Motivic Splitting Lemma

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Abstract. Let M be a Chow motive over a field F. Let X be a smooth projective variety over F and N be a direct summand of the motive of X. Assume that over the generic point of X the motives M and N become isomorphic to a direct sum of twisted Tate motives. The main result of the paper says that if a morphism $f : M \to N$ splits over the generic point of X then it splits over F, i.e., N is a direct summand of M. We apply this result to various examples of motives of projective homogeneous varieties.

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1 Introduction

By a variety X over a field F we always mean a reduced and irreducible scheme of finite type over F. By $F(X)$ we denote the function field of X.

Definition 1.1. Let M be a Chow motive over F. We say M is *split* over F if it is a direct sum of twisted Tate motives over F. We say a motive M is *generically split* if there exists a smooth projective variety X over F and an integer l such that M is a direct summand of the twisted motive $M(X)\{l\}$ of X and M is split over $F(X)$. In particular, a smooth projective variety X is called *generically split* if its Chow motive $M(X)$ is split over $F(X)$.

The classical examples of such varieties are Severi-Brauer varieties, Pfister quadrics and maximal orthogonal Grassmannians. In the present paper we provide useful technical tool to study motivic decompositions of generically split varieties (motives). Namely, we prove the following

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Theorem 1.2. *Let M be a Chow motive over a field F. Let X be a smooth projective variety over F and N be a direct summand of the motive of X. Assume that M and N are split over $F(X)$. Then a morphism $M \to N$ splits, i.e. N is a direct summand of M, if it splits over $F(X)$.*

The paper is organized as follows. In section 2 we introduce the category of Chow motives over a relative base. In section 3 we provide the version of the Rost nilpotence theorem for generically split varieties. In section 4 we prove the main result of this paper (see the above theorem). The last section is devoted to various applications and examples.

2 Chow motives over a relative base

Let X be a variety over a field F. We say X is *essentially smooth* over F if it is an inverse limit of smooth varieties X_i over F taken with respect to open embeddings. Let $\mathrm{CH}^m(X;\Lambda) = \mathrm{CH}^m(X) \otimes_{\mathbb{Z}} \Lambda$ denote the Chow group of codimension m cycles on X with coefficients in a commutative ring Λ. If X is essentially smooth, then $\mathrm{CH}^m(X;\Lambda) = \varinjlim \mathrm{CH}^m(X_i;\Lambda)$, where the limit is taken with respect to the pull-backs induced by open embeddings.
In the present section we introduce the category of *Chow motives* over an essentially smooth variety X with Λ-coefficients. Our arguments follow the paper [9].

I. First, we define the *category of correspondences* $\mathcal{C}(X;\Lambda)$. The objects of $\mathcal{C}(X;\Lambda)$ are smooth projective maps $Y \to X$. The morphisms are given by

$$\mathrm{Hom}([Y \to X],[Z \to X]) = \oplus_i \mathrm{CH}^{\dim(Z_i/X)}(Y \times_X Z_i;\Lambda),$$

where the sum is taken over all irreducible components Z_i of Z of relative dimensions $\dim(Z_i/X)$. The composition of two morphisms is given by the usual correspondence product

$$\psi \circ \phi = (p_{Y,T})_*\big((p_{Y,Z})^*(\phi) \cdot (p_{Z,T})^*(\psi)\big),$$

where $\phi \in \mathrm{Hom}([Y \to X],[Z \to X])$, $\psi \in \mathrm{Hom}([Z \to X],[T \to X])$ and $p_{Y,T}$, $p_{Y,Z}$, $p_{Z,T}$ are projections $Y \times_X Z \times_X T \to Y \times_X T$, $Y \times_X Z$, $Z \times_X T$.
The category $\mathcal{C}(X;\Lambda)$ is a tensor additive category, where the direct sum is given by $[Y \to X] \oplus [Z \to X] := [Y \coprod Z \to X]$ and the tensor product by $[Y \to X] \otimes [Z \to X] := [Y \times_X Z \to X]$ (cf. [9, §2-4]). As usual we denote by $\phi^t \in \mathrm{CH}(Z \times_X Y)$ the transposition of a cycle $\phi \in \mathrm{CH}(Y \times_X Z)$.
The category of *effective* Chow motives $\mathit{Chow}^{\mathrm{eff}}(X;\Lambda)$ can be defined as the pseudo-abelian completion of $\mathcal{C}(X;\Lambda)$. Namely, the objects are pairs (U,ρ), where U is an object of $\mathcal{C}(X;\Lambda)$ and $\rho \in \mathrm{End}_{\mathcal{C}(X;\Lambda)}(U)$ is a projector, i.e. $\rho \circ \rho = \rho$. The morphisms between (U_1,ρ_1) and (U_2,ρ_2) are given by the group $\rho_2 \circ \mathrm{Hom}_{\mathcal{C}(X;\Lambda)}(U_1,U_2) \circ \rho_1$. The composition of morphisms is induced by the correspondence product. In the case $X = \mathrm{Spec}(F)$ and $\Lambda = \mathbb{Z}$ we obtain

the usual category $\mathit{Chow}^{\mathrm{eff}}(F)$ of effective Chow motives over F with integral coefficients (cf. [9, §5]).
Consider the projective line $\mathbb{P}^1$ over F. The projector $\rho = [\mathrm{Spec}(F) \times \mathbb{P}^1] \in \mathrm{CH}^1(\mathbb{P}^1 \times \mathbb{P}^1)$ defines an object $(\mathbb{P}^1, \rho)$ in $\mathit{Chow}^{\mathrm{eff}}(F)$ called the *Tate motive* over F and denoted by $\mathbb{Z}\{1\}$ (cf. [9, §6]).

II. We have two types of restriction functors.

1) For any morphism $f\colon X_1 \to X_2$ of essentially smooth varieties we have a tensor additive functor

$$\mathit{res}_{X_2/X_1}\colon \mathcal{C}(X_2;\Lambda) \to \mathcal{C}(X_1;\Lambda)$$

given on the objects by $[Y_2 \to X_2] \mapsto [Y_2 \times_{X_2} X_1 \to X_1]$ and on the morphisms by $\phi \mapsto (\mathrm{id} \times f)^*(\phi)$, where $\mathrm{id} \times f : (Y_2 \times_{X_2} Z_2) \times_{X_2} X_1 \to Y_2 \times_{X_2} Z_2$ is the natural map. It induces a functor on pseudo-abelian completions

$$\mathit{res}_{X_2/X_1}\colon\ \mathit{Chow}^{\mathrm{eff}}(X_2;\Lambda) \to \mathit{Chow}^{\mathrm{eff}}(X_1;\Lambda).$$

2) For any homomorphism of commutative rings $h\colon \Lambda \to \Lambda'$ we have a tensor additive functor

$$\mathit{res}_{\Lambda'/\Lambda}\colon \mathcal{C}(X,\Lambda) \to \mathcal{C}(X;\Lambda')$$

which is identical on objects and is given by $\mathrm{id} \otimes h\colon \mathrm{CH}(Y \times_X Z;\Lambda) \to \mathrm{CH}(Y \times_X Z;\Lambda')$ on morphisms. Again, it induces a functor on pseudo-abelian completions

$$\mathit{res}_{\Lambda'/\Lambda}\colon\ \mathit{Chow}^{\mathrm{eff}}(X;\Lambda) \to \mathit{Chow}^{\mathrm{eff}}(X;\Lambda').$$

Observe that the functor $\mathit{res}_{\Lambda'/\Lambda}$ commutes with res_{X_2/X_1}. We denote by $\mathit{res}_{X_2/X_1,\Lambda'/\Lambda}$ the composite $\mathit{res}_{X_2/X_1} \circ \mathit{res}_{\Lambda'/\Lambda}$. To simplify the notation we omit X_2 (resp. Λ), if $X_2 = \mathrm{Spec}\, F$ (resp. $\Lambda = \mathbb{Z}$).
Let $f\colon X \to \mathrm{Spec}\, F$ and $h\colon \mathbb{Z} \to \Lambda$ be the structure maps. Then $\mathit{res}_{X,\Lambda}\colon\ \mathit{Chow}^{\mathrm{eff}}(F) \to \mathit{Chow}^{\mathrm{eff}}(X;\Lambda)$. Given a motive N over F we denote by $N_{X,\Lambda}$ its image $\mathit{res}_{X,\Lambda}(N)$ in $\mathit{Chow}^{\mathrm{eff}}(X;\Lambda)$. The image $\mathbb{Z}\{1\}_{X,\Lambda}$ of the Tate motive is denoted by T and is called the *Tate motive over* X. Let M be a motive from $\mathit{Chow}^{\mathrm{eff}}(X;\Lambda)$ and $l \geq 0$ be an integer. The tensor product $M \otimes T^{\otimes l}$ is denoted by $M\{l\}$ and is called the *twist* of M. The *trivial Tate motive* $T^{\otimes 0}$ will be denoted Λ (thus, $T^{\otimes l} = \Lambda\{l\}$).
The same arguments as in the proof of [9, Lemma of §8] show that for any motives U and V from $\mathit{Chow}^{\mathrm{eff}}(X;\Lambda)$ and $l \geq 0$ the natural map

$$\mathrm{Hom}_{\mathit{Chow}^{\mathrm{eff}}(X;\Lambda)}(U,V) \to \mathrm{Hom}_{\mathit{Chow}^{\mathrm{eff}}(X;\Lambda)}(U\{l\},V\{l\}) \tag{1}$$

given by $\phi \mapsto \phi \otimes \mathrm{id}_T$ is an isomorphism.

III. We define the category $Chow(X;\Lambda)$ of Chow motives over X with Λ-coefficients as follows. The objects are pairs (U,l), where U is an object of $Chow^{\mathrm{eff}}(X;\Lambda)$ and l is an integer. The morphisms are given by

$$\mathrm{Hom}((U,l),(V,m)) := \lim_{N\to+\infty} \mathrm{Hom}_{Chow^{eff}(X;A)}(U\{N+l\}, V\{N+m\}).$$

This is again a tensor additive category, where the sum and the product are given by

$$(U,l)\oplus(V,m) := (U\{l-n\}\oplus V\{m-n\}, n), \text{ where } n=\min(l,m),$$

$$(U,l)\otimes(V,m) := (U\otimes V, l+m).$$

Observe that the Tate motive T is isomorphic to $([\mathrm{id}: X\to X],1)$ and, hence, it is invertible in $(Chow(X;\Lambda),\otimes)$. Moreover, we can say that $Chow(X;\Lambda)$ is obtained from $Chow^{\mathrm{eff}}(X;\Lambda)$ by inverting T (cf. [9, §8]).
According to (1) the natural functor $Chow^{\mathrm{eff}}(X;\Lambda)\to Chow(X;\Lambda)$ given by $U\mapsto (U,0)$ is fully faithful and the restriction $res_{X,\Lambda}$ descend to the respective functor $res_{X,\Lambda}\colon Chow(F)\to Chow(X;\Lambda)$.
For a smooth projective morphism $Y\to X$ we denote by $M(Y\to X)$ its effective motive $([Y\to X],\mathrm{id})$ considered as an object of $Chow(X;\Lambda)$. If $X=\mathrm{Spec}\,F$ and $\Lambda=\mathbb{Z}$, then we denote the motive $M(Y\to X)$ simply by $M(Y)$. By definition there is a natural identification

$$\mathrm{Hom}_{Chow(X;\Lambda)}(M(Y\to X)\{i\}, M(Z\to X)\{j\}) = \mathrm{CH}^{\dim(Z/X)+j-i}(Y\times_X Z;\Lambda).$$

IV. Let M be an object of $Chow(X;\Lambda)$. We define the Chow group with low index $\mathrm{CH}_m(M;\Lambda)$ of M as

$$\mathrm{CH}_m(M;\Lambda) := \mathrm{Hom}_{Chow(X;\Lambda)}(\Lambda\{m\}, M)$$

and the Chow group with upper index $\mathrm{CH}^m(M;\Lambda)$ as

$$\mathrm{CH}^m(M;\Lambda) := \mathrm{Hom}_{Chow(X;\Lambda)}(M, \Lambda\{m\}).$$

Observe that if $M=M(Y\to X)$, then we obtain the usual Chow groups $\mathrm{CH}^{\dim(Y/X)-m}(Y;\Lambda)$ and $\mathrm{CH}^m(Y;\Lambda)$ of a variety Y. A composite with a morphism $f: M\to N$ induces a homomorphism between the Chow groups $R_m(f)\colon \mathrm{CH}_m(M)\to \mathrm{CH}_m(N)$ and $R^m(f)\colon \mathrm{CH}^m(N)\to\mathrm{CH}^m(M)$ called the *realization map*.

3 The Rost nilpotence

We will extensively use the following version of the Rost nilpotence (cf. [14, Proposition 9])

Proposition 3.1. *Let N be a generically split motive over a field F. Then for any field extension E/F and any coefficient ring Λ the kernel of the restriction*

$$res_{E/F} : \mathrm{End}_F(N) \to \mathrm{End}_E(N_E)$$

consists of nilpotents.

To simplify the notation we denote by $\mathrm{End}_X(M)$ the endomorphism group $\mathrm{Hom}_{Chow(X;\Lambda)}(M, M)$, where M is a motive over a variety X.

Proof. Recall that (see Definition 1.1) a motive N over F is generically split if there exists a smooth projective variety X and $l \in \mathbb{Z}$ such that N is a direct summand of $M(X)\{l\}$ and $N_K = res_{K/F}(N)$ is split, where $K = F(X)$ denotes the function field of X.
We may assume that N is a direct summand of $M(X)$ (that is, $l = 0$). Since for a split motive M and a field extension E/L, the map $\mathrm{End}_L(M_L) \to \mathrm{End}_E(M_E)$ is an isomorphism, we may assume that $E = K$.
Consider the composite of ring homomorphisms

$$res_{K/F}\colon \mathrm{End}_F(N) \xrightarrow{res_{X/F}} \mathrm{End}_X(N_X) \xrightarrow{res_{K/X}} \mathrm{End}_K(N_K),$$

where the last map is induced by passing to the generic point $\mathrm{Spec}\, K \to X$.
Observe that $\mathrm{End}_K(N_K) = \varinjlim \mathrm{End}_U(N_U)$, where the limit is taken over all open subvarieties $U \subset X$. Then $\ker(res_{K/X}) = \cup_U \ker(res_{U/X})$ and by Lemma 3.2 the kernel of $res_{K/X}$ consists of nilpotents.
On the other hand, the map $res_{X/F}$ is injective. Indeed, since N is a direct summand of $M(X)$, $\mathrm{End}_F(N)$ is a subring of $\mathrm{End}_F(M(X))$ and $\mathrm{End}_X(N_X)$ is a subring of $\mathrm{End}_X(M(X)_X)$. So, it is sufficient to prove the injectivity for the case $N = M(X)$. The restriction $res_{X/F}\colon \mathrm{End}_F(M(X)) \to \mathrm{End}_X(M(X)_X)$ coincides with the pull-back $\pi_{1,2}^*\colon \mathrm{CH}(X\times X;\Lambda) \to \mathrm{CH}(X\times X\times X;\Lambda)$ induced by the projection on the first two coordinates. And $\pi_{1,2}^*$ splits by $(\mathrm{id}_X \times \Delta_X)^*\colon \mathrm{CH}(X \times X \times X;\Lambda) \to \mathrm{CH}(X \times X;\Lambda)$, where $\Delta_X\colon X \to X \times X$ is the diagonal. The proposition is proven. □

Lemma 3.2. *Let X be a smooth projective variety over F and Λ be a commutative ring. Let $U \subset X$ be an open embedding. Then for any motive M from $Chow(X;\Lambda)$ the kernel of the restriction map*

$$res_{U/X} : \mathrm{End}_X(M) \to \mathrm{End}_U(M_U)$$

consists of nilpotents.

Proof. If M is a direct summand of $[Y \to X]\{i\}$, then $\mathrm{End}_X(M)$ is a subring of $\mathrm{End}_X(M(Y \to X))$ and it is sufficient to study the case $M = M(Y \to X)$.
Recall that $\mathrm{End}_X(M(Y \to X)) = \mathrm{CH}^{\dim(Y)-\dim(X)}(Y \times_X Y;\Lambda)$.
Let ϕ be an element from the kernel of $res_{U/X}$. Let $j\colon Z \to X$ be the reduced closed complement to U in X. Then by the localization sequence for Chow groups the cycle ϕ belongs to the image of the induced push-forward

$$(\mathrm{id}_{(Y\times_X Y)} \times j)_*\colon \mathrm{CH}((Y \times_X Y) \times_X Z;\Lambda) \to \mathrm{CH}(Y \times_X Y;\Lambda).$$

Let $\operatorname{codim}(Z)$ be the minimum of codimensions of irreducible components of Z, and $d := [\frac{\dim(X)}{\operatorname{codim}(Z)}]+1$. We claim that the d-th power $\phi^{\circ d}$ of ϕ taken with respect to the correspondence product is trivial. Indeed, $\phi^{\circ d} = (\pi_{1,d+1})_*(\phi_1\cdot\phi_2\cdot\ldots\cdot\phi_d)$, where $\phi_i = \pi^*_{i,i+1}(\phi)$ and the map $\pi_{i,i'}\colon Y^{\times(d+1)} \to Y\times_X Y$ is the projection on the i-th and i'-th components. Since $\pi^*_{i,i'}\circ(\mathrm{id}_{(Y\times_X Y)}\times j)_*$ coincides with $(\mathrm{id}_{Y^{\times(d+1)}}\times j)_*\circ(\pi_{i,i'}\times \mathrm{id}_Z)^*$, all cycles ϕ_i belong to the image of the push-forward

$$(\mathrm{id}_{Y^{\times(d+1)}}\times j)_*\colon\ \mathrm{CH}(Y^{\times(d+1)}\times_X Z)\to \mathrm{CH}(Y^{\times(d+1)}).$$

By Proposition 6.1 applied to the projection $Y^{\times(d+1)}\to X$ and the closed embedding $j\colon Z\hookrightarrow X$ we obtain that the product

$$\phi_1\cdot\ldots\cdot\phi_d\in\big((\mathrm{id}_{Y^{\times(d+1)}}\times j)_*\,\mathrm{CH}(Y^{\times(d+1)}\times_X Z)\big)^d$$

is trivial. Therefore, $\phi^{\circ d}$ is trivial as well. □

We finish this section with the following

Definition 3.3. Given motive M over a field F and a field extension L/F we say a cycle in $\mathrm{CH}(M_L)$ is *rational* if it is in the image of the restriction map $res_{L/F}$.

Observe that the rationality of cycles is preserved by push-forward and pull-back maps. It also respects addition, intersection and correspondence product of cycles.

4 Motivic splitting lemma

In the present section we prove the main result of this paper

Theorem 4.1. *Let M be a Chow motive over a field F. Let X be a smooth projective variety over F and N be a direct summand of the motive of X. Assume that M and N are split over the function field $K = F(X)$. Then a morphism $f\colon M\to N$ splits, i.e. N is a direct summand of M, if it splits over K.*

Proof. To construct a section of f we apply recursively the following procedure starting from $g = 0$ and such m that $\mathrm{CH}^i(N_K) = 0$, for $i < m$.

> For a morphism $g : N \to M$ such that the realization morphism $R^i(f_K\circ g_K)$ is the identity on $\mathrm{CH}^i(N_K)$ for $i < m$, we construct a new morphism $g' : N\to M$ such that $R^i(f_K\circ g'_K)$ is the identity on $\mathrm{CH}^i(N_K)$ for $i\le m$.

Since the motive N_K splits, for the corresponding projector ρ_N over K we may write $(\rho_N)_K = \sum_l \omega_l\times\omega_l^\vee$ for certain $\omega_l\in \mathrm{CH}^*(X_K)$ and $\omega_l^\vee\in\mathrm{CH}_*(X_K)$ such that $\deg(\omega_l \cdot \omega_m^\vee) = \delta_{l,m}$. Elements ω_l form a basis of $\mathrm{CH}^*(N_K) = (\rho_N)_K \circ \mathrm{CH}^*(X_K) \subset \mathrm{CH}^*(X_K)$.

Consider the surjection $\mathrm{CH}^m(X \times X) \twoheadrightarrow \mathrm{CH}^m(K \times_F X) = \mathrm{CH}^m(X_K)$. Let Ω_l be a preimage of an element ω_l of $\mathrm{CH}^m(X_K)$.

Consider the difference $\mathrm{id} - f \circ g$ and denote it by h. Assume that over K it sends a basis element ω_j to a cycle α_j. Since $R^i(h_K)$ is trivial for all $i < m$, the cycle $h_K = h_K \circ (\rho_N)_K$ can be written as

$$h_K = \sum_{\operatorname{codim} \alpha_l = m} \alpha_l \times \omega_l^\vee + \sum_{\operatorname{codim} \alpha_j > m} \alpha_j \times \omega_j^\vee \in \mathrm{CH}^{\dim X}(X_K \times X_K). \quad (2)$$

From (2) we immediately see that

$$\alpha_l = \mathrm{pr}_{1*}(\Omega_{l,K} \cdot h_K) \in \mathrm{CH}^m(X_K) \text{ is rational.} \quad (3)$$

Also, $\alpha_l \circ (\rho_N)_K = \alpha_l$.

The realization $R^m(f_K)$ is a $\mathbb{Z}$-linear map $\mathrm{CH}^m(N_K) \to \mathrm{CH}^m(M_K)$. Let $C = (c_{ij})$ be the respective matrix of coefficients, i.e.,

$$R^m(f_K) : \omega_i \mapsto \sum_j c_{ji} \theta_j,$$

where $\{\theta_i\}$ is a $\mathbb{Z}$-basis of $\mathrm{CH}^m(M_K)$. Let $s : N_K \to M_K$ be a section of f_K. The realization map $R^m(s)$ is a left inverse to $R^m(f_K)$. Hence, for the respective matrix of coefficients $D = (d_{ij})$ we have

$$R^m(s) : \theta_i \mapsto \sum_j d_{ji} \omega_j$$

and $D \cdot C = \mathrm{id}$, i.e., $\sum_j d_{ij} c_{jk} = \delta_{ik}$. For each α_l define the morphism $u_l : N \to M$ as

$$u_l = \sum_i d_{li} \Theta_i^\vee \circ (\mathrm{pr}_1^*(\alpha_l) \cdot \Delta_X) \circ p_N,$$

where $\Theta_i^\vee$ is a preimage of an element $\theta_i^\vee$ of $\mathrm{CH}_m(M_K)$ by means of the canonical surjection $\mathrm{Hom}_F(M(X)(m)[2m], M) \to \mathrm{CH}_m(M_K)$ and $p_N : N \to M(X)$ be the morphism presenting N as a direct summand of $M(X)$. By definition, u_l is a rational morphism and the realization $R^m(u_l)$ is given by

$$\theta_i \mapsto d_{li} \alpha_l$$

Hence, the composite $R^m(f \circ u_l) = R^m(u_l) \circ R^m(f)$ maps ω_i to $\delta_{il} \alpha_l$.

Set $\tilde{g} = g + \sum_l u_l$. By construction, the realization $R(f \circ \tilde{g})$ is the identity on $\mathrm{CH}^i(N_K)$ for $i \le m$. Consider the endomorphism $\mathrm{id} - f \circ \tilde{g}$ of N. Over K its realization $R^i(\mathrm{id} - f \circ \tilde{g})$ is trivial for each $i \le m$.

Recursion step is proven and we obtain map $g' : N \to M$ such that $(f \circ g')_K = id_{N_K}$. Let $q = id - f \circ g'$. By the Proposition 3.1, $q^r = 0$, for some r. Set $g = g' \circ (id + q + q^{\circ 2} + \ldots + q^{\circ (r-1)})$. Then $f \circ g = id_N$ and N is a direct summand of M. □

5 Examples and Applications

Geometric construction of a generalized Rost motive. Let p be a prime and F be a field of characteristic different from p. Let n be a positive integer. To each nonzero cyclic subgroup $\langle\alpha\rangle$ in $K_n^M(F)/p$ consisting of pure symbols one can assign some motive M_α in the category $Chow(F;\mathbb{Z}/p\mathbb{Z})$, which satisfies the following property
For an arbitrary field extension E/F

$\alpha|_E \neq 0 \iff (M_\alpha)_E = res_{E/F}(M_\alpha)$ is indecomposable;
$\alpha|_E = 0 \iff (M_\alpha)_E$ is split.

It follows from the results of V. Voevodsky and M. Rost that for a given subgroup such motive always exists and is unique (see [17, § 5] and [15, Prop. 5.9]). Moreover, when split it is isomorphic to

$$\bigoplus_{i=0}^{p-1} \mathbb{Z}/p\mathbb{Z}\{i \cdot \tfrac{p^{n-1}-1}{p-1}\}.$$

Such a motive is called a *generalized Rost motive with $\mathbb{Z}/p\mathbb{Z}$-coefficients.*

Definition 5.1. A motive with integral coefficients which specializes modulo p into a generalized Rost motive and splits modulo q for every prime q different from p will be called an *integral generalized Rost motive* and denoted by $\mathcal{R}_{n,p}$.

Integral generalized Rost motives, hypothetically, should be parameterized not by the pure cyclic subgroups of $\mathrm{K}_n^M(F)/p$, but by the pure symbols of $\mathrm{K}_n^M(F)/p$ up to a sign. The existence of integral generalized Rost motives is known for $n = 2$ and arbitrary p, for $p = 2$ and arbitrary n, and for the pair $n = 3, p = 3$. All these examples are essentially due to M. Rost.
As the first application of Theorem 4.1 we obtain the construction of the classical integral Rost motive corresponding to a Pfister form.

Corollary 5.2. *(cf. [14, Theorem 17.(9) and Proposition 19]) Let X be a hyperplane section of a n-fold Pfister quadric Y over a field F. Then $M(Y) \simeq M(X)\{1\} \oplus \mathcal{R}_{n,2}$, where $\mathcal{R}_{n,2}$ is an integral Rost motive.*

Proof. In the proof we use several auxiliary facts concerning quadrics and their motives which can be found in [5].
Let ϕ_X and ϕ_Y be the quadratic forms which define X and Y. By definition ϕ_X is a subform of codimension 1 of the Pfister form ϕ_Y. According to [5, Def.5.1.2 and Thm.5.3.4.(a)] Y becomes isotropic over $K = F(X)$. This fact together with [5, Prop.4.2.1] implies that both ϕ_X and ϕ_Y become totally split (hyperbolic) over K. Then by [5, E.10.8] the motives $M(X)_K$ and $M(Y)_K$ are split over K.
Let Γ_e be the graph of the closed embedding $e\colon X \hookrightarrow Y$. The respective correspondence cycle $[\Gamma_e] \in \mathrm{CH}_{\dim X}(X \times Y)$ induces the realization map $R^*(\Gamma_e)$

which coincides with the pull-back $e^*\colon \mathrm{CH}^*(Y) \to \mathrm{CH}^*(X)$ (see §2.IV). It is known that the Chow ring of a hyperbolic quadric is generated by two elements $\langle h, l\rangle$, where h is the class of a hyperplane section and l is the class of a maximal totally isotropic subspace. In this notation the pull-back e_K^* maps $h_Y \mapsto h_X$ and $l_Y \mapsto l_X$, i.e. maps the ring $\mathrm{CH}(Y_K)$ *onto* the ring $\mathrm{CH}(X_K)$. The latter means that $R^*(\Gamma_e)$ and, therefore, the transposed correspondence cycle $[\Gamma_e]^t \in \mathrm{CH}_{\dim Y-1}(Y \times X)$ have a section over K.
Take $f = [\Gamma_e]^t\colon M(Y) \to M(X)\{1\}$ and apply Theorem 1.2. We obtain the decomposition $M(Y) = M(X)\{1\} \oplus N$, where N is such that

$$N_K = \mathbb{Z} \oplus \mathbb{Z}\{2^{n-1} - 1\}.$$

Let E/F be a field extension. The Pfister quadric Y corresponds to some pure symbol $\alpha \in \mathrm{K}_n^M(F)/2$ (see [5, §9.4]) with the property that $\alpha|_E = 0$ if and only if Y_E has a rational point. Consider the specialization $N_{E,\mathbb{Z}/2}$ with $\mathbb{Z}/2$-coefficients. We have the following chain of equivalences: $N_{E,\mathbb{Z}/2}$ is decomposable $\Leftrightarrow$ $N_{E,\mathbb{Z}/2}$ contains $\mathbb{Z}/2$ as a direct summand $\Leftrightarrow$ $M(Y;\mathbb{Z}/2)_E$ contains $\mathbb{Z}/2$ as a direct summand $\Leftrightarrow$ (see [14, §1.4]) Y_E has a zero-cycle of odd degree $\Leftrightarrow$ (Springer Theorem) Y_E has a rational point. At the same time, the specialization $N_{\mathbb{Z}/p}$ is split for any odd prime p, since $M(Y;\mathbb{Z}/p)$ is split. Hence, N is an integral generalized Rost motive corresponding to the symbol α. □

To provide the next application we use several auxiliary facts concerning Albert algebras and Cayley planes which can be found in [4], [8], [11], [12]. We use the notation of [12, §3].
Consider an Albert algebra J defined by means of the first Tits construction. Let $\mathrm{F}_4(J)$ and $\mathrm{E}_6(J)$ denote the respective simple groups of types F_4 and E_6. Let X be the variety of maximal parabolic subgroups of $\mathrm{F}_4(J)$ of type P_4. Let Y be the variety of maximal parabolic subgroups of $\mathrm{E}_6(J)$ of type P_1 Here P_i corresponds to a standard parabolic subgroup generated by the Borel subgroup and all unipotent subgroups corresponding to linear spans of all simple roots with no i-th terms (our enumeration of roots follows Bourbaki). The variety Y is called a *(twisted) Cayley plane*.
Observe that there is a closed embedding $e\colon X \hookrightarrow Y$ such that over the splitting field K of J the class $[X_K] \in \operatorname{Pic} Y_K$ generates the Picard group of Y_K. In other words, X_K is a hyperplane section of Y_K (see [8, 6.3]).

COROLLARY 5.3. *Let X and Y be as above. Then $M(Y) \simeq M(X)\{1\} \oplus \mathcal{R}_{3,3}$, where $\mathcal{R}_{3,3}$ is an integral generalized Rost motive corresponding to the Serre-Rost invariant $g_3(J)$ in $K_3^M(F)/3$.*

Proof. We follow the previous proof step by step.
Let K denote the function field of X. Analyzing the Tits indices of $\mathrm{F}_4(J)$ we conclude that J becomes reduced over K. Moreover, since J is defined by means of the *first* Tits construction, J becomes split over K. By definition it implies that both groups and varieties become split over K.

Consider now the graph Γ_e of the closed embedding $e\colon X \hookrightarrow Y$. As before, the respective correspondence cycle $[\Gamma_e]$ induces the realization map $R^*(\Gamma_e)$ which coincides with the pull-back e^*. The Chow rings $\mathrm{CH}(X_K)$ and $\mathrm{CH}(Y_K)$ are generated by $\langle h, g_1^4\rangle$ (see [10, 4.10]) and $\langle H, \sigma_4', \sigma_8\rangle$ (see [4, 5.1]). By the Lefschetz hyperplane theorem the pull-back e^* has to be an isomorphisms on all graded components of codimensions ≤ 7. This immediately implies that e^* maps $H \mapsto h$ and $\sigma_4' \mapsto g_1^4$, i.e. maps the ring $\mathrm{CH}(Y_K)$ *onto* the ring $\mathrm{CH}(X_K)$. So $R^*(\Gamma_e)$ and, therefore, the transposed cycle $[\Gamma_e]^t$ have a section over K. Take $f = [\Gamma_e]^t\colon M(Y) \to M(X)\{1\}$ and apply Theorem 1.2. We obtain the decomposition $M(Y) = M(X)\{1\} \oplus N$, where the motive N is such that

$$N_K = \mathbb{Z} \oplus \mathbb{Z}\{4\} \oplus \mathbb{Z}\{8\}.$$

Let E/F be a field extension. Let $\alpha = g_3(J) \in K_3^M(F)/3$ be the Serre-Rost invariant of the Jordan algebra J (see [13]). Analyzing the Tits indices of $\mathrm{E}_6(J)$ we see that $\alpha|_E = 0$ if and only if Y_E has a zero-cycle of degree coprime to 3. Consider the specialization $N_{E,\mathbb{Z}/3}$ with $\mathbb{Z}/3$-coefficients. Similar to the quadric case there is a chain of equivalences which says that $N_{E,\mathbb{Z}/3}$ is decomposable $\Leftrightarrow$ Y_E has a zero-cycle of degree coprime to 3. At the same time, the specialization $N_{\mathbb{Z}/p}$ is split for any prime $p \neq 3$, since $M(Y;\mathbb{Z}/p)$ is split. Therefore, N is an integral generalized Rost motive corresponding to the symbol α. □

Remark 1. Observe that in view of the main result of [10] we obtain the following decomposition

$$M(Y) \simeq \bigoplus_{i=0}^{8} \mathcal{R}_{3,3}\{i\}.$$

So from the motivic point of view the variety Y is a 3-analog of a Pfister quadric.

Projective homogeneous varieties of type F_4. As before let J be an Albert algebra defined by means of the first Tits construction. Let $\mathrm{F}_4(J)$ be the respective group of type F_4. Let X be the same as before, i.e. the variety of maximal parabolic subgroups of type P_4 of $\mathrm{F}_4(J)$. Let Y be the variety of maximal parabolic subgroups of type P_3 of $\mathrm{F}_4(J)$. Observe that Y has dimension 20.

Corollary 5.4. *Let X and Y be as above. Then the motive $M(X;\mathbb{Z})$ is isomorphic to a direct summand of the motive $M(Y;\mathbb{Z})$.*

Proof. Since the Albert algebra J splits over the function field K of X, the motives $M(X)$ and $M(Y)$ become split over K as well. By the main result of [10] $M(X)$ splits as

$$M(X) \simeq \bigoplus_{i=0}^{7} \mathcal{R}_{3,3}\{i\},$$

where $\mathcal{R}_{3,3}$ is the integral generalized Rost motive corresponding to $g_3(J)$.
Let Z be the variety of parabolic subgroups of type $P_{3,4}$ of $\mathrm{F}_4(J)$. Observe that Z has dimension 21 and there is a map $\mathrm{pr}_{XY} = (\mathrm{pr}_X, \mathrm{pr}_X)\colon Z \to X \times Y$, where pr_X, pr_Y are the quotient maps. For each $i = 0 \dots 7$ consider the composite

$$f_{\alpha_i} : M(Y) \xrightarrow{\mathrm{pr}_{XY*}(\alpha_i)} M(X) \to \mathcal{R}_{3,3}\{i\}, \text{ where } \alpha_i \in \operatorname{Pic} Z.$$

Set $f = \bigoplus_{i=0}^{7} f_{\alpha_i}\colon M(Y) \to M(X)$. Assume that we can choose $\alpha_i \in \operatorname{Pic} Z$ in such a way that the realization map $R^*(f)$ becomes split injective over K. Then by Theorem 1.2 applied f, the motive $M(X)$ is isomorphic to a direct summand of $M(Y)$.
So to prove the corollary it is enough to find $\alpha_i \in \operatorname{Pic} Z$, $i = 0 \dots 7$, such that $R^*(f)$ is split injective over K.
Observe that the restriction map $res_{K/F}\colon \operatorname{Pic} Z \to \operatorname{Pic} Z_K$ is an isomorphism (see [10, Lemma 4.3]). Therefore, we may assume that $\alpha_i \in \operatorname{Pic} Z_K$. Observe also that the ring structures of $\mathrm{CH}(X_K)$, $\mathrm{CH}(Y_K)$ and $\mathrm{CH}(Z_K)$ are known. We have $R^*(f_{\alpha_i})_K = R^*(\alpha_i)_K \circ R^*(\rho_i)_K$, where ρ_i is an idempotent defining $\mathcal{R}_{3,3}\{i\}$. Both realizations $R^*(\rho_i)_K$ and $R^*(\alpha_i)_K$ can be described explicitly on generators. Indeed, the realization $R^*(\alpha_i)_K$ is given by the composite $\mathrm{CH}(X_K) \xrightarrow{\mathrm{pr}_X^*} \mathrm{CH}(Z_K) \xrightarrow{\cdot\alpha_i} \mathrm{CH}(Z_K) \xrightarrow{\mathrm{pr}_{Y*}} \mathrm{CH}(Y_K)$, where the maps pr_X^* and pr_{Y*} can be described using [10, §3]. The explicit description of the cycles $(\rho_i)_K$ is provided in [10, 5.5].
Let $\{\alpha_i = c_{1i}g_1 + c_{2i}g_2\}_{i=0\dots7}$, $c_{1i}, c_{2i} \in \mathbb{Z}$, be the presentation of the cycles α_i in terms of a fixed $\mathbb{Z}$-basis $\langle g_1, g_2\rangle$ of $\operatorname{Pic} Z_K$. Since all realization maps $R^*(\alpha_i)_K$, $R^*(\rho_i)_K$ are $\mathbb{Z}$-linear, the question of split injectivity of $R^*(f)_K$ translates into the problem of solving certain system of $\mathbb{Z}$-linear equations in 16 variables $\{c_{1i}, c_{2i}\}_{i=0\dots7}$. Direct computations show that this system has a solution. This finishes the proof of the corollary. □

Twisted forms of Grassmannians. Consider a Grassmannian $\mathbb{G}(d,n)$ of d-dimensional planes in a n-dimensional affine space. Its twisted form is called a generalized Severi-Brauer variety and denoted by $\mathrm{SB}_d(A)$, where A is the respective central simple algebra of degree n (see [7, §1.C]). The next corollary relates the motive of a generalized Severi-Brauer variety with the motive of usual Severi-Brauer variety.

Corollary 5.5. *Let A and B be two central division algebras of degree n with $[A] = \pm d[B]$ in the Brauer group $\mathrm{Br}(F)$, where d and n are coprime. Then the motive of the Severi-Brauer variety $\mathrm{SB}(A)$ is a direct summand in the motive of the generalized Severi-Brauer variety $\mathrm{SB}_d(B)$.*

Proof. We construct the morphism $f : M(\mathrm{SB}_d(B)) \to M(\mathrm{SB}(A))$ as follows. Consider the Plücker embedding $pl : \mathrm{SB}_d(B) \to \mathrm{SB}(\Lambda^d B)$. It induces the morphism $M(\mathrm{SB}_d(B)) \to M(\mathrm{SB}(\Lambda^d B))$, where $\Lambda^d B$ is the d-th lambda power of B (see [7, II.10.A]). By [6, Cor. 1.3.2] the motive $M(\mathrm{SB}(\Lambda^d B))$ splits as a direct sum of shifted copies of $M(\mathrm{SB}(A))$, where $[A] = d[B]$ in $\mathrm{Br}(F)$. Take f to be the composite of the Plücker embedding and the projection $M(\mathrm{SB}(\Lambda^d B)) \to M(\mathrm{SB}(A))$.

We claim that f has a section (splits) over the generic point of $\mathrm{SB}(A)$. Indeed, it is equivalent to the fact that for each $m = 0, \ldots, n-1$

$$\underset{i}{g.c.d.}(c_i^{(m)}) = 1$$

where $c_i^{(m)}$ are degrees of the Schubert varieties generating $\mathrm{CH}^m(\mathbb{G}(d,n))$. The latter can be computed using explicit formulas for degrees of Schubert varieties provided for instance in [3, Ch. 14, Ex. 14.7.11.(ii)].

Then by Theorem 1.2 the motive $M(\mathrm{SB}(A))$ is a direct summand in $M(\mathrm{SB}_d(B))$. Observe that the motives $M(\mathrm{SB}(A))$ and $M(\mathrm{SB}(A^{\mathrm{op}}))$ are isomorphic. So replacing $[A]$ by $[A^{op}] = -[A]$ doesn't change anything. □

Compactifications of a Merkurjev-Suslin variety. Here we follow definitions and notation of [16]. Let A be a cubic division algebra over F. Recall that a smooth compactification D of a Merkurjev-Suslin variety $\mathcal{MS}(A,c)$ can be identified with the smooth hyperplane section of the twisted form $X = \mathrm{SB}_3(M_2(A))$ of Grassmannian $\mathbb{G}(3,6)$. Using Theorem 1.2 one obtains a shortened proof of the main result of [16]

Corollary 5.6. *Let D be the smooth projective variety introduced above. Then*

$$M(D) \simeq \bigoplus_{i=1}^{5} M(\mathrm{SB}(A))\{i\} \oplus \mathcal{R}_{3,3},$$

where $\mathcal{R}_{3,3}$ is an integral generalized Rost motive. In other words, from the motivic point of view the variety D can be viewed as a 3-analog of a Norm quadric.

Proof. Let $i : D \hookrightarrow X$ denote the closed embedding. It induces the map $\Gamma_i \colon M(D) \to M(X)$. The variety X is a projective homogeneous PGL_6-variety corresponding to a maximal parabolic subgroup of type P_3. According to the Tits indices for the group $\mathrm{PGL}_{M_2(A)}$ the parabolic subgroup P_3 is defined over F and, hence, X is isotropic. By [2, Thm. 7.5] the motive of X splits as

$$M(X) = \mathbb{Z} \oplus Q\{1\} \oplus Q\{4\} \oplus \mathbb{Z}\{9\},$$

where $Q = M(\mathrm{SB}(A) \times \mathrm{SB}(A^{\mathrm{op}})) = \bigoplus_{i=0}^{2} M(\mathrm{SB}(A))\{i\}$ by the projective bundle theorem. Hence, we obtain

$$M(X) = \mathbb{Z} \oplus \bigoplus_{i=1}^{6} M(\mathrm{SB}(A))\{i\} \oplus \mathbb{Z}\{9\}. \tag{4}$$

We define f to be the composite of Γ_i and the canonical projection from $M(X)$ to the direct summand $\bigoplus_{i=1}^{5} M(\mathrm{SB}(A))\{i\}$ of (4). Observe that the motive

$M(D)$ splits over the generic point of $\mathrm{SB}(A)$. The direct computations (using multiplication tables provided in [16]) show that f has a section over $F(\mathrm{SB}(A))$. By Theorem 1.2 we conclude that

$$M(D) \simeq \bigoplus_{i=1}^{5} M(\mathrm{SB}(A))\{i\} \oplus N$$

for some motive N which splits over $F(\mathrm{SB}(A))$ as $\mathbb{Z} \oplus \mathbb{Z}\{4\} \oplus \mathbb{Z}\{8\}$.
Note that both D and the twisted form of F_4/P_4 (given by the first Tits construction) split the same symbol $\mathfrak{g}_3$ in $K_3^M(F)/3$. This implies that there is a morphism $f : N_{\mathbb{Z}/3} \to \mathcal{R}_{3,3}$ of motives with $\mathbb{Z}/3$-coefficients which becomes an isomorphism over the separable closure of F, where $\mathcal{R}_{3,3}$ is a generalized Rost motive corresponding to $\mathfrak{g}_3$. Since N is split over the generic point of the twisted form of F_4/P_4, $\mathcal{R}_{3,3}$ is a direct summand of $N_{\mathbb{Z}/3}$ which implies that $\mathcal{R}_{3,3} \simeq N_{\mathbb{Z}/3}$. Finally observe that $N_{\mathbb{Z}/p}$ splits if $p \neq 3$. ☐

6 Appendix

Proposition 6.1. *Let X be a smooth quasi-projective variety, $\pi\colon Y \to X$ a smooth morphism and $i\colon Z \hookrightarrow X$ a closed embedding. Consider the Cartesian square*

$$\begin{array}{ccc} Y_Z & \xrightarrow{i'} & Y \\ \downarrow{\scriptstyle \pi_Z} & & \downarrow{\scriptstyle \pi} \\ Z & \xrightarrow{i} & X \end{array}$$

Then $(\mathrm{im}\, i'_*)^d = 0$ for $d = [\frac{\dim(X)}{\mathrm{codim}_X(Z)}] + 1$, where $\mathrm{codim}_X(Z)$ is the minimum of codimensions of irreducible components of Z.

It is sufficient to prove the following:

Lemma 6.2. *Let $\pi\colon Y \to X$ be a smooth morphism, with X smooth quasi-projective, and $i_1\colon Z_1 \hookrightarrow X$, $i_2\colon Z_2 \hookrightarrow X$ closed embeddings.*
Then there exists a closed embedding $i_3\colon Z_3 \hookrightarrow X$ such that

$$\mathrm{codim}(Z_3) \geq \mathrm{codim}(Z_1) + \mathrm{codim}(Z_2) \quad \textit{and} \quad \mathrm{im}(i'_1)_* \cdot \mathrm{im}(i'_2)_* \subset \mathrm{im}(i'_3)_*,$$

where $i'_l\colon Y_{Z_l} \hookrightarrow Y$, $l = 1, 2, 3$ is obtained from the respective Cartesian square.

Proof. We have $(i'_1)_*(a) \cdot (i'_2)_*(b) = \Delta_X^*((i'_1 \times i'_2)_*(a \times b))$. The diagonal map $\Delta_Y\colon Y \to Y \times Y$ is the composition $Y \xrightarrow{\phi} Y \times_X Y \xrightarrow{f_W} Y \times Y$, where ϕ is the relative diagonal and the second map is the natural embedding. By Lemma 6.3 applied to $B = X \times X$, $V = X$, $f = \Delta_X$, $T = Z_1 \times Z_2$ and $W = Y \times Y$ we obtain a closed embedding $h\colon Z \hookrightarrow X$ such that

$$\mathrm{codim}(Z) \geq \mathrm{codim}(Z_1) + \mathrm{codim}(Z_2) \text{ and } \mathrm{im}(f_W^* \circ (i'_1 \times i'_2)_*) \subset \mathrm{im}(h_{W*}).$$

Consider the Cartesian square

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & Y \times_X Y \\ {\scriptstyle h'}\uparrow & & \uparrow{\scriptstyle h_W} \\ Y_Z & \xrightarrow{\phi_Z} & (Y \times_X Y)_Z \end{array}$$

By [3, Theorem 6.2(a)], $\phi^* \circ h_{W*} = h'_* \circ \phi^!$. Hence, $\operatorname{im}(\Delta_X^* \circ (i_1 \times i_2)_*) \subset \operatorname{im}(h'_*)$ and the lemma is proven. $\square$

Lemma 6.3. *Let $V \xrightarrow{f} B \xleftarrow{g} T$ be closed embeddings with regular f, and smooth quasi-projective B. Let $\varepsilon\colon W \to B$ be a smooth morphism. Consider two Cartesian diagrams:*

$$\begin{array}{ccccc} W_V & \xrightarrow{f_W} & W & \xleftarrow{g_W} & W_T \\ {\scriptstyle \varepsilon_V}\downarrow & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon_T} \\ V & \xrightarrow{f} & B & \xleftarrow{g} & T \end{array} \quad \text{and} \quad \begin{array}{ccc} T & \xrightarrow{g} & B \\ {\scriptstyle \tilde{f}}\uparrow & & \uparrow{\scriptstyle f} \\ \tilde{T} & \xrightarrow{\tilde{g}} & V \end{array}$$

There exists a closed embedding $h\colon Z \hookrightarrow V$ such that $\operatorname{codim}(h) \geq \operatorname{codim}(g)$ *and* $\operatorname{im}(f_W^* \circ g_{W*}) \subset \operatorname{im}(h_{W*})$.

Proof. Consider the Cartesian square

$$\begin{array}{ccc} W_T & \xrightarrow{g_W} & W \\ {\scriptstyle \tilde{f}_W}\uparrow & & \uparrow{\scriptstyle f_W} \\ W_{\tilde{T}} & \xrightarrow{\tilde{g}_W} & W_V \end{array}$$

By [3, Theorem 6.2(a)], $f_W^* \circ g_{W*} = \tilde{g}_{W*} \circ f_W^!$. The morphism $f_W^!\colon \operatorname{CH}^*(W_T) \to \operatorname{CH}^*(W_{\tilde{T}})$ is given by the composition:

$$\operatorname{CH}^*(W_T) \xrightarrow{\sigma} \operatorname{CH}^*(\mathcal{C}_W) \xrightarrow{\rho_{W*}} \operatorname{CH}^*(\mathcal{N}_W) \xrightarrow{s^*} \operatorname{CH}^*(W_{\tilde{T}}),$$

where σ is the specialization map from [3, §5.2], $\mathcal{C}_W = C_{W_T}(W_{\tilde{T}}) = C_T(\tilde{T}) \times_B W$ is the normal cone of the morphism $\tilde{f}_W$ and $\mathcal{N}_W = W_{\tilde{T}} \times_{W_V} \mathcal{N}_{f_W} = (\tilde{T} \times_V \mathcal{N}_f) \times_B W$ is the total space of the vector bundle $\tilde{g}_W^*(\mathcal{N}_{f_W}) = (\varepsilon_{\tilde{T}} \circ \tilde{g})^*(\mathcal{N}_f)$ over $W_{\tilde{T}}$, $\rho_W\colon \mathcal{C}_W \hookrightarrow \mathcal{N}_W$ is the closed embedding and $s\colon W_{\tilde{T}} \to \mathcal{N}_W$ is the zero section.

Consider the Cartesian square of projective completions of $\mathcal{C}_W$ and $\mathcal{N}_W$

$$\begin{array}{ccc} \mathbb{P}(\mathcal{C}_W \oplus \mathcal{O}) & \xrightarrow{\overline{\rho}_W} & \mathbb{P}(\mathcal{N}_W \oplus \mathcal{O}) \\ {\scriptstyle e_{\mathcal{C}}}\uparrow & & \uparrow{\scriptstyle e_{\mathcal{N}}} \\ \mathcal{C}_W & \xrightarrow{\rho_W} & \mathcal{N}_W \end{array}$$

By [3, Proposition 3.3] the morphism $s^* \circ \rho_{W_*} \colon \mathrm{CH}^*(\mathcal{C}_W) \to \mathrm{CH}^*(W_{\tilde{T}})$ is given by $s^* \circ \rho_{W_*}(x) = \pi_{W_*}(c_d(\tilde{g}_W^* \mathcal{N}_{f_W} \otimes \mathcal{O}(1)) \cdot \overline{\rho}_{W_*}(y))$, where $e_{\mathcal{C}}^*(y) = x$, $\pi_W \colon \mathbb{P}(\mathcal{N}_W \oplus \mathcal{O}) \to W_{\tilde{T}}$ is the projection and $d = \mathrm{codim}(f_W) = \mathrm{codim}(f)$.
By Lemma 6.4, we can choose a cycle α representing $c_d(\tilde{g}^* \mathcal{N}_f \otimes \mathcal{O}(1))$ in such a way that $|\alpha| \cap \mathbb{P}(\mathcal{C} \oplus \mathcal{O})$ has codimension d in $\mathbb{P}(\mathcal{C} \oplus \mathcal{O})$. Consider $Z := \pi(|\alpha| \cap \mathbb{P}(\mathcal{C} \oplus \mathcal{O}))$ and the closed embedding $j \colon Z \hookrightarrow \tilde{T}$. Then for arbitrary $x \in \mathrm{CH}^*(\mathbb{P}(\mathcal{C}_W \oplus \mathcal{O}))$ we have $|\pi_{W_*}(\varepsilon_{\tilde{T}}^*(\alpha) \cdot \overline{\rho}_{W_*}(x))| \subset \varepsilon^{-1}(Z)$. This implies that $\mathrm{im}(f_W^!) \subset \mathrm{im}(j_{W_*})$ and $\mathrm{im}(\tilde{g}_{W_*} \circ f_W^!) \subset \mathrm{im}(h_{W_*})$, where $h = \tilde{g} \circ j$. At the same time, $\mathrm{codim}(h) \geq \mathrm{codim}(g)$, and the lemma is proven. □

Lemma 6.4. *Let X be a quasi-projective variety, and Z_l, $l = 1, \ldots, n$ be closed irreducible subvarieties of dimensions d_l. Let $\mathcal{V}$ be a vector bundle over X. Then there exists a representative α_d of $c_d(\mathcal{V})$ such that $|\alpha_d| \cap Z_l$ has dimension $\leq d_l - d$.*

Proof. The total Chern class $c_\bullet(\mathcal{V})$ is the inverse of the total Segre class $s_\bullet(\mathcal{V})$, and $s_i(\mathcal{V}) = \pi_*(c_1(\mathcal{O}(1))^{n-1+i})$, where $\pi : \mathbb{P}_X(\mathcal{V}) \to X$ is the projection, and $n = \dim(\mathcal{V})$. Thus, the general case of our statement follows by the inductive application of the one with $d = 1$, and $\mathcal{V}$ - linear bundle. Indeed, since $c_d([X]) = -\sum_{j=1}^{d} \pi_*(c_1(\mathcal{O}(1))^{n-1+j}(\pi^{-1}(c_{d-j}([X]))))$, and α_{d-j} can be chosen with the needed property, it is sufficient to apply the above particular case to the set of irreducible components of $\pi^{-1}(Z_l \cap |\alpha_{d-j}|)$, $l = 1, \ldots, n; j = 1, \ldots, d$ inside $\mathbb{P}_X(\mathcal{V})$. Finally, the case $d = 1$ and linear $\mathcal{V}$ follows from the presentation $\mathcal{V} = \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$, where $\mathcal{L}_i$ have "sufficiently many sections", which is possible, since X is quasi-projective. □

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