbb{Z}_{\geq 0}$ we fix an order on the set

$$\begin{aligned} &Supp\ k = \{i_1,\dots,i_r\}, \text{ such that} \\ &ord_p k_{i_1} \leq \dots \leq ord_p k_{i_r}. \end{aligned}$$

For any ascending partition of $Supp\ k$ into disjoint intervals

$$\mathcal{P} : Supp\ k = I_0 \sqcup I_1 \sqcup \dots \sqcup I_l,$$

such that $I_t \neq \emptyset$ for $1 \leq t \leq l$, we have the p-basic differential

$$\tilde{e}(k,\mathcal{P}) = T^{k_{I_0}}\left(p^{-ord_p k_{I_1}} dT^{k_{I_1}}\right)\cdot\dots\cdot\left(p^{-ord_p k_{I_l}} dT^{I_l}\right). \tag{32}$$

The order on $Supp\ k$ is fixed once for all and therefore not indicated in the notation (compare [LZ] 2.1).
In [LZ] 2.2 we have defined the basic Witt differentials

$$e_n(\xi,k,\mathcal{P}) \in W_n\Omega^l_{A/R}.$$

They are defined for functions $k : [1,d] \to \mathbb{Z}_{\geq 0}[\frac{1}{p}]$, and $\xi \in V^{u(k)}W_{n-u(k)}(R)$, where $u(k)$ is the minimal nonnegative integer, such that the weight $p^{u(k)}k$ takes integral values.
In our case the map (27) is the unique $W_n(R)$-linear map given by

$$\begin{aligned} \Omega^l_{\tilde{A}/W_n(R)} &\longrightarrow W_n\Omega^l_{A/R}. \\ \tilde{e}(k,\mathcal{P}) &\longmapsto e_n(1,k,\mathcal{P}). \end{aligned} \tag{33}$$

The map $\tilde{F}$ looks as follows

$$\begin{aligned} \tilde{F} : I_R \otimes_{W_n(R)} \Omega^l_{\tilde{A}/W_n(R)} &\longrightarrow W_{n-1}\Omega^l_{A/R,[F]} \\ {}^V\xi \otimes \tilde{e}(k,\mathcal{P}) &\longmapsto e_{n-1}(\xi,pk,\mathcal{P}). \end{aligned}$$

For each weight $k : [1,d] \longrightarrow \mathbb{Z}_{\geq 0}[\frac{1}{p}]$, we consider the subgroup $W_n\Omega^l_{A/R}(k)$ of $W_n\Omega^l_{A/R}$, which is generated by basic Witt-differentials $e_n(\xi, k, \mathcal{P})$ of fixed weight k. The complex $\mathcal{N}^m W_n\Omega$ splits into a direct sum of subcomplexes $\mathcal{N}^m(k)$:

$$W_{n-1}\Omega^0_{[F]}(pk) \xrightarrow{d} \cdots \xrightarrow{d} W_{n-1}\Omega^{m-1}_{[F]}(pk) \xrightarrow{dV} W_n\Omega^m_{[F]}(k) \to \cdots .$$

Similarly let $\Omega^l_{\tilde{A}/W_n(R)}(k) \subset \Omega^l_{\tilde{A}/W_n(R)}$ the $W_n(R)$-submodule generated by the p-basic differentials $\tilde{e}(k, \mathcal{P})$ of fixed integral weight k. Then $\mathcal{F}^m\tilde{\Omega}^{\cdot}$ is the direct sum of the following subcomplexes $\mathcal{F}^m(k)$:

$$I_R \otimes_{W_n(R)} \tilde{\Omega}^0_n(k) \xrightarrow{pd} \cdots \xrightarrow{pd} I_R \otimes_{W_n(R)} \tilde{\Omega}^{m-1}_n(k) \xrightarrow{d} \tilde{\Omega}^m_n(k) \to \cdots .$$

It is obvious that for integral weight k the map

$$\mathcal{F}^m(k) \longrightarrow \mathcal{N}^m(k) \tag{34}$$

is an isomorphism of complexes. Therefore the proposition follows if we show that for k not integral the complexes $\mathcal{N}^m(k)$ are acyclic. This follows in degrees not equal to $m-1$ or m from the corresponding statement for the de Rham-Witt complex (see [LZ] Proof of thm. 3.5).
For non-integral k consider a cycle $\omega \in W_{n-1}\Omega^{m-1}_{[F]}(k)$, i.e. $dV\omega = 0$. Because of the relation $FdV = d$, it follows that ω is also a cycle in the de Rham-Witt complex $W_{n-1}\Omega^{\cdot}$ and consequently a boundary, because k is not integral.
Finally consider a cycle $\omega \in W_n\Omega^m(k)$. It may be uniquely written as a sum

$$\omega = \sum_{\mathcal{P}} e_n(\xi_{\mathcal{P}}, k, \mathcal{P}).$$

By [LZ] Prop. 2.6 ω is a cycle, iff $\mathcal{P} = \emptyset \sqcup \mathcal{P}'$, i.e. iff the first interval I_0 of the partition $\mathcal{P}$ is empty, for all $e_n(\xi_{\mathcal{P}}, k, \mathcal{P}) \neq 0$ which appear in the sum. Since k is not integral the coefficient $\xi_{\mathcal{P}}$ is of the form $\xi_{\mathcal{P}} = {}^V\tau_{\mathcal{P}}$ and

$$d\ {}^V e_{n-1}(\tau_{\mathcal{P}}, pk, \mathcal{P}) = e_n(\xi_{\mathcal{P}}, k, \mathcal{P}).$$

Q.E.D.

We make n variable. We set $A = R[T_1, \ldots, T_d]$, $A_n = W_n(R)[T_1 \ldots T_d]$. We extend the Frobenius homomorphism $F : W_n(R) \longrightarrow W_{n-1}(R)$ to a map

$$\begin{aligned} \phi_n : & A_n \longrightarrow A_{n-1}, \\ & T_i \longmapsto T_i^p. \end{aligned} \tag{35}$$

We denote $\delta_n : A_n \longrightarrow W_n(A)$ the $W_n(R)$-algebra homomorphism, such that $\delta_n(T_i) = [T_i]$.
Assume we are given an étale homomorphism $A \longrightarrow B$ of R-algebras. Then we find a unique set of lifting B_n of B, which are étale over A_n and morphisms

$$\psi_n : B_n \longrightarrow B_{n-1} \text{ and } \varepsilon_n : B_n \longrightarrow W_n(B),$$

which are compatible with ϕ_n and δ_n, compare [LZ] Prop. 3.2.

Corollary 4.3 *The map ε_n defines a quasi-isomorphism of complexes:*

$$\begin{array}{ccccccc}
I_R \otimes \Omega^0_{B_n/W_n(R)} & \xrightarrow{pd} & \dots & \xrightarrow{pd} & I_R \otimes \Omega^{m-1}_{B_n/W_n(R)} & \xrightarrow{d} & \Omega^m_{B_n/W_n(R)} \cdots \\
\downarrow{\scriptstyle \tilde{F}} & & & & \downarrow{\scriptstyle \tilde{F}} & & \downarrow \\
W_{n-1}\Omega^0_{B/R,[F]} & \xrightarrow{d} & \dots & \xrightarrow{d} & W_{n-1}\Omega^{m-1}_{B/R,[F]} & \xrightarrow{dV} & W_n\Omega^m_{B/R} \cdots
\end{array}$$

Proof: For the given number n, we find a number m such that $p^m W_n(R) = 0$. Let us denote by $\phi^m : A_{m+n} \longrightarrow A_n$ the composite of m morphisms of type (35). It is clear from the definition that

$$d\phi^m : A_{m+n} \longrightarrow \Omega^1_{A_n/W_n(R)}$$

is zero. Consider the commutative diagram

$$\begin{array}{ccc}
B_{m+n} & \overset{d\psi^m}{\rightarrow} & \Omega^1_{B_n/W_n(R)} \\
\uparrow & & \uparrow \\
A_{m+n} & \overset{d\phi^m}{\rightarrow} & \Omega^1_{A_n/W_n(R)}.
\end{array}$$

The derivation $A_{m+n} \longrightarrow \Omega^1_{B_n/W_n(R)}$ is zero. Since B_{m+n}/A_{m+n} is étale, the extension $d\psi^m$ is zero too.
Consider the commutative diagram

$$\begin{array}{ccc}
B_{m+n} & \xrightarrow{\psi^m} & B_m \\
\uparrow & & \uparrow \\
A_{m+n} & \overset{\phi^m}{\rightarrow} & A_n.
\end{array}$$

It induces a morphism of algebras which are étale over A_n:

$$B_{m+n} \otimes_{A_{m+n},\phi^m} A_n \longrightarrow B_n. \tag{36}$$

This is an isomorphism. Indeed since $A_n \longrightarrow A/pA$ has nilpotent kernel it is enough to show that (36) becomes an isomorphism after tensoring with $\otimes_{A_n} A/pA$. But then we obtain the well-known isomorphism

$$\begin{array}{rcl}
B/pB \otimes_{A/pA,Frob^m} A/pA & \longrightarrow & B/pB \\
b \otimes a & \longmapsto & b^{p^m} \cdot a.
\end{array}$$

From the isomorphism (36) we deduce an isomorphism

$$\begin{array}{rcl}
B_{m+n} \otimes_{A_{m+n},\phi^m} \Omega^{\cdot}_{A_n/R} & \xrightarrow{\sim} & \Omega^{\cdot}_{B_n/R} \\
b \quad \otimes \quad \omega & \longmapsto & \psi^m(b) \cdot \omega.
\end{array} \tag{37}$$

We note that (37) becomes an isomorphism of complexes if we take $1 \otimes d$ as a differential on the left hand side. Hence the first row of (4.3) is obtained by tensoring the corresponding complex for $B_n = A_n$ with B_{n+m}.
Let us consider the complex

$$W_{n-1}\Omega^0_{A/R,[F]} \xrightarrow{d} \cdots \xrightarrow{d} W_{n-1}\Omega^{m-1}_{A/R,[F]} \xrightarrow{dV} W_n\Omega^m_{A/R} \xrightarrow{d} \cdots . \qquad (38)$$

We consider it as a complex of $W_{n+m}(A)$-modules via $F^m : W_{n+m}(A) \longrightarrow W_n(A)$. Then all differentials become linear (compare [LZ] Remark 1.8).
This shows that we obtain the second row of diagram of Corollary 4.3 if we tensorize (38) with $W_{n+m}(B) \otimes_{W_{n+m}(A),F^m}$. Because of the obvious isomorphism ([LZ] (3.2))

$$B_{n+m} \otimes_{A_{n+m},\delta} W_{n+m}(A) \tilde{\rightarrow} W_{n+m}(B),$$

the result is the same if we tensorize (38) by

$$B_{n+m} \otimes_{A_{n+m},\delta\phi^m} .$$

Therefore the whole diagram of Corollary 4.3 is obtained from the corresponding diagram for $B = A$ by tensoring with $B_{n+m} \otimes_{A_{n+m},\phi^m}$. Since this tensor product is an exact functor we obtain the corollary from the proposition. *Q.E.D.*

Let X/R be a smooth scheme. We assume that R is reduced and $p \cdot R = 0$. Then we consider still another complex derived from the de Rham-Witt complex. We set $W\Omega^l = W\Omega^l_{X/R}$ and define $\mathcal{I}^m W_n\Omega_{X/R}$ starting in degree 0.

$$p^{m-1}VW_{n-1}\Omega^0 \xrightarrow{d} p^{m-2}VW_{n-1}\Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} VW_{n-1}\Omega^{m-1} \xrightarrow{d} W_n\Omega \ldots . \qquad (39)$$

We recall the relation $pd\ {}^V\omega = {}^V d\omega$ of [LZ] 1.17. For varying n we obtain a procomplex $\mathcal{I}^m W.\Omega_{X/R}$.

PROPOSITION 4.4 *Let R be a reduced ring of char p. The procomplexes $\mathcal{I}^m W.\Omega$ and $\mathcal{N}^m W.\Omega$ are isomorphic in the pro-category of the category of complexes of abelian sheaves on X_{zar}.*

PROOF: We have an obvious morphism of procomplexes

$$\mathcal{N}^m W.\Omega \longrightarrow \mathcal{I}^m W.\Omega \qquad (40)$$

$$\begin{array}{ccccccccc}
W_{n-1}\Omega^0_{[F]} & \xrightarrow{d} & W_{n-1}\Omega^1_{[F]} & \ldots & W_{n-1}\Omega^{m-1}_{[F]} & \xrightarrow{dV} & W_n\Omega^m & \xrightarrow{d} & \ldots \\
p^{m-1}V \downarrow & & p^{m-2}V \downarrow & & V \downarrow & & id \downarrow & & \\
p^{m-1}VW_{n-1}\Omega^0 & \xrightarrow{d} & p^{m-2}VW_{n-1}\Omega^1 & \ldots & VW_{n-1}\Omega^{m-1} & \xrightarrow{d} & W_n\Omega^m & \xrightarrow{d} & \ldots .
\end{array}$$

We have to prove that this induces an isomorphism of proobjects. We set $W\Omega = \varprojlim W_n\Omega$. On $W\Omega$ the multiplication by p and the Verschiebung are injective. Therefore we have an inverse $p^iVW\Omega \stackrel{p^{-i}V^{-1}}{\longrightarrow} W\Omega_{[F]}$.

Lemma 4.5 *Let $n > k \geq i+1$. Then there is a map $p^i VW_n\Omega^l \longrightarrow W_{n-k}\Omega^l$, which makes the following diagram commutative*

$$\begin{array}{ccc} p^i VW_n\Omega^l_{X/R} & \longrightarrow & W_{n-k}\Omega^l_{X/R,[F]} \\ \uparrow & & \uparrow \\ p^i VW\Omega^l_{X/R} & \overset{p^{-i}V^{-1}}{\longrightarrow} & W\Omega^l_{X/R,[F]}. \end{array} \tag{41}$$

Proof of the lemma: Let $n > k \geq i$. For $\xi \in W_n(R)$ we denote by $\bar{\xi}$ its restriction to $W_{n-k}(R)$. Then we have a well-defined map

$$\begin{array}{rcc} p^i VW_n(R) & \longrightarrow & W_{n-k}(R) \\ p^{i\ V}\xi & \longmapsto & \bar{\xi}. \end{array} \tag{42}$$

Indeed, write $\xi = (x_0, \ldots, x_{n-1})$. Then

$$p^{i\ V}\xi = (0, \ldots, 0, x_0^{p^i}, \ldots, x_{n-i-1}^{p^i}) \in W_{n+1}(R).$$

Therefore the vector $(x_0, \ldots, x_{n-i-1}) \in W_{n-i}(R)$ is uniquely determined by $p^{i\ V}\xi$. We view $W_{n-i}(R)$ as a $W_{n+1}(R)$-module via

$$W_{n+1}(R) \xrightarrow{F} W_n(R) \xrightarrow{Res} W_{n-i}(R).$$

Then we obtain a morphism of $W_{n+1}(R)$-modules because of the following commutative diagram

$$\begin{array}{ccc} p^i VW(R) & \overset{p^{-i}V^{-1}}{\longrightarrow} & W(R) \\ \downarrow & & \downarrow \\ p^i VW_n(R) & \longrightarrow & W_{n-i}(R). \end{array}$$

The existence of the diagram (41) is clearly local for the Zariski-topology on X.

We begin with the case, where $X = \operatorname{Spec} A$ and $A = R[T_1, \ldots, T_d]$ is a polynomial algebra. In this case an element of $p^i VW\Omega^l_{A/R}$ may be expressed, in terms of basic Witt-differentials:

$$\omega = \sum p^{i\ V} e_n(\xi_{\mathcal{P},k}, k, \mathcal{P}), \quad \xi_{\mathcal{P},k} \in V^{u(k)} W_{n-u(k)}(R). \tag{43}$$

Note that $e_n(\xi_{\mathcal{P},k}, k, \mathcal{P}) = 0$, when $u(k) \geq n$.

The terms of the sum (43) are uniquely determined by [LZ] Prop.2.5 because of the direct decomposition

$$W_{n+1}\Omega^l_{A/R} = \oplus_{k,\mathcal{P}} W_{n+1}\Omega^l_{A/R}(\frac{k}{p}, \mathcal{P}).$$

Using loc. cit. we find:

$$p^{i\ V} e(\xi_{\mathcal{P},k}, k, \mathcal{P}) = p^{i\ V} e(\xi'_{\mathcal{P},k}, k, \mathcal{P}), \tag{44}$$

iff $p^i\ {}^V\xi_{\mathcal{P},k} = p^i\ {}^V\xi'_{\mathcal{P},k}$, except in the case where k/p is not integral and $I_0 = \emptyset$. In the latter case the equality (44) holds, iff $p^{i+1}\ {}^V\xi_{\mathcal{P},k} = p^{i+1}\ {}^V\xi'_{\mathcal{P},k}$.
With the lemma above this shows that the following map is well-defined:

$$\begin{aligned} p^iVW_n\Omega^l \longrightarrow & \quad W_{n-(i+1)}\Omega^l \\ \omega \longmapsto & \quad \sum e_{n-(i+1)}(\bar{\xi}_{k,\mathcal{P}}, k, \mathcal{P}). \end{aligned}$$

This proves the lemma in the case of a polynomial algebra A. Assume now that $A \longrightarrow B$ is a étale morphism.
The image of the canonical injection

$$W_{n+1}(B) \otimes_{W_{n+1}(A)} p^iVW_n\Omega_{A/R} \to W_{n+1}(B) \otimes_{W_{n+1}(A)} W_n\Omega_{A/R} \simeq W_{n+1}\Omega_{B/R}$$

coincides with $p^iVW_n\Omega_{B/R}$. This follows from the following commutative diagram

$$\begin{array}{ccc} W_{n+1}(B) \otimes_{W_{n+1}(A),F} W_n\Omega_{A/R} & \xrightarrow{\sim} & W_n\Omega_{B/R} \\ {\scriptstyle id\otimes p^iV}\Big\downarrow & & {\scriptstyle p^iV}\Big\downarrow \\ W_{n+1}(B) \otimes_{W_{n+1}(A)} W_{n+1}\Omega_{A/R} & \xrightarrow{\sim} & W_{n+1}\Omega_{B/R}. \end{array}$$

The top horizontal arrow is given by $\xi \otimes \omega \mapsto^F \xi\omega$ and the lower horizontal arrow is multiplication.
Now we find the desired map by tensoring $p^iVW_n\Omega_{A/R} \longrightarrow W_{n-(i+1)}\Omega_{A/R}$:

$$\begin{array}{ccc} W_{n+1}(B) \otimes_{W_{n+1}(A)} p^iVW_n\Omega_{A/R} & \longrightarrow & W_{n-i}(B) \otimes_{W_{n-i}(A),F} W_{n-(i+1)}\Omega_{A/R} \\ \wr\downarrow & & \wr\downarrow \\ p^iVW_n\Omega_{B/R} & \longrightarrow & W_{n-(i+1)}\Omega_{B/R}. \end{array}$$

The composition of the last map with $p^iV : W_{n-(i+1)}\Omega_{B/R} \longrightarrow W_{n-i}\Omega_{B/R}$ is just the restriction. This proves the lemma. *Q.E.D.*
The proposition follows immediately because we obtain an inverse to the map (40):

$$\begin{array}{ccccccc} p^{m-1}VW_{n-1}\Omega^0 & \xrightarrow{d} & p^{m-2}VW_{n-1}\Omega^1 \ldots & VW_{n-1}\Omega^{m-1} & \xrightarrow{d} & W_n\Omega^m \ldots \\ \downarrow & & \downarrow & \downarrow & & \downarrow{\scriptstyle Res} \\ W_{n-m-1}\Omega^0_{[F]} & \xrightarrow{d} & W_{n-m-1}\Omega^1_{[F]} \ldots & W_{n-m-1}\Omega^{m-1}_{[F]} & \xrightarrow{dV} & W_{n-m-1}\Omega^m \ldots \end{array}$$

The first m vertical maps defined by the lemma are equivariant with respect to

$$W_n(R) \xrightarrow{Res} W_{n-m}(R) \xrightarrow{F} W_{n-m-1}(R)$$

The remaining maps are equivariant with respect to $W_n(R) \to W_{n-m}(R)$. The commutativity of the diagram follows, since it is a homomorphic image of a corresponding diagram for $W\Omega$ without level. This proves the proposition. *Q.E.D.*

Let X/R be a smooth scheme. Let us denote by $\mathcal{J}_{X/W_n(R)} \subset \mathcal{O}_{X/W_n(R)}$ the sheaf of pd-ideals. We denote by $\mathcal{J}^{[m]}_{X/W_n(R)}$ its m-th divided power. Let

$$u_n : Crys(X/W_n(R)) \longrightarrow X_{zar}$$

be the canonical morphism of sites. We are going to define a morphism in $D(X_{zar})$ the derived category of abelian sheaves on X_{zar} for $m < p$:

$$Ru_{n*}\mathcal{J}^{[m]}_{X/W_n(R)} \longrightarrow \mathcal{I}^m W_n\Omega^{\cdot}_{X/R} \tag{45}$$

In order to define (45) we begin with the case, where X admits an embedding in a smooth scheme Y/R, such that Y has a Witt-lift: $\tilde{Y}/W_n(R)$ and $\mathcal{O}_{\tilde{Y}} \longrightarrow W_n(\mathcal{O}_Y)$.

The left hand side of (45) may be computed with the filtered Poincaré lemma [BO] Theorem 7.2: Let D be the divided power hull of X in $\tilde{Y}$. Let $I_D \subset \mathcal{O}_D$ be the pd-ideal. The pd-de Rham-complex $\breve{\Omega}_{D/W_n(R)}$ has the following subcomplex $Fil^m\breve{\Omega}_{D/W_n(R)}$:

$$I_D^{[m]}\breve{\Omega}^{\circ}_{D/W_n(R)} \xrightarrow{d} I_D^{[m-1]}\breve{\Omega}^{1}_{D/W_n(R)} \xrightarrow{d} \ldots I_D\breve{\Omega}^{m-1}_{D/W_n(R)} \xrightarrow{d} \breve{\Omega}_{D/W_n(R)} \ldots \tag{46}$$

Then the left hand side of (45) is isomorphic to the hypercohomology of (46). The Witt-lift defines a morphism

$$\mathcal{O}_{\tilde{Y}} \longrightarrow W_n(\mathcal{O}_Y) \longrightarrow W_n(\mathcal{O}_X).$$

It maps the ideal sheaf of $X \subset \tilde{Y}$ to the ideal sheaf $I_X = VW_{n-1}(\mathcal{O}_X) \subset W_n(\mathcal{O}_X)$. Since I_X is endowed with divided powers, we obtain

$$\mathcal{O}_D \longrightarrow W_n(\mathcal{O}_X), \tag{47}$$

mapping I_D to I_X. The homomorphism (47) induces a map of the pd-de Rham complexes

$$\breve{\Omega}_{D/W_n(R)} \longrightarrow \breve{\Omega}_{W_n(X)/W_n(R)} \longrightarrow W_n\Omega_{X/R}.$$

Taking into account that $I_X^{[h]} = p^{h-1}I_X$ for $h < p$, we obtain the desired morphism from (46) to the complex $\mathcal{I}^m W_n\Omega$ if $m < p$:

$$p^{m-1}I_X W_n\Omega^0_{X/R} \longrightarrow \ldots \longrightarrow I_X W_n\Omega^{m-1}_{X/R} \xrightarrow{d} W_n\Omega_{X/R} \to \ldots.$$

We note that $I_X W_n\Omega^l_{X/R} = VW_{n-1}\Omega^l_{X/R}$ follows from the formula

$${}^V(\eta d\omega_1 \ldots d\omega_r) = {}^V\eta d\, {}^V\omega_1 \ldots d\, {}^V\omega_r.$$

Hence we obtain a morphism

$$Ru_{n*}\mathcal{J}^{[m]}_{X/W_n(R)} \tilde{\to} Fil^m\breve{\Omega}_{D/W_n(R)} \longrightarrow \mathcal{I}^m W_n\Omega_{X/R}. \tag{48}$$

The independence of the last arrow from the embedding of X into a Witt lift $(Y, \tilde{Y})$ is proved in a standard manner: Let $X \hookrightarrow Y'$ be an embedding into a second Witt lift $(Y', \tilde{Y}')$. Then we obtain a Witt lift of the product $Y \times_{\operatorname{Spec} R} Y'$: Indeed, $\tilde{Y} \times_{\operatorname{Spec} W_n(R)} \tilde{Y}'$ is a lifting of $Y \times Y'$ and the two given Witt lifts induce a morphism:

$$\mathcal{O}_{\tilde{Y}} \otimes_{W_n(R)} \mathcal{O}_{\tilde{Y}'} \longrightarrow W_n(\mathcal{O}_Y) \otimes_{W_n(R)} W_n(\mathcal{O}_{Y'}) \longrightarrow W_n(\mathcal{O}_Y \otimes \mathcal{O}_{Y'}).$$

If P denotes the pd-hull of X in $\tilde{Y} \times_{\operatorname{Spec} W_n(R)} \tilde{Y}'$. We obtain a commutative diagram

$$\begin{tikzcd}
Fil^m\breve{\Omega}_{D/W_n(R)} \arrow[d] \arrow[dr] & \\
& \mathcal{I}^m W_n\Omega_{X/R} \\
Fil^m\breve{\Omega}_{P/W_n(R)} \arrow[ur] &
\end{tikzcd}$$

Since the vertical arrow induces by [BO] the identity on $Ru_{n*}\mathcal{J}^{[m]}_{X/W_n(R)}$ the independence of (45) of the chosen Witt lift follows.

If X admits no embedding in a smooth scheme Y which has a Witt lift, one can proceed by simplicial methods [I] or [LZ] §3.2, but we omit the details here.

THEOREM 4.6 *For each $m < p$ and n the map in $D^+(X_{zar}, W_n(R))$*

$$Ru_{n*}\mathcal{J}^{[m]}_{X/W_n(R)} \longrightarrow \mathcal{I}^m W_n\Omega_{X/R} \tag{49}$$

is a quasi-isomorphism.

PROOF: Clearly the question is local for the Zariski-topology on X. We may therefore assume that $X = \operatorname{Spec} B$, where the R-algebra B is étale over $R[T_1, \ldots, T_d]$. From the discussion above we know that any Witt-lift of B leads to the same morphism (49). We choose a Frobenius lift $\{B_n\}_{n \in \mathbb{N}}$ of the algebra B as in the corollary 4.3. We begin with the reduction to the case $B = R[T_1, \ldots, T_d]$. Let J be the kernel of $B_n \longrightarrow B$. Then $\mathcal{J}^{[i]} = p^{i-1} I_R B_n$, where $I_R = VW_{n-1}(R) \subset W_n(R)$. Hence we have to show that the following morphism of complexes induces a quasi-isomorphism:

$$\begin{array}{ccccccc}
p^{m-1}I_R\Omega^0_{B_n/W_n(R)} & \xrightarrow{d} & \dots I_R\Omega^{m-1}_{B_n/W_n(R)} & \xrightarrow{d} & \Omega^m_{B_n/W_n(R)} & \xrightarrow{d} \\
\downarrow & & \downarrow & & \downarrow & \\
p^{m-1}VW_{n-1}\Omega^0_{B/R} & \xrightarrow{d} & \dots VW_{n-1}\Omega^{m-1}_{B/R} & \xrightarrow{d} & W_n\Omega^m_{B/R} & \xrightarrow{d}
\end{array} \tag{50}$$

We choose a number s, such that $p^sW_n(R) = 0$. We consider the groups in the first complex as B_{n+s} modules via $\psi^s : B_{n+s} \to B_n$. As shown in the proof of Corollary 4.3 we obtain a complex of B_{n+s}-modules. The same is true if we consider the groups in the second complex as B_{n+s}-modules by $\psi^s : B_{n+s} \to B_n \to W_n(B)$.

We obtain the diagram above from the corresponding diagram for $B = A$ by tensoring with $B_{n+s}\otimes_{A_{n+s}}$. Since B_{n+s} is étale over A_{n+s}, we have reduced our statement to the case where $B = R[T_1, \dots, T_d]$ and where the Witt-lift is a standard one.

In the case of a polynomial algebra we have a decomposition of the de Rham Witt complex according to weights [LZ] 2.17.

Because the operator V is homogeneous, we have a similar decomposition for the complex $\mathcal{I}^mW_n\Omega_{A/R}$. In fact, by [LZ] Prop. 2.5 an element of $p^{m-l-1}VW_{n-1}\Omega^l$, for $l \leq m-1$ may be uniquely written as a sum of elements of the following types

$e_n(p^{m-l-1}\ {}^V\xi, k, I_0, \dots, I_l)$ for k integral
$e_n(p^{m-l-1}\ {}^V\xi, k, I_0, \dots, I_l)$ for $I_0 \neq \emptyset$, k not integral
$e_n(p^{m-l}\ {}^V\xi, k, I_0, \dots, I_l)$ for $I_0 = \emptyset$, k not integral.

Here $\xi \in W_{n-1}(R)$ for k integral and $\xi \in V^{u(k)-1}W_{n-u(k)}(R)$ for k nonintegral. Clearly the elements of the first type span a subcomplex of $\mathcal{I}^mW_n\Omega_{A/R}$ which is isomorphic to the complex in the first row of (50). Indeed, the p-basic differentials of this complex are mapped to basic Witt-differentials of the first type above. The last two types of Witt-differentials above span an acyclic subcomplex because of the formula

$$de_n(p^{m-l-1}\ {}^V\xi, k, I_0, \dots, I_l) = e_n(p^{m-l-1}\ {}^V\xi, k, \phi, I_0, \dots, I_l),$$

for $I_0 \neq \emptyset$ and k not integral. The exactness of the non integral part at $W_n\Omega^m_{B/R}$ follows in the same way. *Q.E.D.*

Let $X_n/W_n(R)$ be a compatible system of smooth liftings of X/R for $n \in \mathbb{N}$. The Theorem 4.6 provides an isomorphism in the derived category between $\mathcal{I}^mW_n\Omega_{X/R}$ and

$$p^{m-1}I_R\Omega^0_{X_n/W_n(R)} \to p^{m-2}I_R\Omega^1_{X_n/W_n(R)} \to \dots I_R\Omega^{m-1}_{X_n/W_n(R)} \to \Omega^{m-1}_{X_n/W_n(R)} \dots . \tag{51}$$

We know by Proposition 4.4 that $\{\mathcal{I}^mW_n\Omega_{X/R}\}$ is isomorphic to the procomplex $\{\mathcal{N}^mW_n\Omega_{X/R}\}$. The same argument shows that the procomplex (51) is quasi-isomorphic to $\{\mathcal{F}^m\Omega^{\cdot}_{X_n/W_n(R)}\}_{n\in\mathcal{N}}$. Passing to the projective limit we obtain:

COROLLARY 4.7 *Let R be a reduced ring. Let X/R be a smooth and proper scheme. Assume that $X_n/W_n(R)$ is a compatible system of smooth liftings of X. Then there is for each number $m < p$ a natural isomorphism in the derived category $D^+(X_{zar,W(R)})$:*

$$\mathcal{N}^m W\Omega^{\cdot}_{X/R} \cong \mathcal{F}^m \Omega^{\cdot}_{\mathcal{X}/W(R)},$$

where $\mathcal{X} = \varinjlim X_n$ in the sense of EGA I Prop. 10.6.3.

This is a weak form of the Conjecture 4.1 which asserts this for every level separately.

5 DISPLAY STRUCTURE ON CRYSTALLINE COHOMOLOGY

Let R be a ring such that p is nilpotent in R. Let (A, σ, α) be a frame for R [Z1]. This means that A is a torsion free a p-adic ring with an endomorphism $\sigma : A \to A$, which induces the Frobenius endomorphism $A/pA \to A/pA$. The map $\alpha : A \to R$ is a surjective ring homomorphism, such that the ideal $\mathfrak{a} = \operatorname{Ker} \alpha$ has divided powers.

DEFINITION 5.1 *An A-window consists of*

1) *A finitely generated projective A-module P_0.*

2) *A descending filtration of P_0 by A-submodules*

$$\dots P_{i+1} \subset P_i \subset \dots \subset P_2 \subset P_1 \subset P_0. \tag{52}$$

3) *σ-linear homomorphisms*

$$F_i : P_i \to P_0.$$

The following conditions are required.

(i) *$\mathfrak{a}P_i \subset P_{i+1}$ and the factor module $P_{i+1}/\mathfrak{a}P_i$ is a finitely generated projective R-module E_{i+1} for $i \geq 0$. We set $E_0 = P_0/\mathfrak{a}P_0$.*

(ii) *The inclusions $P_{i+1} \to P_i$ induce injective R-module morphisms*

$$\dots \to E_{i+1} \to E_i \to \dots \to E_0,$$

such that E_{i+1} is a direct summand of E_i.

(iii) *$\mathfrak{a}P_i = P_{i+1}$ if i is big enough.*

(iv) *$F_i(x) = pF_{i+1}(x)$ for $x \in P_{i+1}$.*

(v) *The union of the images $F_i(P_i)$ for $i \in \mathbb{Z}_{\geq 0}$ generate P_0 as an A-module.*

A window is called standard if it arises in the following way. Let $L_0, \ldots, L_d$ be finitely generated projective A-modules. Let

$$\Phi_i : L_i \to \bigoplus_{j=0}^{d} L_j$$

be σ-linear homomorphisms, such that the determinant of $\Phi_0 \oplus \cdots \oplus \Phi_d$ is a unit. Then we set for $i \geq 0$

$$P_i = \mathfrak{a}^i L_0 \oplus \mathfrak{a}^{i-1} L_1 \oplus \ldots \oplus \mathfrak{a} L_{i-1} \oplus L_i \oplus \cdots \oplus L_d.$$

We define F_i on this direct sum as follows: The restriction of F_i to $\mathfrak{a}^{i-k} L_k$ for $k < i$ resp. L_k for $k \geq i$ to is defined by

$$\begin{array}{rcll} F_i(ax) & = & \frac{\sigma(a)}{p^{i-k}} \Phi_k(x) & \text{for } 0 \leq k < i, \;\; x \in L_k, \;\; a \in \mathfrak{a}^{i-k} \\ F_i(x) & = & p^{k-i} \Phi_k(x) & \text{for } i \leq k \;\; x \in L_k. \end{array}$$

It is clear that (P_i, F_i) form a window.

Each window is isomorphic to a standard window. Indeed, let $E_0 = \oplus \bar{L}_j$ be a splitting of the filtration (52) in the definition:

$$E_i = \oplus_{j \geq i} \bar{L}_j.$$

Let L_i be a finitely generated projective A-module which lifts $\bar{L}_i$. We find homomorphisms $L_i \to P_i$ which make the following diagrams commutative:

$$\begin{array}{ccc} \bar{L}_i & \longrightarrow & E_i \\ \uparrow & & \uparrow \\ L_i & \longrightarrow & P_i. \end{array}$$

It follows from the lemma of Nakayama that $\oplus L_i \to P_0$ is an isomorphism, since it is modulo $\mathfrak{a}$. By induction we obtain

$$P_i = \mathfrak{a}^i L_0 \oplus \cdots \oplus \mathfrak{a} L_{i-1} \oplus L_i \oplus \cdots \oplus L_d. \tag{53}$$

We set $\Phi_i = F_i | L_i$. The condition (v) implies that $\oplus \Phi_i : \oplus L_j \to \oplus L_j$ is a σ-linear epimorphism and therefore an isomorphism.

Remark: A window (P_i) is of degree d, if $P_{i+1} = \mathfrak{a} P_i$ for $i \geq d$. To give a window of degree d it is enough to give only the modules $P_0, \ldots, P_d$. The axioms may be formulated in the same way for this finite chain of modules. The axiom (v) then requires that the union of $F_0(P_0), F_1(P_1) \ldots, F_d(P_d)$ generates P_0 as an A-module.

We will now see that an A-window induces a display over R. There is a natural ring homomorphism $\delta : A \to W(A)$, such that for the Witt-polynomials $\mathbf{w}_n$ there is the identity

$$\mathbf{w}_n(\delta(a)) = \sigma^n(a), \quad a \in A.$$

Consider the composite ring homomorphism.

$$\varkappa : A \rightarrow W(A) \rightarrow W(R).$$

We have by [Z1] Prop. 1.5:

$$\begin{aligned} \varkappa(\sigma(a)) &= {}^{F}\varkappa(a) \quad \text{for } a \in A \\ \varkappa(\tfrac{\sigma(a)}{p}) &= {}^{V^{-1}}\varkappa(a) \quad \text{for } a \in \mathfrak{a}. \end{aligned}$$

The last equation makes sense because $\varkappa(a) \in VW(R)$ for $a \in \mathfrak{a}$.
It is clear that a datum (L_i, Φ_i) for a standard window over A induces the datum $(W(R) \otimes_{W(A)} L_i, F \otimes \Phi_i)$ for a standard display over R. We will show that the resulting display does not depend on the decomposition $P_0 = \oplus L_i$ we have used.
We give an invariant construction of a display $(Q_i, \iota_i, \alpha_i, F_i)$ from a window (P_i, F_i). The display comes with morphisms $\tau_i : P_i \rightarrow Q_i$ such that the following diagrams commute

$$\begin{tikzcd} P_i \arrow[r, "\tau_i"] & Q_i \\ P_{i+1} \arrow[u] \arrow[r, "\tau_{i+1}"] & Q_{i+1} \arrow[u, "\iota_i"'] \end{tikzcd} \qquad \begin{tikzcd} P_{i+1} \arrow[r, "\tau_{i+1}"] & Q_{i+1} \\ \mathfrak{a} \otimes P_i \arrow[u] \arrow[r] & I_R \otimes_{W(R)} Q_i \arrow[u, "\alpha_i"'] \end{tikzcd} \qquad \begin{tikzcd} P_0 \arrow[r, "\tau_0"] & Q_0 \\ P_{i+1} \arrow[u, "F_i"] \arrow[r, "\tau_i"] & Q_i . \arrow[u, "F_i"'] \end{tikzcd} \tag{54}$$

We construct Q_i and τ_i inductively, such that the diagrams (54) commute. We set $Q_0 = W(R) \otimes_{\varkappa, A} P_0$ and we let $\tau_0 : P_0 \rightarrow Q_0$ be the canonical map.
Assume that $\tau_k : P_k \rightarrow Q_k$ was constructed for $k \leq i$. Then we consider the following commutative diagrams:

$$\begin{tikzcd} P_i \arrow[r, "\tau_i"] & Q_i \arrow[r, "F_i"] & Q_0 \\ & & Q_0 \arrow[u, "p"] \\ P_{i+1} \arrow[uu, "\iota"] \arrow[rr, "F_{i+1}"] & & P_0 \arrow[u, "\tau_0"] \end{tikzcd} \qquad \begin{tikzcd} Q_i \arrow[r, "F_i"] & Q_0 \\ I_R \otimes Q_i \arrow[u] \arrow[r, "\tilde{F}_i"] & Q_0 \arrow[u, "p"] \end{tikzcd}$$

We obtain a morphism to the fibre product

$$(W(R) \otimes_A P_{i+1}) \oplus (I_R \otimes Q_i) \rightarrow Q_i \times_{F_i, Q_0, p} Q_0. \tag{55}$$

We define Q_{i+1} as the image of (55). This gives a map $P_{i+1} \stackrel{\tau_{i+1}}{\rightarrow} Q_{i+1}$. We define $\iota : Q_{i+1} \rightarrow Q_i$ and $F_{i+1} : Q_{i+1} \rightarrow Q_0$ and $\alpha_i : I_R \otimes Q_i \rightarrow Q_{i+1}$ as the canonical maps determined by these data. A routine verification shows that this construction gives the same result as the construction via standard windows.

Moreover the following universal property holds. Let $(Q'_i, \iota'_i, \alpha'_i, F'_i)$ be a display over R and let $\tau'_i : P_i \rightarrow Q'_i$ be maps such that the diagrams (54) for τ'_i commute. Then the maps τ'_i are the composition of τ_i and a morphism of displays $(Q_i, \iota_i, \alpha_i F_i) \rightarrow (Q'_i, \iota'_i, \alpha'_i, F'_i)$.

Let $A \xrightarrow{\alpha} R, \sigma, \mathfrak{a}$ as before. Let $X \rightarrow \operatorname{Spec} R$ be a scheme which is projective and smooth. Let $\mathcal{Y} \xrightarrow{f} \operatorname{Spf} A$ be a smooth pA-adic formal scheme, which lifts X. We set $A_n = A/p^n$ and $Y_n = \mathcal{Y} \times_{\operatorname{Spf} A} \operatorname{Spec} A_n$. For big n the map α factors through $A_n \xrightarrow{\alpha_n} R$. The kernel $\mathfrak{a}_n$ inherits a pd-structure. We consider the crystalline topos $(X/A)_{crys}$. Let $\mathcal{J}_{X/A_n} \subset \mathcal{O}_{X/A_n}$ be the pd-ideal sheaf. We are interested in the cohomology groups:

$$H^i(X, \mathcal{J}^{[m]}_{X/A}) = \varprojlim_n H^i_{crys}(X/A_n, \mathcal{J}^{[m]}_{X/A_n}). \tag{56}$$

Remark: It would be more accurate to consider the cohomology groups of $R \varprojlim_n R\Gamma(X/A_n, \mathcal{J}^{[m]}_{X/A_n})$. But under the Assumptions 5.2 and 5.3 we are going to make these groups will coincide.

By [BO] 7.2 the groups $H^i_{crys}(X/A_n, \mathcal{J}^{[m]}_{X/A_n})$ are the hypercohomology groups of the following complex $Fil^{[m]}\Omega^{\cdot}_{Y_n/A_n}$:

$$\mathfrak{a}_n^{[m]} \otimes_{A_n} \Omega^0_{Y_n/A_n} \rightarrow \mathfrak{a}_n^{[m-1]} \otimes_{A_n} \Omega^1_{Y_n/A_n} \cdots \rightarrow \mathfrak{a}_n \otimes_{A_n} \Omega^{m-1}_{Y_n/A_n} \rightarrow \Omega^m_{Y_n/A_n} \cdots \tag{57}$$

We will make the following assumptions:

Assumption 5.2 *The cohomology groups $H^q(Y_n, \Omega^p_{Y_n/A_n})$ are for each n locally free A_n-modules of finite type.*

Assumption 5.3 *The de Rham spectral sequence degenerates at E_1*

$$E_1^{pq} = H^q(Y_n, \Omega^p_{Y_n/A_n}) \Rightarrow \mathbb{H}^{p+q}(Y_n, \Omega^{\cdot}_{Y_n/A_n}).$$

Since Y_n is quasicompact and separated by assumption the cohomology sheafs $\mathbb{R}^m f_{n*}\Omega^{\cdot}_{Y_n/A_n}$ are quasicoherent. From the assumption we see that these sheaves are locally free of finite type. Hence the complex $\mathbb{R}f_{n*}\Omega^{\cdot}_{Y_n/A_n}$ is quasi-isomorphic to the direct sum of its cohomology groups. This implies that the cohomology groups $\mathbb{R}^m f_{n*}\Omega^{\cdot}_{Y_n/A_n}$ commute with arbitrary base change. The same applies to the cohomology groups $R^q f_{n*}\Omega^p_{Y_n/A_n}$. By Proposition 3.2 and the projection formula (Proposition 3.1) we obtain a degenerating spectral sequence

$$\begin{array}{rl} E_1^{ij} = H^j(Y_n, \Omega^i_{Y_n/A_n}) \otimes_{A_n} \mathfrak{a}^{[m-i]} \Rightarrow & \mathbb{H}^{i+j}(Y_n, Fil^{[m]}\Omega^{\cdot}_{Y_n/A_n}) \\ & \quad \| \\ & H^{i+j}_{crys}(X/A_n, \mathcal{J}^{[m]}_{X/A_n}) \end{array}$$

If we pass to the projective limit we obtain a degenerating spectral sequence

$$E_1^{ij} = \mathfrak{a}^{[m-i]} \otimes H^j(\mathcal{Y}, \Omega^i_{\mathcal{Y}/A}) \Rightarrow H^{i+j}_{crys}(X/A, \mathcal{J}^{[m]}_{X/A}). \tag{58}$$

The groups involved have no p-torsion.
We set $\bar{X} = X \times_{\operatorname{Spec} R} \operatorname{Spec} \bar{R}$, where $\bar{R} = R/pR$. By [BO] 5.17 there is a canonical isomorphism

$$H^i_{crys}(X/A, \mathcal{O}_{X/A}) \simeq H^i_{crys}(\bar{X}/A, \mathcal{O}_{\bar{X}/A}). \tag{59}$$

The absolute Frobenius on $\bar{X}$ and σ on A induce an endomorphism on the right hand side of (59) and therefore an endomorphism

$$F : H^i_{crys}(X/A, \mathcal{O}_{X/A}) \to H^i_{crys}(X/A, \mathcal{O}_{X/A}).$$

LEMMA 5.4 *Let $p^{[m]}$ be the maximal power of p which divides $p^m/m!$ Then the image of the following composition*

$$H^i_{crys}(X/A, \mathcal{J}^{[m]}_{X/A}) \to H^i_{crys}(X/A), \mathcal{O}_{X/A}) \xrightarrow{F} H^i_{crys}(X/A, \mathcal{O}_{X/A})$$

is contained in $p^{[m]} H^i_{crys}(X/A, \mathcal{O}_{X/A})$.

PROOF: The argument is well known [K], but we repeat it in the generality we need. We may replace A by A_n. We embed X into a smooth and projective A_n-scheme Z, such that there is an endomorphism $\sigma : Z \to Z$ which lifts the absolute Frobenius modulo p and which is compatible with σ on A_n. We may take for Z the projective space. Consider the pd-hull D of X in Z. It is also the pd-hull of $\bar{X}$ in Z. Therefore σ extends to D/A_n and to the pd-differentials $\breve{\Omega}_{D/A_n}$. We obtain by [BO] an isomorphism

$$\mathbb{H}^i(X, \breve{\Omega}^{\cdot}_{D/A_n}) \xrightarrow{\sim} H^i_{crys}(X/A, \mathcal{O}_{X/A_n}),$$

which is equivariant with respect to the action of σ on the left hand side and F on the right hand side.
Consider the morphisms

$$\bar{X} \to D \to Z.$$

Let $I(\bar{X})$ be the ideal of $\bar{X}$ in Z and $\bar{\mathcal{J}}_D$ be the ideal of $\bar{X}$ in D. Consider the diagram

$$\begin{tikzcd} (\mathcal{O}_Z, I(\bar{X})) \arrow[r] \arrow[d, "\sigma"] \arrow[dr, "\kappa"] & (\mathcal{O}_D, \bar{\mathcal{J}}_D) \arrow[d, "\sigma_D"] \\ (\mathcal{O}_Z, I(\bar{X})) \arrow[r] & (\mathcal{O}_D, \bar{\mathcal{J}}_D) \end{tikzcd}$$

The composite κ maps $I(\bar{X})$ to $p \cdot \mathcal{O}_D$. This follows because

$$\sigma(z) \equiv z^p \mod p \text{ for } z \in \mathcal{O}_Z. \tag{60}$$

If $z \in I(\bar{X})$ the image of z^p in $\bar{\mathcal{J}}_D$ becomes divisible by p, because we have divided powers. Therefore the induced map σ_D on the divided power envelope maps $\bar{\mathcal{J}}_D$ to $p\mathcal{O}_D$. Therefore

$$\sigma(\bar{\mathcal{J}}_D^{[m]}) \subset p^{[m]}\mathcal{O}_D.$$

For $z \in \mathcal{O}_Z$ we find from (60) that in $\breve{\Omega}^1_{D/A_n}$:

$$d\sigma(z) \equiv 0 \mod p.$$

The composite map of the lemma is induced by a map of complexes:

$$\begin{array}{ccccccc} \mathcal{J}_D^{[m]}\breve{\Omega}^\circ_{D/A_n} & \longrightarrow & \dots & \longrightarrow & \mathcal{J}_D^{[m-i]}\breve{\Omega}^i_{D/A_n} & \longrightarrow & \dots \\ \sigma\big\downarrow & & & & \sigma\big\downarrow & & \\ \breve{\Omega}^\circ_{D/A_n} & \longrightarrow & \dots & \longrightarrow & \breve{\Omega}^i_{D/A_n} & \longrightarrow & \dots . \end{array} \tag{61}$$

The image of this map lies in $p^{[m]} \cdot \breve{\Omega}^{\cdot}_{D/A_n} = p^{[m]}A_n \otimes^{\mathbb{L}}_{A_n} \breve{\Omega}_{D/A_n}$. The last equality follows since by [BO] 3.32 the sheaf $\mathcal{O}_D$ is flat over A_n. The hypercohomology of the last complex is by the projection formula

$$\begin{aligned} p^{[m]}A_n \otimes^{\mathbb{L}} R\Gamma(X, \breve{\Omega}_{D/A_n}) &= p^{[m]}A_n \otimes^{\mathbb{L}} \mathbb{R}\Gamma_{crys}(X/A_n, \mathcal{O}_{X/A_n}) \\ &= p^{[m]}A_n \otimes^{\mathbb{L}} R\Gamma(Y_n, \Omega^{\cdot}_{Y_n/A_n}) \end{aligned}$$

But the cohomology of the last complex is $p^{[m]}\mathbb{H}^i(Y_n, \Omega^{\cdot}_{Y_n/A_n})$, since we assumed that the cohomology is locally free. This shows that (61) factors on the hypercohomology through $p^{[m]}\mathbb{H}_{crys}(X/A_n, \mathcal{O}_{X/A_n}) = p^{[m]}\mathbb{H}^i(Y_n, \Omega^{\cdot}_{Y_n/A_n})$. *Q.E.D.*

Theorem 5.5 *Let R be a ring, such that p is nilpotent in R. Let X be a scheme which is projective and smooth over R. Let $A \to R$ be a frame. We assume that X lifts to a projective and smooth p-adic formal scheme $\mathcal{Y}/\operatorname{Spf} A$ such that the assumptions 5.2 and 5.3 are fullfilled. Then for each number $n < p$ the canonical maps*

$$H^n_{crys}(X/A, \mathcal{J}^{[m]}_{X/A}) \to H^n_{crys}(X/A, \mathcal{J}^{[m-1]}_{X/A}) \to \dots \to H^n_{crys}(X/A, \mathcal{O}_{X/A})$$

are injective. The A-modules $P_m = H^n_{crys}(X/A, \mathcal{J}^{[m]}_{X/A})$ for $m \leq n$ together with the maps

$$\frac{1}{p^m}F = F_m : P_m \to P_0$$

given by Lemma 5.4 form a window of degree n.

Proof: We consider a number $m \leq n$. Then we have $\mathcal{J}^m_{X/A} = \mathcal{J}^{[m]}_{X/A}, \mathfrak{a}^m = \mathfrak{a}^{[m]}$. We write $Fil^{[m]}\Omega^{\cdot}_{\mathcal{Y}/A} = \varprojlim_n Fil^{[m]}\Omega^{\cdot}_{Y_n/A_n}$. Then we find a canonical isomorphism

$$P_m = \mathbb{H}^n(X, Fil^{[m]}\Omega_{\mathcal{Y}/A}) \cong H^n_{crys}(X/A, \mathcal{J}^m_{X/A}) \tag{62}$$

From the degenerating spectral sequence (58) we obtain the injectivity of $P_m \to P_{m-1}$, since we have injectivity on the associated graded groups.
In the following considerations m, n can be arbitrary natural number, without the restriction $m \leq n < p$. Then $Fil^{[m]}_{\mathcal{Y}/A}$ will be the complex $Fil^m_{\mathcal{Y}/A}$

$$\mathfrak{a}^m \Omega^0_{\mathcal{Y}/A} \to \mathfrak{a}^{m-1}\Omega^1_{\mathcal{Y}/A} \to \cdots \to \mathfrak{a}\Omega^{m-1}_{\mathcal{Y}/A} \to \Omega^m_{\mathcal{Y}/A} \to \ldots$$

Consider the following morphism:

$$\mathfrak{a} \otimes \mathbb{H}^n(X, Fil^m \Omega^{\cdot}_{\mathcal{Y}/A}) \to \mathbb{H}^n(X, \mathfrak{a} Fil^m \Omega^{\cdot}_{\mathcal{Y}/A}). \tag{63}$$

We have for $\mathfrak{a} Fil^m \Omega^{\cdot}_{\mathcal{Y}/A}$ a degenerating spectral sequence as (58). Therefore the right hand side of (63) is a subgroup of $H^n(X, Fil^m \Omega^{\cdot}_{\mathcal{Y}/A})$.
We claim that the induced inclusion is an equality

$$\mathfrak{a}\mathbb{H}^n(X, Fil^m \Omega^{\cdot}_{\mathcal{Y}/A}) = \mathbb{H}^n(X, \mathfrak{a} Fil^m \Omega^{\cdot}_{\mathcal{Y}/A}). \tag{64}$$

This equality holds for $m = 0$ by the projection formula. Indeed, consider the canonical map:

$$Fil^m \Omega_{\mathcal{Y}/A} \to \mathfrak{a}^m \Omega^0_{\mathcal{Y}/A} \to 0.$$

The kernel is the following complex C:

$$0 \to \mathfrak{a}^{m-1}\Omega^1_{\mathcal{Y}/A} \to \cdots \to \mathfrak{a}\Omega^{m-1}_{\mathcal{Y}/A} \to \Omega^m_{\mathcal{Y}/A} \to \ldots.$$

This complex C is of the same nature as $Fil^m \Omega^{\cdot}_{\mathcal{Y}/A}$ but with less ideals involved. By an induction we may assume that

$$\mathfrak{a}\mathbb{H}^n(X, C) = \mathbb{H}^n(X, \mathfrak{a}C).$$

By the projection formula we find

$$\mathfrak{a}H^n(X, \mathfrak{a}^m \Omega^0_{\mathcal{Y}/A}) = \mathfrak{a}^{m+1} H^n(X, \Omega^0_{\mathcal{Y}/A}).$$

The assertion (64) follows from the diagram

$$\begin{array}{ccccc}
\mathbb{H}^n(X, \mathfrak{a}C) & \longrightarrow & \mathbb{H}^n(X, \mathfrak{a} Fil^m \Omega^{\cdot}_{\mathcal{Y}/A}) & \longrightarrow & H^n(X, \mathfrak{a}^{m+1}\Omega^0_{\mathcal{Y}/A}) \\
\| \uparrow & & \cup \uparrow & & \| \uparrow \\
\mathfrak{a}\mathbb{H}^n(X, C) & \longrightarrow & \mathfrak{a}\mathbb{H}^n(X, Fil^m \Omega^{\cdot}_{\mathcal{Y}/A}) & \longrightarrow & \mathfrak{a}H^n(X, \mathfrak{a}^m \Omega^0_{\mathcal{Y}/A})
\end{array} \tag{65}$$

The upper line is a short exact sequence by a spectral sequence argument as above. The lower line is a complex. The first arrow is injective and the second surjective but it is a priori not exact in the middle term. One sees that the upper and lower line in (63) must be isomorphic. This proves (65).
We have already seen that the following maps are injective

$$\mathbb{H}^n(X, \mathfrak{a} Fil^m \Omega^{\cdot}_{\mathcal{Y}/A}) \to \mathbb{H}^n(X, Fil^{m+1}\Omega^{\cdot}_{\mathcal{Y}/A}) \to \mathbb{H}^n(X, Fil^m \Omega^{\cdot}_{\mathcal{Y}/A}).$$

Therefore we obtain an exact sequence

$$0 \to \mathbb{H}^n(X, \mathfrak{a}Fil^m\Omega^{\cdot}_{\mathcal{Y}/A}) \to \mathbb{H}^n(X, Fil^{m+1}\Omega^{\cdot}_{\mathcal{Y}/A}) \to \mathbb{H}^n(X, \sigma^{\geq m+1}\Omega^{\cdot}_{X/R}) \to 0.$$

Since by (64) the map $\mathfrak{a} \otimes \mathbb{H}^n(X, Fil^m\Omega_{\mathcal{Y}/A}) \to \mathbb{H}^n(X, \mathfrak{a}Fil^m\Omega^{\cdot}_{\mathcal{Y}/A})$ is surjective, we see that

$$P_m = \mathbb{H}^n(X, Fil^m\Omega^{\cdot}_{\mathcal{Y}/A}) \text{ and } E_m = \mathbb{H}^n(X, \sigma^{\geq m}\Omega^{\cdot}_{X/R})$$

fulfill the conditions (i)-(iii) for a window without any restriction on m and n. We note that for fixed n we have $P_{m+1} = \mathfrak{a}P_m$ for $m \geq n$. As explained after the definition of a window, we can obtain a decomposition

$$P_m = \mathfrak{a}^m L_0 \oplus \mathfrak{a}^{m-1}L_1 \oplus \cdots \oplus \mathfrak{a}^{m-n}L_n,$$

with the convention that $\mathfrak{a}^k = A$ if $k \leq 0$.
Concretely we can find the liftings L_i as follows. We consider the maps:

$$\mathbb{H}^n(X, Fil^m\Omega^{\cdot}_{\mathcal{Y}/A}) \to \mathbb{H}^n(X, \sigma^{\geq m}\Omega^{\cdot}_{\mathcal{Y}/A}) \to H^{(n-m)}(X, \Omega^m_{\mathcal{Y}/A})$$

Then L_m is obtained by splitting the last surjection. This construction gives isomorphisms:

$$L_m \cong H^{(n-m)}(X, \Omega^m_{\mathcal{Y}/A})$$

We now impose the condition $m \leq n < p$ of the theorem. By lemma 5.4 and (62) the Frobenius endomorphism $F : P_0 \to P_0$ is divisible by p^m when restricted to P_m. We set

$$\Phi_m = \frac{1}{p^m}F_{|L_m}.$$

The assertion that $\{P_m\}$ is a window is then equivalent with the condition that

$$\oplus_{i=0}^n \Phi_i : \oplus_{i=0}^n L_i \to \oplus_{i=0}^u L_i$$

is a σ-linear isomorphism, or in other words that $\det(\oplus_{i=0}^n \Phi_i)$ is a unit in $W(A)$. Clearly it suffices to show that for any homomorphism $R \to k$ to a perfect field k the image of $\det(\oplus\Phi_i)$ by the morphism

$$A \xrightarrow{\varkappa} W(R) \to W(k) \to k$$

is a nonzero. The compositum map $A \to W(k)$ respects the Frobenius and induces a map on crystalline cohomology

$$H^n_{crys}(X/A, \mathcal{O}_{X/A}) \to H^n_{crys}(X_k/W(k), \mathcal{O}_{X_k/W(k)})$$

which respects the Frobenius. It is induced by the base change map for de Rham cohomology.

$$\mathbb{H}^n(X, \Omega^{\cdot}_{\mathcal{Y}/A}) \to \mathbb{H}^n(X_k, \Omega^{\cdot}_{Y \otimes_A W(k)/W(k)}).$$

The special decomposition we have chosen

$$\mathbb{H}^n(X, \Omega^{\cdot}_{\mathcal{Y}/A}) = \oplus L_i,$$

induces a similar decomposition

$$\mathbb{H}^n(X_k, \Omega^{\cdot}_{\mathcal{Y}_{W(k)}/W(k)}) = \mathbb{H}^n(X, \Omega^{\cdot}_{\mathcal{Y}/A}) \otimes_A W(k) = \oplus L_i \otimes_A W(k).$$

Therefore we have reduced our assertion to the case $R = k$ a perfect field and $A = W(k)$. This case was proved by Mazur (Compare [Fo] p.91 and Kato [K] Prop.2.5). We give an argument in the case $n < p - 2$ which is based on the comparison Corollary 4.7 but doesn't use gauges.
For any complex $\mathcal{A}^{\cdot}$ of abelian sheaves on X consider the exact sequence induced by the naive filtration.

$$0 \to \sigma_{>i}\mathcal{A}^{\cdot} \to \mathcal{A}^{\cdot} \to \sigma_{\leq i}\mathcal{A}^{\cdot} \to 0,$$

where i is an arbitrary integer. If $n + 1 \leq i$ we obtain an isomorphism

$$\mathbb{H}^n(X, \mathcal{A}^{\cdot}) \cong \mathbb{H}^n(X, \sigma_{\leq i}\mathcal{A}^{\cdot}).$$

We apply this to the Nygaard complex $\mathcal{N}^m W\Omega^{\cdot}_{X/k}$ and to the de Rham-Witt complex $W\Omega^{\cdot}_{X/k}$. For $i \leq m-1$ the operator $\hat{F}_m$ (5) induces clearly a bijection of the truncated complexes

$$\hat{F}_m : \sigma_{\leq i}\mathcal{N}^m W\Omega^{\cdot}_{X/k} \to \sigma_{\leq i} W\Omega^{\cdot}_{X/k}$$

Therefore if $n + 1 \leq i \leq m - 1$ we obtain a bijection

$$F_m : \mathbb{H}^n(X, \mathcal{N}^m W\Omega^{\cdot}_{X/k}) \to \mathbb{H}^n(X, W\Omega^{\cdot}_{X/k})$$

We set $m = n + 2$. Since $m < p$ by assumption (and because k is reduced) there are canonical isomorphisms in the derived category:

$$\mathcal{N}^m W\Omega^{\cdot}_{X/k} \cong \mathcal{F}^m\Omega_{\mathcal{Y}/W(k)} \cong Fil^m\Omega_{\mathcal{Y}/W(k)}$$

But since $m > n$ the map F_m is identified with the linearization of $\oplus\Phi_i$. This says that the last map is a Frobenius linear isomorphism. *Q.E.D.*
Remark: The proof shows that $H^n_{DR}(\mathcal{Y})$ with its Hodge filtration is strongly divisible (compare [Fo] 1.2 Prop.) for $n < p - 2$. If we knew that $\mathcal{N}^m W\Omega^{\cdot}_{X/k}$ and $\mathcal{F}^m\Omega_{\mathcal{Y}/W(k)}$ are quasi-isomorphic, the last argument would imply that $H^n_{DR}(\mathcal{Y})$ is strongly divisible without restriction on n. We note also that the last argument works directly over any reduced ring k.

Corollary 5.6 *Let X be a smooth and projective scheme over a ring R such that p is nilpotent in R.*
Let us assume that there is a frame $A \to R$ and a smooth and projective p-adic lifting $\mathcal{Y}/\operatorname{Spf} A$ of X, which satisfies the conditions of the theorem.
Then we obtain for $n < p$ by base change a display structure of degree n on $H^n_{crys}(X/W(R), \mathcal{O}_{X/W(R)})$. This display structure is independent of the frame A and the formal lifting $\mathcal{Y}$ we have chosen if $p \cdot R = 0$.

Proof: For a given frame A the independence of the lifting $\mathcal{Y}$ is clear, because the window structure is purely defined in terms of the crystalline cohomology of X/A.
If we have a morphism of frames $B \to A$ and a formal lifting $\mathcal{Z}$ of X to B, then we set $\mathcal{Y} = \mathcal{Z}_A$. Then the window associated to $\mathcal{Y}$ is obtained from the window associated to $\mathcal{Z}$ by base change (one should think in terms of decompositions (53)). Therefore the induced displays are the same.
If $p \cdot R = 0$ and A' and A'' are 2 frames, we obtain a new frame $A' \times_R A'' \to R$. Then $\sigma' \times \sigma''$ is an endomorphism of $A' \times_R A''$ because σ' and σ'' induce the same endomorphism on R. If $\mathcal{Y}'/\operatorname{Spf} A'$ and $\mathcal{Y}''/\operatorname{Spf} A''$ are formal liftings, we obtain a formal lifting $\mathcal{Y}' \times_\kappa \mathcal{Y}''$ of X over $A' \times_R A''$. Therefore we obtain the same display structure by base change.

Theorem 5.7 *Let R be a reduced ring of characteristic p. Let X/R be a smooth projective scheme. Assume that there is a compatible system of smooth and projective liftings $Y_n/W_n(R)$. We assume that the assumptions 5.2 and 5.3 are satisfied with $A_n = W_n(R)$*
Then there is a display structure on $H^n_{crys}(X/W(R), \mathcal{O}_{X/W(R)})$ for $n < p$, where
$$P_m = \mathbb{H}^n(X, \mathcal{N}^m W\Omega_{X/R}) = H^n_{crys}(X/W(R), \mathcal{J}^{[m]}_{X/W(R)}).$$

Proof: The second equality is the filtered comparison theorem. If we had a p-adic lifting $\mathcal{Y}/\operatorname{Spf} W(R)$, the theorem would follow from the last one because $W(R) \to R$ is a frame. The slightly more general statement follows by the same reasoning as the last theorem. *Q.E.D.*
We make the following conjecture:

Conjecture 5.8 *Let R be a ring such that p is nilpotent in R. Let X/R be a smooth projective scheme. Let us assume that the crystalline cohomology groups $H^i_{crys}(X/W_n(R))$ are locally free $W_n(R)$-modules for $i \geq 0$ and $n > 1$, and that the de Rham spectral sequence*

$$E_1^{p,q} = H^q(X, \Omega^p_{X/R}) \Rightarrow \mathbb{H}^{p+q}(X, \Omega^{\cdot}_{X/R})$$

degenerates.
Then the canonical predisplay structure on $P_m = \mathbb{H}^n(X, \mathcal{N}^m W\Omega_{X/R})$ is a display structure.

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