Divisibility of the Dirac Magnetic Monopole as a Two-Vector Bundle over the Three-Sphere

Christian Ausoni, Bjørn Ian Dundas and John Rognes

Received: September 10, 2008
Revised: October 18, 2008

Communicated by Lars Hesselholt

Abstract. We show that when the gerbe μ representing a magnetic monopole is viewed as a virtual 2-vector bundle, then it decomposes, modulo torsion, as two times a virtual 2-vector bundle ς. We therefore interpret ς as representing half a magnetic monopole, or a semipole.

2000 Mathematics Subject Classification: 19D50, 55P43, 81S10, 81T40.
Keywords and Phrases: magnetic monopole, gerbe, two-vector bundle, higher algebraic K-theory, topological Hochschild homology.

§1. Introduction

Let A be a connective S-algebra, where S is the sphere spectrum, and let $K(A) = K_0(\pi_0(A)) \times BGL_\infty(A)^+$ be its algebraic K-theory space. The natural map $w\colon BGL_1(A) \to K(A)$ is given by the inclusion of 1×1 matrices $BGL_1(A) \to BGL_\infty(A)$, followed by the canonical map into the plus construction. Let ku denote the connective complex K-theory spectrum, with $\pi_* ku = \mathbb{Z}[u]$, $|u| = 2$, and let $\pi\colon ku \to H\mathbb{Z}$ be the unique 2-connected S-algebra map to the integral Eilenberg–Mac Lane spectrum. Its homotopy fiber is bu, with $\pi_* bu = (u) \subset \mathbb{Z}[u]$. We define $BSL_1(ku)$ and $K(\pi)$ as the homotopy fibers of the induced maps $\pi\colon BGL_1(ku) \to BGL_1(\mathbb{Z})$ and $\pi\colon K(ku) \to K(\mathbb{Z})$,

The authors thank Andy Baker and Birgit Richter for organizing the workshop "New Topological Contexts for Galois Theory and Algebraic Geometry" where this project started, and the Banff International Research Station for its support and hospitality.

respectively, so that we have the following commutative diagram of horizontal homotopy fiber sequences

$$
\begin{array}{ccccc}
BSL_1(ku) & \longrightarrow & BGL_1(ku) & \xrightarrow{\pi} & BGL_1(\mathbb{Z}) \\
\downarrow & & \downarrow{\scriptstyle w} & & \downarrow{\scriptstyle w} \\
K(\pi) & \longrightarrow & K(ku) & \xrightarrow{\pi} & K(\mathbb{Z})\,.
\end{array} \tag{1.1}
$$

We have a map from the Eilenberg–Mac Lane complex $K(\mathbb{Z}, 3)$ to the upper left hand corner of this diagram, induced by the infinite loop space inclusion $BU(1) \to BU_\otimes$ and the equivalences $K(\mathbb{Z}, 3) \simeq BBU(1)$ and $BBU_\otimes \simeq BSL_1(ku)$. Recall that the space $K(\mathbb{Z}, 3)$ represents gerbes with band $U(1)$ [Br93, Ch. V], whereas $K(ku)$ represents virtual 2-vector bundles [BDR04, Thm. 4.10], [BDRR]. A 2-vector bundle of rank 1 is the same as a gerbe, and the composite map

$$K(\mathbb{Z}, 3) \to K(ku) \tag{1.2}$$

represents the construction that views a gerbe as a virtual 2-vector bundle.

We now consider gerbes and 2-vector bundles over the base space S^3. There is a map $\mu\colon S^3 \to K(\mathbb{Z}, 3)$ representing a generator of $H^3(S^3) = \mathbb{Z}$, or dually, corepresenting a generator of $\pi_3 K(\mathbb{Z}, 3) = \mathbb{Z}$. It also represents a $U(1)$-gerbe over S^3, which is interpreted in [Br93, Ch. VII] as a mathematical model for a magnetic monopole, stationary in time and localized at one point.

Parallel transport in this gerbe, around closed loops in S^3, defines a holonomy line bundle over the free loop space of S^3 [Br93, Ch. VI]. Its complex 1-dimensional fibers can be interpreted as the state spaces of these free loops, viewed as strings in S^3. Parallel transport over compact surfaces in S^3, between tensor products of copies of this line bundle, defines an action functional that makes these state spaces part of a field theory. Here the compact surfaces are viewed as world sheets in S^3. In a quantized theory one would consider Hilbert spaces of sections in the holonomy line bundle, rather than its individual fibers, as the state spaces.

Following the point of view explained in [AR, §5.5], we also view 2-vector bundles over a base space as data defining (virtual) state spaces and action functionals for strings in that base. More field theories arise this way, since the state spaces are no longer restricted to being 1-dimensional, hence it is also possible to model more kinds of particles by 2-vector bundles than those arising from gerbes.

In particular we may ask, as the second author did, how the magnetic monopole μ over S^3 decomposes when viewed as a virtual 2-vector bundle. Does it remain a single particle?

The addition in the abelian group $\pi_3 K(ku)$ is induced by the H-group multiplication of $K(ku)$, which represents the direct sum of virtual 2-vector bundles, or in the above terms, the superposition of two particles. Therefore, a mathematical formulation of the question stated above is: "What is the structure of $\pi_3 K(ku) = K_3(ku)$, and what is the image of $\mu \in \pi_3 K(\mathbb{Z}, 3)$ in that group?" The surprising answer, which the title of this paper refers to, is that modulo torsion, μ becomes divisible by two as a virtual 2-vector bundle. In more detail, there are virtual 2-vector bundles ς and ν over S^3, with $24\nu = 0$, such that

$$\mu + \nu = 2\varsigma \tag{1.3}$$

in $K_3(ku)$. Modulo torsion, ς is therefore half a magnetic monopole. Ignoring torsion is justified in the physical interpretation, since the numerical invariants of a field theory traditionally take torsion-free values, and will send ν to zero. On the other hand, both μ and ς have infinite order in $K_3(ku)$.

§2. Statement of results

Let $i \colon S \to K(ku)$ be the unit map, and recall that $\pi_3(S) = \mathbb{Z}/24\{\nu\}$ and $K_3(\mathbb{Z}) \cong \mathbb{Z}/48\{\lambda\}$ [LS76]. The composite map $\pi i \colon S \to K(\mathbb{Z})$ induces the injection $\pi_3(S) \to K_3(\mathbb{Z})$ that takes ν to 2λ.
By [Wa78, Prop. 1.2], as generalized in [BM94, Prop. 10.9], the homotopy fiber $K(\pi)$ is 2-connected. Hence $K_i(ku) \to K_i(\mathbb{Z})$ is an isomorphism for $i \leq 2$. Here is what happens in dimension three:

Theorem 2.1. *(a) The maps $K(\mathbb{Z}, 3) \to BSL_1(ku) \to K(\pi)$ induce isomorphisms*

$$\mathbb{Z}\{\mu\} = \pi_3 K(\mathbb{Z}, 3) \xrightarrow{\cong} \pi_3 BSL_1(ku) \xrightarrow{\cong} K_3(\pi)\,.$$

(b) The unit map $i \colon S \to K(ku)$ induces a homomorphism

$$\mathbb{Z}/24\{\nu\} = \pi_3(S) \xrightarrow{i_*} K_3(ku)$$

that identifies the source with the torsion subgroup in the target. We abbreviate $i_*(\nu)$ to $\nu \in K_3(ku)$.
(c) The homotopy fiber sequence $K(\pi) \to K(ku) \to K(\mathbb{Z})$ induces a short exact sequence

$$0 \to K_3(\pi) \to K_3(ku) \xrightarrow{\pi_*} K_3(\mathbb{Z}) \to 0$$

which is isomorphic to the nontrivial extension

$$0 \to \mathbb{Z}\{\mu\} \to \mathbb{Z}\{\varsigma\} \oplus \mathbb{Z}/24\{\nu\} \xrightarrow{\pi_*} \mathbb{Z}/48\{\lambda\} \to 0\,.$$

Here the first map takes μ to $2\varsigma - \nu$, and the second map takes ς to λ and ν to 2λ.

Corollary 2.2. *The map $K(\mathbb{Z}, 3) \to K(ku)$ that represents viewing $U(1)$-gerbes as virtual 2-vector bundles induces the homomorphism*

$$\mathbb{Z}\{\mu\} = \pi_3 K(\mathbb{Z}, 3) \to K_3(ku) = \mathbb{Z}\{\varsigma\} \oplus \mathbb{Z}/24\{\nu\}$$

that takes μ to $2\varsigma - \nu$, where $24\nu = 0$. The image of $\varsigma \in K_3(ku)$ in $K_3(\mathbb{Z})$ is the generating element λ of order forty-eight.

Corollary 2.3. *There is no "determinant map"*

$$\det\colon K(ku) \to BGL_1(ku)$$

such that the composite $\det \circ w$ *is an equivalence.*

Corollary 2.2 is readily extracted from Theorem 2.1(a) and (c). Corollary 2.3 follows, since $\det\colon K_3(ku) \to \pi_3 BGL_1(ku) \cong \mathbb{Z}\{\mu\}$ cannot map $2\varsigma - \nu$ to μ.

Remark 2.4. For commutative rings R there is a determinant map $\det\colon K(R) \to BGL_1(R)$, which is left inverse to w. On the other hand, it follows from [Wa82, Cor. 3.7] that $w\colon BGL_1(S) \to K(S)$ admits no such retraction up to homotopy. In [AR, §5.2], the first and third authors used the existence of a rational determinant map $\det_{\mathbb{Q}}\colon K(ku) \to BSL_1(ku)_{\mathbb{Q}} \simeq (BBU_{\otimes})_{\mathbb{Q}}$ to define the rational anomaly bundle of a 2-vector bundle, generalizing the definition of the anomaly line bundle of a gerbe. Corollary 2.3 shows that no such generalization can be integrally defined on all of $K(ku)$. This suggests that an integral anomaly bundle will only be defined on a space covering $K(ku)$, classifying 2-vector bundles with some form of higher orientation.

§3. Proofs

Proof of Thm. 2.1(a). In view of the infinite loop space splitting $BU_{\otimes} \simeq BU(1) \times BSU_{\otimes}$ it is clear that $K(\mathbb{Z}, 3) \simeq BBU(1) \to BBU_{\otimes} \simeq BSL_1(ku)$ is 4-connected. For the second part, we refer to the proof of [BM94, Prop. 10.9] to see that there is an isomorphism

$$\operatorname*{colim}_n M_n(\pi_2(bu))/[GL_n(\pi_0(ku)), M_n(\pi_2(bu))] \cong K_3(\pi)\,. \tag{3.1}$$

Here M_n denotes the ring of $n \times n$ matrices, and GL_n acts on M_n by conjugation. Furthermore, under the isomorphism (3.1), $\pi_3 BSL_1(ku) \to K_3(\pi)$ factors as

$$\pi_3 BSL_1(ku) \cong \pi_2(bu) = M_1(\pi_2(bu))/[GL_1(\pi_0(ku)), M_1(\pi_2(bu))]\,, \tag{3.2}$$

followed by the canonical map from the term $n = 1$ into the colimit in (3.1). For each $n \geq 1$ the matrix trace induces an isomorphism [Ka83, Prop. 1.3]

$$M_n(\pi_2(bu))/[GL_n(\pi_0(ku)), M_n(\pi_2(bu))] \xrightarrow{\cong} \pi_2(bu)/[\pi_0(ku), \pi_2(bu)] = \pi_2(bu)\,,$$

hence each structure map in the colimit is an isomorphism, and therefore the canonical map from (3.2) to $K_3(\pi)$ is also an isomorphism. □

To proceed, we make use of the natural trace map $tr\colon K(A) \to THH(A)$ to topological Hochschild homology [BHM93]. We define $THH(\pi)$ as the homotopy fiber of $\pi\colon THH(ku) \to THH(\mathbb{Z})$, so as to obtain the following commutative diagram of horizontal homotopy fiber sequences

$$\begin{array}{ccccc} K(\pi) & \longrightarrow & K(ku) & \xrightarrow{\pi} & K(\mathbb{Z}) \\ \downarrow & & \downarrow{\scriptstyle tr} & & \downarrow{\scriptstyle tr} \\ THH(\pi) & \longrightarrow & THH(ku) & \xrightarrow{\pi} & THH(\mathbb{Z})\,. \end{array} \tag{3.3}$$

Proof of Thm. 2.1(b) and (c). Passing to homotopy groups, we get the following vertical map of short exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K_3(\pi) & \longrightarrow & K_3(ku) & \xrightarrow{\pi_*} & K_3(\mathbb{Z}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (\cong)} & & \downarrow{\scriptstyle tr_*} & & \downarrow{\scriptstyle tr_*} & & \\
0 & \longrightarrow & THH_3(\pi) & \longrightarrow & THH_3(ku) & \xrightarrow{\pi_*} & THH_3(\mathbb{Z}) & \longrightarrow & 0\,.
\end{array}
\tag{3.4}
$$

Here $K_3(\pi) \to K_3(ku)$ is injective because $K_4(\mathbb{Z}) = 0$ [Ro00], and $K_3(ku) \to K_3(\mathbb{Z})$ is surjective because $K_2(\pi) = 0$. Furthermore, $THH_3(\pi) \to THH_3(ku)$ is injective because $THH_4(\mathbb{Z}) = 0$ [Bö], [FP98, Cor. 3.2] and $THH_3(ku) \to THH_3(\mathbb{Z})$ is surjective because

$$\mathbb{Z}/48\{\lambda\} = K_3(\mathbb{Z}) \xrightarrow{tr_*} THH_3(\mathbb{Z}) = \mathbb{Z}/2\{e\}$$

takes λ to e [BM94, Thm. 10.14], [Ro98, Thm. 1.1] and the right hand square commutes. The left hand vertical map $K_3(\pi) \to THH_3(\pi)$ is split injective, by [BM94, Thm. 10.12]. We shall soon see that it is in fact an isomorphism.

The 2-primary homotopy of $THH(ku)$ is fully computed in [AHL], but in low dimensions the following direct argument suffices. The homotopy cofiber ku/S of $S \to ku$ is 1-connected, with $\pi_2(ku/S) \cong \mathbb{Z}$. By construction, $THH(ku)$ is the geometric realization of a simplicial spectrum, and the map from the $(n-1)$-skeleton to the n-skeleton has cofiber $\Sigma^n ku \wedge (ku/S) \wedge \cdots \wedge (ku/S)$, with n copies of ku/S, which is $(3n-1)$-connected. By induction, the map from the 1-skeleton to all of $THH(ku)$ is 5-connected. Furthermore, the 0-simplices ku split off from the 1-skeleton of $THH(ku)$ since ku is commutative, so $THH_3(ku) \cong \pi_3(ku) \oplus \pi_3(\Sigma ku \wedge (ku/S)) \cong \mathbb{Z}\{\epsilon\}$, for some choice of generator ϵ.

Diagram (3.4) is therefore isomorphic to

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}\{\mu\} & \longrightarrow & K_3(ku) & \xrightarrow{\pi_*} & \mathbb{Z}/48\{\lambda\} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle tr_*} & & \downarrow{\scriptstyle tr_*} & & \\
0 & \longrightarrow & \mathbb{Z}\{2\epsilon\} & \longrightarrow & \mathbb{Z}\{\epsilon\} & \xrightarrow{\pi_*} & \mathbb{Z}/2\{e\} & \longrightarrow & 0\,,
\end{array}
\tag{3.5}
$$

where the split injection $\mathbb{Z}\{\mu\} \to \mathbb{Z}\{2\epsilon\}$ must be an isomorphism. (We assume that we have chosen our orientations so that μ maps to 2ϵ, rather than -2ϵ.) The right hand square is a pullback, so there is a short exact sequence

$$0 \to \mathbb{Z}/24\{\nu\} \xrightarrow{i_*} K_3(ku) \xrightarrow{tr_*} \mathbb{Z}\{\epsilon\} \to 0\,, \tag{3.6}$$

where the image of the injective homomorphism $i_* \colon \pi_3(S) \to K_3(ku)$ is identified under $\pi_* \colon K_3(ku) \to K_3(\mathbb{Z})$ with the kernel of $tr_* \colon K_3(\mathbb{Z}) \to THH_3(\mathbb{Z})$. Hence the image of i_* equals the kernel of $tr_* \colon K_3(ku) \to THH_3(ku)$.

To fix a splitting of (3.6), we let $\varsigma \in K_3(ku)$ be the class mapping to ϵ in $THH_3(ku)$ and to λ in $K_3(\mathbb{Z})$. This is admissible, since both classes map to e in $THH_3(\mathbb{Z})$. Then

$$K_3(ku) \cong \mathbb{Z}\{\varsigma\} \oplus \mathbb{Z}/24\{\nu\}\,,$$

and $\mu \in K_3(\pi)$ maps to 2ς in $K_3(ku)$ modulo the image of i_*. Since μ continues to 0 in $K_3(\mathbb{Z})$, the exact formula must be $2\varsigma - \nu$. □

References

[AHL] Vigleik Angeltveit, Michael Hill and Tyler Lawson, *Topological Hochschild homology of ℓ and ko*, arXiv:0710.4368v1 preprint.

[AR] Christian Ausoni and John Rognes, *Rational algebraic K-theory of topological K-theory*, arXiv:0708.2160v1 preprint.

[BDRR] Nils A. Baas, Bjørn Ian Dundas, Birgit Richter and John Rognes, *Two-vector bundles define a form of elliptic cohomology*, arXiv:0706.0531v2 preprint.

[BDR04] Nils A. Baas, Bjørn Ian Dundas and John Rognes, *Two-vector bundles and forms of elliptic cohomology*, Topology, geometry and quantum field theory, London Math. Soc. Lecture Note Ser., vol. 308, Cambridge Univ. Press, Cambridge, 2004., pp. 18–45.

[Bö] Marcel Bökstedt, *Topological Hochschild homology of* $\mathbf{F}_p$ *and* $\mathbf{Z}$ (University of Bielefeld preprint (ca 1986)).

[BHM93] Marcel Bökstedt, Wu Chung Hsiang and Ib Madsen, *The cyclotomic trace and algebraic K-theory of spaces*, Invent. Math. **111** (1993), 465–539.

[BM94] Marcel Bökstedt and Ib Madsen, *Topological cyclic homology of the integers*, K-theory (Strasbourg, 1992), Astérisque, vol. 226, 1994, pp. 57–143.

[Br93] Jean-Luc Brylinski, *Loop spaces, characteristic classes and geometric quantization*, Progress in Mathematics, vol. 107, Birkhäuser Boston, Inc., Boston, MA, 1993.

[FP98] Vincent Franjou and Teimuraz Pirashvili, *On the Mac Lane cohomology for the ring of integers*, Topology **37** (1998), 109–114.

[Ka83] Christian Kassel, *Calcul algébrique de l'homologie de certains groupes de matrices*, J. Algebra **80** (1983), 235–260.

[LS76] Ronnie Lee and Robert H. Szczarba, *The group* $K_3(\mathbf{Z})$ *is cyclic of order forty-eight*, Ann. of Math. (2) **104** (1976), 31–60.

[Ro00] John Rognes, $K_4(\mathbf{Z})$ *is the trivial group*, Topology **39** (2000), 267–281.

[Ro98] John Rognes, *Trace maps from the algebraic K-theory of the integers (after Marcel Bökstedt)*, J. Pure Appl. Algebra **125** (1998), 277–286.

[Wa78] Friedhelm Waldhausen, *Algebraic K-theory of topological spaces. I*, Algebraic and geometric topology (Stanford Univ., Stanford, Calif., 1976), Part 1, Proc. Sympos. Pure Math., vol. XXXII, Amer. Math. Soc., Providence, RI, 1978, pp. 35–60.

[Wa82] Friedhelm Waldhausen, *Algebraic K-theory of spaces, a manifold approach*, Current trends in algebraic topology, Part 1, CMS Conf. Proc., vol. 2, Amer. Math. Soc., Providence, R.I., 1982, pp. 141–184.

Christian Ausoni
Institute of Mathematics
University of Bonn
DE-53115 Bonn
Germany

Bjørn Ian Dundas
Department of Mathematics
University of Bergen
NO-5008 Bergen
Norway

John Rognes
Department of Mathematics
University of Oslo
NO-0316 Oslo
Norway