A note on the p-adic Galois representations attached to Hilbert modular forms

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Abstract. We show that the p-adic Galois representations attached to Hilbert modular forms of motivic weight are potentially semistable at all places above p and are compatible with the local Langlands correspondence at these places, proving this for those forms not covered by the previous works of T. Saito and of D. Blasius and J. Rogawski.

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1 Introduction

Let F be a totally real extension of $\mathbf{Q}$ of degree d. Let $\overline{F}$ be an algebraic closure of F and let $G_F := \mathrm{Gal}(\overline{F}/F)$. Let $I := \mathrm{Hom}_{\mathbf{Q}}(F, \mathbf{C})$ be the set of embeddings of F into $\mathbf{C}$. The set I indexes the archimedean places of F. For each finite place v of F let $\overline{F}_v$ be an algebraic closure of F_v and fix an F-embedding $\overline{F} \hookrightarrow \overline{F}_v$. These determine a choice of a decomposition group $D_v \subset G_F$ for each v and an identification of D_v with $\mathrm{Gal}(\overline{F}_v/F_v)$. Let p be a rational prime and fix an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$ and an isomorphism $\iota : \mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}}_p$. Via composition with ι the set I is identified with the embeddings of F into $\overline{\mathbf{Q}}_p$.

Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A}_F)$. Then π is a restricted tensor product $\pi = \otimes' \pi_v$ with v running over all places of F. Assume that each π_i, $i \in I$, is a discrete series representation with Blattner parameter $k_i \geq 2$ and central character $x \mapsto \mathrm{sgn}(x)^{k_i} |x|_i^{-w}$ with w an integer independent of i. We say that π has infinity type $(\mathbf{k}, w)$, $\mathbf{k} := (k_i)_{i \in I}$. Assume also that each $k_i \equiv w \pmod 2$. In this case, π is an automorphic representation associated with a Hilbert modular eigenform of weight $\mathbf{k}$. We recall that attached to π (and ι) is a two-dimensional semisimple Galois representation

$$\rho_\pi : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$$

such that

$$\mathrm{WD}(\rho_\pi|_{D_v})^{\text{Fr-ss}} \cong \iota\mathrm{Rec}_v(\pi_v \otimes |\cdot|_v^{-1/2}) \quad \forall v \nmid p\infty. \tag{1}$$

Here $\mathrm{WD}(\sigma)$ denotes the Weil-Deligne representation over $\overline{\mathbf{Q}}_p$ associated to a continuous representation $\sigma : D_v \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ for a place $v \nmid p\infty$ (see [Ta, (4.2.1)]), and the superscript 'Fr-ss' denotes its Frobenius semi-simplification. Also, $\mathrm{Rec}_v(\tau)$ denotes the Frobenius semi-simple Weil-Deligne representation over $\mathbf{C}$ associated with an irreducible admissible representation τ of $\mathrm{GL}_n(F_v)$ by the local Langlands correspondence, and $\iota\mathrm{Rec}_v(\tau)$ is the Weil-Deligne representation over $\overline{\mathbf{Q}}_p$ obtained from $\mathrm{Rec}_v(\tau)$ by change of scalars via the isomorphism ι. We choose Rec_v so that when $n = 1$, Rec_v is the inverse of the Artin map of local class field theory normalized so that uniformizers correspond to geometric frobenius elements. The existence of a ρ_π satisfying (1) was established by Carayol [Ca2], Wiles [W], Blasius and Rogawski [BR], and Taylor [Tay1], following the work of Eichler, Shimura, Deligne, Langlands, and others on the Galois representations associated with elliptic modular eigenforms.

The purpose of this note is to complete the proof of the analog of (1) at places $v \mid p$:

Theorem 1 *Let $v \mid p$ be a place of F. The representation $\rho_\pi|_{D_v}$ is potentially semistable with Hodge-Tate type $(\mathbf{k}, w)$ and satisfies*

$$\mathrm{WD}(\rho_\pi|_{D_v})^{\text{Fr-ss}} \cong \iota\mathrm{Rec}_v(\pi_v \otimes |\cdot|_v^{-1/2}). \tag{2}$$

We recall that $\rho_v := \rho_\pi|_{D_v}$ is potentially semistable if

$$D_{pst}(\rho_v) := \bigcup_{L/F_v} (\rho_v \otimes_{\mathbf{Q}_p} B_{st})^{\mathrm{Gal}(\overline{F}_v/L)}$$

is a free $\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F_{v,0}^{ur}$-module of rank 2, where here L is ranging over all finite extensions of F_v, $F_{v,0}^{ur}$ is the union of all absolutely unramified subfields of $\overline{F}_v$, and B_{st} is Fontaine's ring of semistable p-adic periods (the latter has a continuous action of $D_v = \mathrm{Gal}(\overline{F}_v/F_v)$ with the property that $B_{st}^{\mathrm{Gal}(\overline{F}_v/L)} = L_0$, the maximal absolutely unramified subfield of L). We also recall that the module $D_{HT}(\rho_v) := (V \otimes_{\mathbf{Q}_p} B_{HT})^{D_v}$ is a graded $\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F_v$-module (recall that $B_{HT} := \oplus_{n\in\mathbf{Z}} \mathbf{C}_{F_v}(n)$, $\mathbf{C}_{F_v} := \widehat{\overline{F}}_v$, with the obvious action of D_v). By $\rho_\pi|_{D_v}$ having Hodge-Tate type $(\mathbf{k}, w)$, we mean that for $j \in \mathrm{Hom}_{\mathbf{Q}_p}(F_v, \overline{\mathbf{Q}}_p)$ the induced graded module $D_{HT}(\rho_v) \otimes_{\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F, j} \overline{\mathbf{Q}}_p$ is non-zero in degrees $(w - k_{i(j)})/2$ and $(w + k_{i(j)} - 2)/2$, where $i(j) \in I$ is the induced embedding of F into $\overline{\mathbf{Q}}_p$. To make sense of the left-hand side of (2) we recall that Fontaine has defined an action of the Weil-Deligne group on $D_{pst}(\rho_v)$. Given an embedding $\tau : F_{v,0}^{ur} \hookrightarrow \overline{\mathbf{Q}}_p$ we obtain a Weil-Deligne representation over $\overline{\mathbf{Q}}_p$ on $\mathrm{WD}(\rho_v)_\tau := D_{pst}(\rho_v) \otimes_{\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F_{v,0}^{ur}, \tau} \overline{\mathbf{Q}}_p$. This representation is independent of τ up to equivalence, and we have denoted an element of its equivalence class by $\mathrm{WD}(\rho_v)$. The right-hand side of (2) has the same meaning as in (1).

Saito proved that Theorem 1 holds when either d is odd or there exists a finite place w such that π_w is square-integrable [Sa1, Sa2]; this builds on the aforementioned work of Carayol. Under the same hypotheses or when d is even and some k_i is strictly larger than 2, Blasius and Rogawski proved that $\rho|_{D_v}$ is potentially semistable of Hodge-Tate type $(\mathbf{k}, w)$, and when additionally $\pi_p = \otimes_{v|p}\pi_v$ is unramified they essentially showed that the full conclusion of the theorem holds [BR] (some additional, albeit minor, observations are required to extend their arguments to all such cases). The theorem is of course also known for those π that are the automorphic induction of a (necessarily) algebraic Hecke character of an imaginary quadratic extension of F (such representations are often called CM representations). In this case, Theorem 1 follows from the results in [Se]. These results account for the cases where ρ_π is known to arise from a motive; the conclusion of the theorem then follows from various deep comparison theorems between suitable cohomology theories.

It remains to deal with the cases where ρ_π is not known to arise from a motive, namely those cases where each $k_i = 2$, each π_v is a principal series representation, and π is not a CM representation. In [Tay2] it is shown that if ρ_π is residually irreducible and π_v, $v|p$, is unramified, then $\rho_\pi|_{D_v}$ is crystalline with the predicted Hodge-Tate weights. For $p > 2$ unramified in F, the same result is proved in [Br] without the hypothesis that ρ_π be residually irreducible. For those ρ_π that are residually irreducible, Kisin [Ki1] deduced Theorem 1 from his results on potentially semistable deformation rings, Taylor's construction of the representations ρ_π, and Saito's results. In this paper, we prove Theorem 1 by a different approach. A simple base change argument reduces the theorem, in the cases not covered by Saito's results, to that where d is even and each π_v, $v|p$, is unramified. From the automorphy of the symmetric square $\mathrm{Sym}^2\pi$ and the results of [Mo] it follows that $\mathrm{Sym}^2\rho_v$ is crystalline[1] and even that $\mathrm{WD}(\mathrm{Sym}^2\rho_v) \cong \iota\mathrm{Rec}_v(\mathrm{Sym}^2\pi_v \otimes |\cdot|_v^{-1})$. From results of Wintenberger [Win1, Win2] we then deduce that ρ_v is crystalline up to a (possibly trivial) quadratic twist and hence that $\mathrm{WD}(\rho_v)$ is isomorphic to a (possibly trivial) quadratic twist of $\iota\mathrm{Rec}_v(\pi_v \otimes |\cdot|_v^{-1/2})$. There exists a suitable p-adic analytic family of eigensystems of cuspidal representations of $\mathrm{GL}_2(\mathbf{A}_F)$ (essentially due to Buzzard [Bu1] in the cases needed) that contains an eigensystem attached to ρ_π. For members of this family with sufficiently regular weights Theorem 1 is known by the work of Blasius and Rogawski. An appeal to a result of Kisin then shows that $\mathrm{WD}(\rho_v)$ has at least one D_v-eigenspace predicted by (2), from which we then conclude that (2) holds.

After completing the first draft of this paper, the author learned that Tong Liu [L] has also proven Theorem 1, at least for $p > 2$, by an argument that is a generalization of that of Kisin [Ki1].

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[1] As remarked at the end of 2.4.1, a similar use of the symmetric square yields a proof of the Ramanujan conjecture for π. This conjecture has previously been established in [B2].

about what was known regarding Theorem 1 asked by Henri Darmon at the summer school on the stable trace formula, automorphic forms, and Galois representations held at BIRS in August of 2008. The referee prodded the author to write a note with more details. The author's research is supported by grants DMS-0701231 and DMS-0803223 from the National Science Foundation and by a fellowship from the David and Lucile Packard Foundation.

2 The proof of Theorem 1

We keep to the notation from the introduction. We assume some familiarity on the part of the reader with p-adic Hodge theory, particularly the theory of Hodge-Tate weights and the notions of crystalline and semistable representations. A good reference is [Fo]. While p-adic Hodge theory is usually applied to continuous representations of $\mathrm{Gal}(\overline{F}_v/F_v)$, $v|p$, defined over a finite extension of $\mathbf{Q}_p$, we apply it to continuous representations over $\overline{\mathbf{Q}}_p$. This should cause no confusion as the latter are always defined over a finite extension of $\mathbf{Q}_p$. While this is well-known, references seem rare, so we provide a quick proof.

Let Γ be a compact group and $\rho : \Gamma \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ a continuous representation. The subfields L of $\overline{\mathbf{Q}}_p$ that are finite over $\mathbf{Q}_p$ form a countable set, and as each $\mathrm{GL}_n(L)$ is closed in $\mathrm{GL}_n(\overline{\mathbf{Q}}_p)$, the subgroups $\Gamma_L := \rho^{-1}(\mathrm{GL}_n(L))$ form a countable set of closed subgroups of Γ whose union is Γ. Since Γ is compact, it carries a Haar measure with total measure finite and non-zero. As the countable union of measurable sets each having measure zero also has measure zero, it follows that some Γ_L must have non-zero measure and hence have finite index in Γ. Write $\Gamma = \sqcup_{i=1}^m g_i\Gamma_L$. Then ρ takes values in $\mathrm{GL}_n(L')$ where L' is the finite extension of $\mathbf{Q}_p$ generated by L and the entries of the $\rho(g_i)$.

2.1 Weil-Deligne representations over $\overline{\mathbf{Q}}_p$ for $v|p$

Let $v|p$ be a place of F. Let $B_{HT} := \oplus_{n\in\widehat{\mathbf{Z}}}\mathbf{C}_{F_v}(n)$ with the obvious action of D_v. Let $B_{cris} \subset B_{st}$ be Fontaine's rings of crystalline and semistable p-adic periods, respectively. Recall that the latter are naturally $F_{v,0}^{ur}$-algebras equipped with a continuous action of D_v such that $B_?^{\mathrm{Gal}(\overline{F}_v/L)} = L_0$ for any finite extension L/F_v, $? = cris, st$, and that furthermore they are both equipped with a compatible $F_{v,0}^{ur}$-semilinear Frobenius morphism $\varphi : B_? \to B_?$ (that is, $\varphi(ax) = \mathrm{frob}_p(a)\varphi(x)$ for all $a \in F_{v,0}^{ur}$, where $\mathrm{frob}_p \in \mathrm{Gal}(F_{v,0}^{ur}/\mathbf{Q}_p)$ is the absolute arithmetic Frobenius). Additionally, B_{st} is equipped with an $F_{v,0}^{ur}$-linear and D_v-equivariant monodromy operator $N : B_{st} \to B_{st}$ such that $B_{cris} = B_{st}^{N=0}$.

For a finite-dimensional $\overline{\mathbf{Q}}_p$-vector space V with a continuous $\overline{\mathbf{Q}}_p$-linear action of D_v we put

$$D_{HT}(V) := (V \otimes_{\mathbf{Q}_p} B_{HT})^{D_v}, \quad D_{cris}(V) := (V \otimes_{\mathbf{Q}_p} B_{cris})^{D_v},$$

and
$$D^L_{st}(V) := (V \otimes_{\mathbf{Q}_p} B_{st})^{\mathrm{Gal}(\overline{F}_v/L)}, \quad D_{pst}(V) := \bigcup_{L/F_v} D^L_{st}(V),$$

where L/F_v is a finite extension. Then $D_{HT}(V)$ is a finite, graded $\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F_v$-module. Also, $D_{cris}(V)$ is a finite $\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F_{v,0}$-module, $D^L_{st}(V)$ is a finite $\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} L_0$-module, and $D_{pst}(V)$ is a finite $\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F^{ur}_{v,0}$-module, each of rank at most $\dim_{\overline{\mathbf{Q}}_p}(V)$. The action of φ induces a $\overline{\mathbf{Q}}_p$-linear, $F_{v,0}$-semilinear (resp. L_0-semilinear) Frobenius operator on $D_{cris}(V)$ (resp. $D^L_{st}(V)$) that we also denote by φ. The action of the monodromy operator N on B_{st} induces a $\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} L_0$-linear nilpotent operator on $D^L_{st}(V)$ that we also denote by N and which satisfies $N \circ \varphi = p\varphi \circ N$. These are compatible with varying L, so φ and N are defined on $D_{pst}(V)$ as well. Note that $D_{cris}(V) = D^{F_v}_{st}(V)^{N=0}$.

Let $W_v \subset D_v$ be the Weil group of F_v. The action of D_v on V and B_{st} induces a $\overline{\mathbf{Q}}_p$-linear, $F^{ur}_{v,0}$-semilinear action r_{sl} of W_v on $D_{pst}(V)$. We define another action r of W_v on $D_{pst}(V)$: for $w \in W_K$ we let $r(w) = r_{sl}(w) \circ \varphi^{\nu(w)}$ with $\nu(w) \in \mathbf{Z}$ such that w acts on $F^{ur}_{v,0}$ as $\mathrm{frob}_p^{-\nu(w)}$. This also defines an action on $D_{cris}(V)$. The action r is $\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F^{ur}_{v,0}$-linear, and we have

$$N \circ r(w) = N \circ r_{sl}(w) \circ \varphi^{\nu(w)} \circ N = r_{sl}(w) \circ N \circ \varphi^{\nu(w)} = p^{\nu(w)} r(w) \circ N.$$

It follows that the pair (r, N) defines an action of the Weil-Deligne group W'_v of F_v on $D_{pst}(V)$. Moreover, if $\tau : F^{ur}_{v,0} \hookrightarrow \overline{\mathbf{Q}}_p$ is any embedding, then it also follows that the induced action on

$$\mathrm{WD}(V)_\tau := D_{pst}(V) \otimes_{\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F^{ur}_{v,0}, \tau} \overline{\mathbf{Q}}_p$$

is a Weil-Deligne representation over $\overline{\mathbf{Q}}_p$ (the subscript τ on the tensor sign means that we consider $\overline{\mathbf{Q}}_p$ as a $\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F^{ur}_{v,0}$-algebra via the homomorphism $id \otimes \tau$). Furthermore, $d \otimes x \mapsto \varphi(d) \otimes x$ defines an isomorphism $\mathrm{WD}(V)_{\tau \circ \mathrm{frob}_p} \xrightarrow{\sim} \mathrm{WD}(V)_\tau$ of Weil-Deligne representations over $\overline{\mathbf{Q}}_p$, hence the equivalence class of $\mathrm{WD}(V)_\tau$ is independent of the choice of τ. We let $\mathrm{WD}(V)$ be any member of this equivalence class.

We recall that V is potentially semistable if $D_{pst}(V)$ is a free $\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F^{ur}_{v,0}$-module of rank equal to $\dim_{\overline{\mathbf{Q}}_p} V$ or, equivalently, $\dim_{\overline{\mathbf{Q}}_p} \mathrm{WD}(V) = \dim_{\overline{\mathbf{Q}}_p} V$. Similarly, V is crystalline if $D_{cris}(V)$ is a free $\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F_{v,0}$-module of rank $\dim_{\overline{\mathbf{Q}}_p} V$. This is equivalent to $(V \otimes_{\mathbf{Q}_p} B_{cris})^{I_v}$ being a free $\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F^{ur}_{0,v}$-module of rank equal to $\dim_{\overline{\mathbf{Q}}_p} V$, where $I_v \subset D_v$ is the inertia subgroup. Thus, V is crystalline if and only if V is potentially semistable and both N and I_v act trivially on $D_{pst}(V)$. In particular, V is crystalline if and only if $\dim_{\overline{\mathbf{Q}}_p} \mathrm{WD}(V) = \dim_{\overline{\mathbf{Q}}_p}(V)$, $\mathrm{WD}(V)$ is unramified (i.e., $N = 0$ and the inertia group I_v acts trivially). Consequently, for V crystalline the eigenvalues of $w \in W_v$ on $\mathrm{WD}(V)^{\text{Fr-ss}}$ are just the roots of the characteristic polynomial of the $\overline{\mathbf{Q}}_p$-endomorphism induced by $\varphi^{\nu(w)}$. We also recall that for a crystalline representation V there is $\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F_v$-filtration on $D_{cris}(V) \otimes_{F_{v,0}} F_v$ whose associated graded module is just $D_{HT}(V)$.

Suppose now that π_v is unramified. From the preceding paragraph it follows that (2) holds if $\rho_v = \rho_\pi|_{D_v}$ is crystalline and if for all $w \in W_v$

$$\det(1 - T\varphi^{\nu(w)}|D_{cris}(V) \otimes_{\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F_{v,0},\tau} \overline{\mathbf{Q}}_p) = \det(1 - Tw|\iota\mathrm{Rec}_v(\pi_v \otimes |\cdot|_v^{-1/2})) \quad (3)$$

for some (equivalently, each) embedding $\tau : F_{v,0} \hookrightarrow \overline{\mathbf{Q}}_p$.

2.2 Reduction to d even and π_v unramified

As mentioned in the introduction, Saito has proven Theorem 1 when the degree d of F is odd or some π_v is square-integrable [Sa1],[Sa2]. We may therefore assume that d is even and that π_v is a principal series representation for finite places v. Theorem 1 then asserts that each ρ_v is potentially crystalline with predicted Hodge-Tate weights. Clearly, this is true for $\rho_v = \rho_\pi|_{D_v}$ if and only if there is a finite extension F'/F such that it is true for $\rho_\pi|_{D_{v'}}$, $v'|v$ the place of F' determined by the fixed embedding $\overline{F} \hookrightarrow \overline{F}_v$. Additionally, if ρ_v is potentially crystalline with the predicted Hodge-Tate weights, then to establish (2) it is enough to show that

$$\mathrm{trace}(w|\mathrm{WD}(\rho_v)) = \mathrm{trace}(w|\iota\mathrm{Rec}_v(\pi_v \otimes |\cdot|_v^{-1/2})) \quad (4)$$

for all $w \in W_v$ with $\nu(w) > 0$.

Let $v|p$. For a given $w \in W_v$ such that $\nu(w) > 0$ there exists an abelian extension F'/F such that (a) the base change π' of π to $\mathrm{GL}_2(\mathbf{A}_{F'})$ is cuspidal and unramified at each place over p and (b) $w \in W_{v'} \subseteq W_v$ for $v'|v$ the place of F' determined by the fixed embedding $\overline{F} \hookrightarrow \overline{F}_v$. That (a) can be satisfied is a consequence of each local constituent of π being a principal series representation (we are, of course, using that base change is known for GL_2 for abelian extensions). That (b) can be simultaneously satisfied with (a) is a simple consequence of $\nu(w) > 0$. Note that the extension F'/F may depend on w. As $\rho_{\pi'} \cong \rho_\pi|_{G_{F'}}$, it follows that $\mathrm{WD}(\rho_{\pi'}|_{D_{v'}}) \cong \mathrm{WD}(\rho_\pi|_{D_v})|_{W'_{v'}}$. Similarly, $\mathrm{Rec}_{v'}(\pi_{v'} \otimes |\cdot|_{v'}^{-1/2}) \cong \mathrm{Rec}_v(\pi_v \otimes |\cdot|_v^{-1/2})|_{W'_{v'}}$. Therefore if Theorem 1 holds for π', then ρ_v is potentially crystalline with the predicted Hodge-Tate weights and (4) holds for the given w. This shows that if Theorem 1 holds whenever the representation is unramifed at all primes above p then it also holds for π. Consequently, it suffices to prove Theorem 1 under the assumption that each π_v, $v|p$, is unramified.

2.3 Galois representations in the cohomology of certain Shimura varieties

As mentioned in the introduction, Blasius and Rogawski have essentially proved Theorem 1 in the case where some $k_i > 2$ and each π_v, $v|p$, is unramified [BR].

We explain this here, giving the necessary modifications required to make their argument cover all such cases. We also record some additional consequences for Galois representations associated with essentially self-dual representations of $\mathrm{GL}_3(\mathbf{A}_F)$.

2.3.1 The Shimura varieties

Let $E_0 \subseteq \overline{F}$ be an imaginary quadratic extension of $\mathbf{Q}$ in which p splits and set $E = FE_0$. Fix a place v_0 of E_0 above p. For convenience we assume that for each place $v|p$ of F the fixed embedding $\overline{F} \hookrightarrow \overline{F}_v$ induces the valuation v_0 on E_0. Fix an embedding $E_0 \hookrightarrow \mathbf{C}$ such that - again for convenience - composition with ι also induces the valuation v_0. Let ϕ be the CM type of E consisting of those embeddings $E \hookrightarrow \mathbf{C}$ extending the fixed embedding of E_0. For $\tau \in \phi$ we write $\bar{\tau}$ for the composition of τ with complex conjugation. Restriction to F determines a bijection between ϕ and I, and we write τ_i for the element of ϕ extending $i \in I$. Via composition with ι, ϕ determines a place of E above each place $v|p$ of F; the fixed decomposition group D_v is also a decomposition group for the place of E above p so determined, hence we also denote this place by v, writing $\bar{v}$ for its conjugate (note that each place $v|p$ of F splits in E). If M is an $\mathcal{O}_E$-module, then $M_\infty := M \otimes \mathbf{C}$ decomposes as $M_\infty \cong \prod_{\tau\in\phi} M_\tau \oplus M_{\bar{\tau}}$ with $M_\sigma := M \otimes_{\mathcal{O}_E,\sigma} \mathbf{C}$ for any embedding $\sigma : E \hookrightarrow \mathbf{C}$. Similarly, $M_p := M \otimes \mathbf{Z}_p$ decomposes as $M_p \cong \prod_v M_v \oplus M_{\bar{v}}$ with $M_w := M \otimes_{\mathcal{O}_E} \mathcal{O}_{E,w}$ for a place $w|p$ of E.

Fix $i_0 \in I$. Let Φ be the Hermitian E-pairing on $V := E^3$ (viewed as column vectors) defined by the diagonal matrix $J := \mathrm{diag}(\alpha, 1, 1)$ with $\alpha \in F^\times$ such that $\tau_{i_0}(\alpha) < 0$ and $\tau_i(\alpha) > 0$ for $i \neq i_0$: $\Phi(x,y) = {}^t\bar{x}Jy$. Then Φ has signature $(2,1)$ with respect to τ_{i_0} and signature $(3,0)$ with respect to all other τ_i. Let $U(\Phi)_{/\mathbf{Q}}$ be the unitary group of Φ and $G := GU(\Phi)_{/\mathbf{Q}}$ its similitude group. We note that $G(\mathbf{C}) \cong \mathbf{C}^\times \times \prod_{\tau\in\phi} \mathrm{GL}_{\mathbf{C}}(V_\tau)$, where the projection to the $\mathbf{C}^\times$-factor is the similitude character, and the projection to the second factor is via the corresponding projection of $\mathrm{GL}_{E\otimes\mathbf{C}}(V_\infty)$. Similarly, $G(\mathbf{Q}_p) \cong \mathbf{Q}_p^\times \times \prod_v \mathrm{GL}_{E_v}(V_v)$, where v runs over the place of F dividing p (or the fixed places of E over these). Let $\psi := \mathrm{trace}_{E/\mathbf{Q}}\beta\Phi$ with β a totally imaginary element of E_0. Then there exists an $\mathcal{O}_E$-lattice $\Lambda \subset V$ such that ψ identifies Λ_p with its $\mathbf{Z}_p$-dual.

Let $\mathbf{S} := \mathrm{Res}_{\mathbf{C}/\mathbf{R}}\mathbf{G}_m$, so $\mathbf{S}(R) = (\mathbf{C}\otimes_{\mathbf{R}} R)^\times$ for any $\mathbf{R}$-algebra R. We identify $\mathbf{S}(\mathbf{C}) = (\mathbf{C}\otimes\mathbf{C})^\times$ with $\mathbf{C}^\times \times \mathbf{C}^\times$ via $z \otimes w \mapsto (zw, \bar{z}w)$. Let $h : \mathbf{S} \to G_{/\mathbf{R}}$ be the homomorphism such that for $(z,w) \in \mathbf{S}(\mathbf{C})$

$$h(z,w) = (zw) \times \prod_{\tau\in\phi} \begin{cases} \mathrm{diag}(z,w,w) & \tau = \tau_{i_0} \\ \mathrm{diag}(w,w,w) & \tau \neq \tau_{i_0}. \end{cases}$$

Let $h(z) = h(z,\bar{z})$. We assume that β is such that $\psi(x, h(i)x)$ is positive definite for $x \in V \otimes \mathbf{R}$. As explained in [Ko], associated with E, V, ψ, and h

is a family of PEL moduli spaces S_K over[2] E, $K \subset G(\mathbf{A}_f)$ being a neat open compact subgroup: in the notation of [Ko, §5] we take[3] $B = E$ with $*$ the non-trivial automorphism fixing F and $(V,(-,-)) = (V,\psi)$; then $C = \mathrm{End}_E(V)$ and the G of *loc. cit.* is the group G defined above, and we take for the $*$-homomorphism $\mathbf{C} \to C \otimes \mathbf{R}$ the $\mathbf{R}$-linear extension of $z \mapsto h(z)$. The varieties S_K are smooth over E and, being solutions to PEL moduli problems, are equipped with 'universal' abelian varieties A_K/S_K. As explained in [Ko, §8], S_K is naturally identified with a disjoint union of a finite number of copies of the canonical model Sh_K over E of the Shimura variety associated with G, h^{-1}, and K, indexed by the isomorphism classes of Hermitian E-spaces (V',ψ') that are everywhere locally isomorphic to (V,ψ). We identify Sh_K with the copy corresponding to the class of (V,ψ) and let A_K/Sh_K be the restriction of the universal abelian variety.

Suppose $K = K_pK^p$ with $K^p \subseteq G(\mathbf{A}_f^p)$ and $K_p \subset G(\mathbf{Q}_p)$ identified with a subgroup $\mathbf{Z}_p^\times \times \prod_{v|p} K_v \subseteq \mathbf{Z}_p^\times \times \prod_v \mathrm{GL}_{\mathcal{O}_{E,\bar v}}(\Lambda_{\bar v})$. Let $v|p$ be a fixed place. If $K_v = \mathrm{GL}_{\mathcal{O}_{E,v}}(\Lambda_v)$, then an argument of Carayol [Ca1, §5] shows that A_K and S_K have good reduction at v. A model of S_K over $\mathcal{O}_{F,v} = \mathcal{O}_{E,v}$ is obtained by considering a moduli problem as in [Ca1, 5.2.2]. To be be precise, one considers the functor from the category of locally Noetherian $\mathcal{O}_{F,v}$-schemes to the category of sets that sends an $\mathcal{O}_{F,v}$-scheme R to the set of isomorphism classes of quadruples $(A,i,\theta,\bar k^v)$ where (a) A is an abelian scheme over R of relative dimension $3d$ and $i : \mathcal{O}_E \hookrightarrow \mathrm{End}_R(A)$ is an embedding such that $\mathrm{Lie}(A)_v$ is a locally free $\mathcal{O}_R$-module of rank one on which $\mathcal{O}_{F,v} = \mathcal{O}_{E,v}$ acts via the structure map $\mathcal{O}_{F,v} \to \mathcal{O}_R$ and such that $\mathrm{Lie}(A)_{v'} = 0$ for all $v'|p$, $v' \neq v$; (b) θ is a prime-to-p polarization of A satisfying $\theta \circ i(x) = i(\bar x)^\vee \circ \theta$ for all $x \in \mathcal{O}_E$; (c) $\bar k^v$ is a K-level structure as[4] in [Ca1, 5.2.2(c)] but with $V_{\mathbf{Z}}$ in the definition of W there replaced by Λ. That this functor is isomorphic over $F_v = E_v$ to that in [Ko, §5] defining S_{K/E_v} follows from the arguments in [Ca1, 2.4-2.6,5.2.2]. That it is representable by a smooth, projective scheme $\mathcal{S}_K$ over $\mathcal{O}_{F,v}$ follows from the arguments in [Ca1, 5.3-5.5]. The p-divisible group A_p of A decomposes under the action of $\mathcal{O}_{E,p} = \mathcal{O}_E \otimes \mathbf{Z}_p$ as $A_p = \prod_{v'|p} A_{v'} \times A_{\bar v'}$. The condition on $\mathrm{Lie}(A)_{v'}$ in (a) then implies that $A_{v'}$ is ind-étale if $v' \neq v$, and part of the level structure $\bar k^v$ is a class modulo $\prod_{v'\neq v} K_{v'}$ of $\mathcal{O}_{E,p}$-linear R-isomorphisms $k_p^v : \prod_{v'\neq v} A[p^n]_{v'} \xrightarrow{\sim} \prod_{v'\neq v}(p^{-n}\Lambda/\Lambda)_{v'}$ with n any integer so large that $K_{v'}$ contains the kernel of the reduction map $\mathrm{GL}_{\mathcal{O}_{E,v'}}(\Lambda_{v'}) \to \mathrm{GL}_{\mathcal{O}_{E,v'}}(\Lambda_{v'}/p^n\Lambda_{v'})$ (see [Ca1, 5.2.3(ii)]). The condition that Λ_p is self-dual ensures that over F_v this moduli problem is equivalent to one with a usual

[2] The reflex field in this case is $\tau_{i_0}(E) \subset \mathbf{C}$ which we identify with E via τ_{i_0}.

[3] As we are only defining the moduli spaces over E at this point, the conditions at p in [Ko, §5] are superfluous.

[4] When adapting the arguments of [Ca1] to the setting of this paper, the roles of the superscripts 1 and 2 in *loc. cit.* are switched. This is a result of our choice of the homomorphism h and the identification of E with the reflex field. A homomorphism $\mathbf{S} \to G_{/R}$ more naturally generalizing that in *loc. cit.* would be $(z,w) \mapsto h(w,z)$. We have chosen h here so that Sh_K is the Shimura variety in [BR].

K-level structure. The representability of this moduli problem by a scheme $\mathcal{S}_K$ over $\mathcal{O}_{F,v}$ follows from the arguments in [Ca1, 5.3] and the properness from those in [Ca1, 5.5]. The smoothness of this scheme follows exactly as in [Ca1, 5.4]. The key point is that for R a local artinian $\mathcal{O}_{F,v}$-module, the conditions on the dimension of A and on $\mathrm{Lie}(A)_v$ in (a) imply that A_v is a divisible $\mathcal{O}_{F,v}$-module of height 3 whose formal (or connected) part has height 1 (we are keeping to the terminology in the Appendix of [Ca1]). The smoothness then follows by the deformation argument given in *loc. cit.* Over E_v, $\mathcal{S}_K$ is just S_K, and A_K is the base change of the universal abelian scheme $\mathcal{A}_K/\mathcal{S}_K$. Hence S_K, Sh_K, and A_K have good reduction at v.

2.3.2 Theorem 1 when some $k_i > 2$ and each π_v unramified

We can now explain how the arguments in [BR] yield Theorem 1 when $d > 1$, some $k_i > 2$, and each π_v, $v|p$, is unramified. Without loss of generality we may assume that $w = \max_{i \in I} k_i$; choosing a different w amounts to replacing ρ_π by a Tate-twist. We may assume that E_0 has been chosen so that the base change π_E of π to $\mathrm{GL}_2(\mathbf{A}_E)$ is cuspidal (equivalently, π is not a CM representation associated to a Hecke character of E). Fix an algebraic Hecke character μ of $\mathbf{A}_E^\times$ satisfying $\mu|_{\mathbf{A}_F^\times} = \omega_{E/F}$, the quadratic character of the extension E/F, and such that μ is unramified at each place over p. As explained[5] in [BR, Prop. 4.1.2], there exists a global L-packet τ on the quasi-split unitary group $U(2)_{/F}$ such that its non-standard base change to $\mathrm{GL}_2(\mathbf{A}_E)$ (with respect to μ) is $\pi_E \otimes \eta|\cdot|_E^{1/2}$ with η an algebraic Hecke character of $\mathbf{A}_E^\times$ that is unramified at each place above p. It follows from [BR, Lem. 4.2.1] that there exists a global character θ of $U(1)_{/F}$ unramified at all places above p for which the L-packet $\rho = \tau \otimes \theta$ of $U(2) \times U(1)$ is such that the endoscopic L-packet $\Pi(\rho_f)$ for $U(\Phi)$ contains an element σ_f with $d(\sigma_f) := \#\{\sigma_\infty \in \Pi(\rho_\infty) \ : \ \epsilon(\sigma_\infty)\epsilon(\sigma_f) = 1\} = 2$. Let χ be an algebraic character of the center of G extending the central character of $\Pi(\rho)$ and unramified at all places above p (cf. [BR, §1.2]). The pair (σ_f, χ) defines an admissible representation $\pi(\sigma_f, \chi)$ of $G(\mathbf{A}_{\mathbf{Q}}^\infty)$. From the definition of σ_f it follows that σ_p is an unramified representation of $U(\mathbf{Q}_p) \cong \prod_v \mathrm{GL}_{E_{\bar{v}}}(V_{\bar{v}})$ in the sense that it is a tensor product of unramified principal series representations of each factor. In particular, as χ is unramified at each place above p, $\pi(\sigma_f, \chi)^K \neq 0$ for $K = K_p K^p$ with K_p identified with $\mathbf{Z}_p^\times \times \prod_{v|p} \mathrm{GL}_{\mathcal{O}_{E,v}}(\Lambda_v)$ and K^p sufficiently small.

As explained in the proof of [BR, Thm. 3.3.1], associated with $\pi(\sigma_f, \chi)$ is a motive $M = (A_K^n, e)$ with coefficients in a number field $T \subset \mathbf{C}$ (this motive is denoted M_0 in *loc. cit.*; n is some integer depending on the weights of π, μ, θ, and χ, A_K^n is the n-fold self-product over the Shimura variety Sh_K, and e is an idempotent in $Z_h(A_K^n \times A_K^n)$) such that for any prime ℓ and any isomorphism

[5] The representations of $\mathrm{GL}_2(\mathbf{A}_F)$ in [BR] are normalized so that what is denoted by π there equals $\pi \otimes |\cdot|_F^{1/2}$ in this paper.

$\iota' : \mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}}_\ell$, the ℓ-adic realization M_ℓ of M satisfies

$$M_{\ell,\iota'} := M_\ell \otimes_{T\otimes\mathbf{Q}_\ell,\iota'} \overline{\mathbf{Q}}_\ell \cong \rho_{\pi,\iota'}|_{G_E} \otimes \rho_{\eta\psi,\iota'}, \tag{5}$$

where the subscript ι' denotes that the objects on the right-hand side are the ℓ-adic Galois representations[6] associated with the embedding ι'. Here ψ is the Hecke character $z \mapsto \chi(N_{E/E_0}(\bar z))$ of $\mathbf{A}_E^\times$. Equivalently,

$$\mathrm{WD}(M_{\ell,\iota'}|_{D_w}) \cong \iota'\mathrm{Rec}_w(\pi_{E,w} \otimes \eta_w\psi_w|\cdot|_w^{-1/2}) \tag{6}$$

for all places $w \nmid \ell$ of E, D_w being any decomposition group for w. More precisely, (6) is only shown in [BR, Thm. 3.3.1] for those $w \nmid \ell$ coprime to the conductor of π and the absolute discriminant of E. But this together with the existence of the ℓ-adic representations associated with π, $\eta\psi$, and ι' implies (5), from which (6) follows for all places $w \nmid \ell$. This relies on more than is proved in *loc. cit.*; it also requires the work of Carayol and Taylor on the existence of the ℓ-adic representations.

As A_K has good reduction at $v|p$, it follows - from the theorems of Faltings and of Katz and Messing cited in [BR, §5] together with (5) and (6) - that for a place $v|p$ of F the representation $M_{p,\iota}$ is crystalline at v and for all $w \in W_v$

$$\begin{aligned}\det(1 - X\varphi^{\nu(w)}|D_{cris}(M_{p,\iota}|_{D_v}) \otimes_{\overline{\mathbf{Q}}_p\otimes_{\mathbf{Q}_p}F_{v,0},\tau} \overline{\mathbf{Q}}_p) \\ = \det(1 - Xw|\iota\mathrm{Rec}_v(\pi_v \otimes \eta_v\psi_v|\cdot|_v^{-1/2}).\end{aligned} \tag{7}$$

As η and ψ are both unramified at all places above p, $\rho_{\eta\psi|\cdot|_E}$ is crystalline at v. It then follows that $\rho_v \cong (M_{p,\iota} \otimes \rho_{\eta\psi}^{-1})|_{D_v}$ is crystalline, and so (3) follows from (7). That ρ_v has Hodge-Tate type $(\mathbf{k}, w)$ is immediate from [BR, Thm. 2.5.1(ii)] and Faltings' proof of the deRham conjecture.

2.3.3 Essentially self-dual representations of $\mathrm{GL}_3(\mathbf{A}_F)$

Let $\Pi = \otimes'\Pi_v$ be a cuspidal automorphic representation of $\mathrm{GL}_3(\mathbf{A}_F)$ for which each Π_i, $i \in I$, is such that its corresponding representation $\mathrm{Rec}_i(\Pi_i)$ of the Weil group of F_i satisfies

$$\mathrm{Rec}_i(\Pi_i)|_{\mathbf{C}^\times} \cong z^{a_i}\bar z^{b_i} \oplus z^{b_i}\bar z^{a_i} \oplus (z\bar z)^{(a_i+b_i)/2}, \quad a_i \neq b_i \in \mathbf{Z}, a_i + b_i \in 2\mathbf{Z}. \tag{8}$$

Suppose also that $\Pi^\vee \cong \Pi\otimes\psi$ for some Hecke character ψ (then ψ is necessarily algebraic). As explained in [B1, 4.1-4.6], it is a consequence of the results in [Mo] that for each prime ℓ and each isomorphism $\iota' : \mathbf{C} \xrightarrow{\sim} \overline{\mathbf{Q}}_\ell$ there is an ℓ-adic Galois representation $\rho_{\Pi,\iota'} : G_F \to \mathrm{GL}_3(\overline{\mathbf{Q}}_\ell)$ satisfying $\mathrm{WD}(\rho_{\Pi,\iota'}|_{D_v}) \cong$

[6] For an algebraic Hecke character ψ of a number field, we denote by $\rho_{\psi,\iota'}$ the ℓ-adic Galois representation associated with ψ and ι', normalized so that the restriction of the Galois character to the decomposition group at a place $w \nmid \ell$ is just the image of the local character ψ_w under the inverse of the Artin map, composed with ι'.

$\iota'\mathrm{Rec}_v(\Pi_v)$ for all places $v \nmid \ell$ that are prime to the conductor of Π and the absolute discriminant of F.

The proof of the existence of $\rho_{\Pi,\iota'}$ follows the arguments in [BR]. In particular, letting E be as in 2.3.2, if the base change of Π to E is still cuspidal then, as explained in the proof of [B1, Thm. 4.2], there is a motive $M = (A_K^m, e)$, K small enough, such that the ℓ-adic realizations of M yield $\rho_\Pi|_{G_E}$ twisted by a representation associated with an algebraic Hecke character of $\mathbf{A}_E^\times$. If Π is unramified at each $v|p$ then one can take $K = K_pK^p$ with K_p identified with $\mathbf{Z}_p^\times \times \prod_{v|p} \mathrm{GL}_{\mathcal{O}_{E,v}}(\Lambda_v)$ and the Hecke character can be taken unramified at each $v|p$. Then arguing as in 2.3.2 shows that $\rho_\Pi := \rho_{\Pi,\iota}$ is crystalline at each $v|p$ and such that $D_{HT}(\rho_\Pi|_{D_v}) \otimes_{\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F_v, j} \overline{\mathbf{Q}}_p$, $j \in \mathrm{Hom}_{\mathbf{Q}_p}(F_v, \overline{\mathbf{Q}}_p)$, is non-zero in degrees $-a_{i(j)}, -b_{i(j)}$, and $-(a_{i(j)} + b_{i(j)})/2$, $i(j) \in I$ being the induced embedding of F. Furthermore, if $\mathrm{WD}(\rho_{\Pi,\iota'}|_{D_v}) \cong \iota'\mathrm{Rec}_v(\Pi_v)$ for some $\ell \neq p$ (only an additional condition if p is not prime to the absolute discriminant of F), then these arguments also show that $\mathrm{WD}(\rho_\Pi|_{D_v}) \cong \iota\mathrm{Rec}_v(\Pi_v)$.

Remark. Suppose Π_v is unramified at each $v|p$. From the good reduction of the Shimura variety Sh_K with K_p as in 2.3.2 or 2.3.3, it follows easily from the Weil conjectures that the Frobenius-at-v eigenvalues of any ℓ-adic representation $\rho_{\Pi,\iota'}$, $\ell \neq p$, have absolute value as predicted by the Ramanujan conjecture for Π_v when considered as elements of $\mathbf{C}$ via ι'. Therefore, if $\mathrm{WD}(\rho_{\Pi,\iota'}|_{D_v}) \cong \iota'\mathrm{Rec}_v(\Pi_v)$, then the Ramanujan conjecture is true for Π_v. This argument shows (at least) that if q is a prime such that Π_w is unramified for all $w|q$, then the Ramanujan conjecture is true for Π_w, $w|q$, provided there is some prime $\ell \neq q$ such that the ℓ-adic representation $\rho_{\Pi,\iota'}$ satisfies $\mathrm{WD}(\rho_{\Pi,\iota'}|_{D_w}) \cong \iota'\mathrm{Rec}_w(\Pi_w)$.

2.4 Theorem 1 for the remaining cases

As a consequence of the work of Saito [Sa1, Sa2], the remarks in 2.2, and the results of [BR] as described in 2.3.2, to complete the proof of Theorem 1 it remains to consider the case where d is even, each $k_i = 2$, each π_v, $v|p$, is unramified, and π is not a CM representation. Replacing π by a twist by an integral power of $|\cdot|_F$ if necessary (which corresponds to twisting ρ_π by a power of the cyclotomic character), we may also assume that $w = 2$. Hereon we assume we are in this case.

2.4.1 An application of the symmetric square

Let $\Pi := \mathrm{Sym}^2\pi \otimes |\cdot|_F^{-1}$, with $\mathrm{Sym}^2\pi$ the symmetric square lift of π to $\mathrm{GL}_3(\mathbf{A}_F)$ (cf. [GJ]). As π is not a CM representation, Π is cuspidal. Since $\mathrm{Rec}_i(\pi_i)|_{\mathbf{C}^\times} \cong (\bar{z}/z)^{1/2} \bar{\oplus} (z/\bar{z})^{1/2}$, $\mathrm{Rec}_i(\Pi_i)|_{\mathbf{C}^\times} \cong \mathrm{Sym}^2\mathrm{Rec}_i(\pi_i|\cdot|_i^{-1/2})|_{\mathbf{C}^\times}$ satisfies (8) with $a_i = -2$ and $b_i = 0$. Furthermore, as $\pi^\vee \cong \pi \otimes \omega^{-1}$, ω the central character of π, it follows that $\Pi^\vee \cong \Pi \otimes \omega^{-2}|\cdot|_F^2$. Therefore, Π satisfies all the hypotheses in 2.3.3. In particular, there exist associated ℓ-adic representations

$\rho_{\Pi,\iota'}$. Clearly $\rho_{\Pi,\iota'} \cong \mathrm{Sym}^2\rho_{\pi,\iota'}$, so $\mathrm{WD}(\rho_{\Pi,\iota'}|_{D_w}) \cong \iota'\mathrm{Rec}_w(\mathrm{Sym}^2\pi_w \otimes |\cdot|_w^{-1})$ for all $w \nmid \ell$. Since π_v, and therefore Π_v, is unramified at each $v|p$, as explained in 2.3.3 we can conclude from this that for each $v|p$: (i) $\mathrm{Sym}^2\rho_\pi|_{D_v}$ is crystalline for $v|p$, (ii) $\mathrm{WD}(\mathrm{Sym}^2\rho_\pi|_{D_v}) \cong \iota\mathrm{Rec}_v(\mathrm{Sym}^2\pi_v \otimes |\cdot|_v^{-1})$, and (iii) $D_{HT}(\mathrm{Sym}^2\rho_\pi|_{D_v}) \otimes_{\overline{\mathbf{Q}}_p\otimes_{\mathbf{Q}_p}F_v,j} \overline{\mathbf{Q}}_p$, $j \in \mathrm{Hom}_{\mathbf{Q}_p}(F_v, \overline{\mathbf{Q}}_p)$, is non-zero in degrees 2, 1, and 0.

Let $v|p$. By conclusion (iii) of the preceding paragraph, the graded module $D_{HT}(\mathrm{Sym}^2\rho_v)$ is the symmetric square of the expected graded module for ρ_v. It then follows from results of Wintenberger[7] - Thm. 1.1.3, Prop. 1.2, and Remarks 1.1.4 of [Win1] or Thm. 2.2.2 of [Win2], applied to the isogeny $\mathrm{GL}_2 \to \mathrm{GL}_2/\pm 1$ - that there is a crystalline representation $\rho : D_v \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ such that $\mathrm{Sym}^2\rho = \mathrm{Sym}^2\rho_v$. From this it follows that ρ_v is isomorphic to a (possibly trivial) quadratic twist of ρ. In particular, ρ_v is potentially crystalline. Therefore $\mathrm{WD}(\mathrm{Sym}^2\rho_v) \cong \mathrm{Sym}^2\mathrm{WD}(\rho_v)$, and it then follows from conclusion (ii) of the preceding paragraph that $\mathrm{Sym}^2\mathrm{WD}(\rho_v) \cong \mathrm{Sym}^2\iota\mathrm{Rec}_v(\pi_v \otimes |\cdot|_v^{-1/2})$. From this it follows that $\mathrm{WD}(\rho_v)$ is isomorphic to a (possibly trivial) quadratic twist of $\iota\mathrm{Rec}_v(\pi_v \otimes |\cdot|_v^{-1/2})$. It also follows that ρ_v is of Hodge-Tate type $(\mathbf{k}, w)$ $(= ((2)_{i\in I}, 2)$ in this case).

Remark. We can also use $\mathrm{Sym}^2\pi$ to show that the Ramanujan conjecture holds for π. We may assume that π is not a CM representation. Let q be a prime. It then follows from the remark at the end of 2.3.3 that if π_w is unramified at each $w|q$, then the Ramanujan conjecture holds for each $\mathrm{Sym}^2\pi_w$ and hence for π_w. A simple base change argument like that in 2.2 then shows that the Ramanujan conjecture holds at all places where π is a principal series. In particular, this establishes the Ramanujan conjecture for those π for which there is no finite place v with π_v square-integrable. That the Ramanujan conjecture is known when such a v exists follows from Carayol's work [Ca2]. The Ramanujan conjecture has already been established for π by Blasius [B2].

2.4.2 The existence of a crystalline period

Recall that we are assuming that for each $v|p$, $\pi_v \cong \pi(\alpha_v, \beta_v)$ is an unramified principal series[8]. As $\mathrm{WD}(\rho_v)$ is isomorphic to a (possibly trivial) quadratic twist of $\iota\mathrm{Rec}_v(\pi_v \otimes |\cdot|_v^{-1/2})$, to prove (2) it suffices to show that $\mathrm{WD}(\rho_v)^{\mathrm{frob}_v=\alpha_v(\varpi_v)q_v^{1/2}} \neq 0$, where frob_v is a geometric frobenius at v, ϖ_v is a uniformizer at v, and q_v is the order of the residue field at v. This is equivalent to showing that $D_{cris}(\rho_v|_{D_v})^{\varphi^{f_v}=q_v^{1/2}\alpha_v(\varpi_v)}$ is a $\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F_{v,0}$-module of rank

[7] Note that 'weakly admissible = admissible' has been proved by Colmez and Fontaine, and 'de Rham = potentially semistable' has been proved (independently in some cases) by André, Berger, Kedlaya, and Mebkhout, and so the hypotheses on which these results depend are known to hold.

[8] By $\pi(\alpha,\beta)$ we mean the usual principal series representation that is the induction to $\mathrm{GL}_2(F_v)$ of the character $\left(\begin{smallmatrix} a & * \\ 0 & d\end{smallmatrix}\right) \mapsto \alpha(a)\beta(d)|a/d|_v^{1/2}$ of the upper-triangular Borel.

at least one. To establish a lower bound on this rank, we make use of p-adic analytic families of cuspidal representations.

Let $\mathcal{O}$ denote the integer ring of F and let $\mathcal{O}_p := \mathcal{O} \otimes \mathbf{Z}_p \xrightarrow{\sim} \prod_{v|p} \mathcal{O}_v$. Let $S_p := \{v|p\}$ be the set of places of F over p and let S_π be the set of finite places of F at which π is ramified. Let $S := S_\pi \cup S_p$ and $K^S := \prod_{v \notin S, v\nmid\infty} \mathrm{GL}_2(\mathcal{O}_v)$. Let $\mathcal{H}^S$ be the abelian Hecke algebra

$$\mathcal{H}^S := C_c(\mathrm{GL}_2(\mathbf{A}_{F,f}^S)//K^S).$$

For each $v \in S_p$ let

$$I_v := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathcal{O}_v) \ : \ \varpi_v | c\}, \quad I_p := \prod_{v|p} I_v,$$

and let $\mathcal{U}_v \subset C_c(\mathrm{GL}_2(\mathcal{O}_v)//I_v)$ be the abelian subalgebra generated by the characteristic functions

$$U_v := \mathrm{char}(I_v \mathrm{diag}(\varpi_v, 1) I_v).$$

Put

$$\mathcal{U}_p := \otimes_{v|p} \mathcal{U}_p \quad \text{and} \quad \mathbf{T}^S := \mathcal{U}_p \otimes \mathcal{H}^S.$$

Then there exists an $f_\pi \in \pi^{K^S I_p}$ that is an eigenvector for the (usual) action of the Hecke ring $\mathbf{T}^S$ such that $\mathrm{char}(I_v \mathrm{diag}(\varpi_v, 1) I_v)$ acts with eigenvalue $q_v^{1/2} \alpha_v(\varpi_v)$.

Let $K \subset \overline{\mathbf{Q}}_p$ be a finite extension of $\mathbf{Q}_p$ containing each $i(F)$, $i \in I$, and the eigenvalues for the action of $\mathbf{T}^S$ on f_π. Let $|K^\times| = \{|x|_p \ : \ x \in K^\times\}$. For $r \in |K^\times|$, we denote by B_r the usual closed rigid ball over K of radius r (so $B_r(\mathbf{C}_p) = \{x \in \mathbf{C}_p \ : \ |x|_p \le r\}$, where $\mathbf{C}_p := \widehat{\overline{\mathbf{Q}}}_p$). Then $\mathcal{O}(B_1) = K < T >$. Let $A_r := \mathcal{O}(B_r)$; this is an affinoid K-algebra. From the work of Buzzard [Bu1, Bu2] one can deduce that if $r_0 \in |K^\times|$ is sufficiently small, then there exists a reduced finite torsion-free A_{r_0}-algebra $\mathcal{R}$ (so also an affinoid K-algebra) and a homomorphism $\phi : \mathbf{T}^S \to \mathcal{R}$ satisfying (i)-(iii) below. For $x \in \mathrm{Hom}_K(\mathcal{R}, \overline{\mathbf{Q}}_p)$ put $\phi_x := x \circ \phi$. Then:

(i) if x is such that $x(1+T) = (1+q)^{n_x}$, $n_x \in p(p-1)\mathbf{Z}_{>0}$ ($q = p$ if p odd and $q = 4$ if $p = 2$), then there exists a cuspidal representation π_x of $\mathrm{GL}_2(\mathbf{A}_F)$ with infinity type $(\mathbf{k}_x, w_x) = ((n_x+2)_{i\in I}, n_x+2))$ and which is unramified at all $v|p$ and such that $\phi_x : \mathbf{T}^S \to \overline{\mathbf{Q}}_p$ gives the eigenvalues of the action of $\mathbf{T}^S$ on an eigenvector $f_x \in \pi_x^{K^S I_p}$;

(ii) there exists $x_0 \in \mathcal{X}(K)$ with $x_0(1+T) = 1$ such that ϕ_{x_0} gives the eigenvalues of the action of $\mathbf{T}^S$ on f_π;

(iii) if $\phi_v := \phi(U_v) \in \mathcal{R}^\times$, then $|x(\phi_v)|_p$ is constant for all x;

(iv) there exists a continuous representation

$$\rho_{\mathcal{R}} : G_F \to \mathrm{GL}_2(\mathcal{R})$$

unramified away from S and such that for x as in (i) the representation $\rho_x : G_F \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ induced from $\rho_{\mathcal{R}}$ by x is equivalent to ρ_{π_x} and that induced by x_0 is equivalent to ρ_π.

Assuming the existence of $\mathcal{R}$ and ϕ, we can complete the proof of Theorem 1. Let $\Sigma \subset \mathrm{Hom}_K(\mathcal{R}, \overline{\mathbf{Q}}_p)$ be the set of x as in (i). Then Σ is Zariski dense by the finiteness of $\mathcal{R}$ over A_r. As explained in 2.3.2 we know that Theorem 1 holds for each π_x, $x \in \Sigma$. Let $v|p$ and $x \in \Sigma$. Then $\pi_{x,v} \cong \pi(\mu_x, \lambda_x)$, an unramified principal series with $x(\phi_v) = \mu_x(\varpi_v)q_v^{1/2}$. In particular, as Theorem 1 holds for $\rho_x \cong \rho_{\pi_x}$ we have that $D_{cris}(\rho_x|_{D_v})^{\varphi^{f_v}=x(\phi_v)}$ is a $\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F_{v,0}$-module of rank at least one for all $x \in \Sigma$, where f_v is the residue class degree of F_v (so $q_v = p^{f_v}$). As the Hodge-Tate type of ρ_x, $x \in \Sigma$, is $(\mathbf{k}_x, w_x)$, each $D_{HT}(\rho_x|_{D_v}) \otimes_{\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F_v, j} \overline{\mathbf{Q}}_p$ is non-zero in degrees 0 and $n_x + 1$. It then follows easily from [Ki2, (5.15)] that[9]

$$D_{cris}(\rho_\pi|_{D_v})^{\varphi^{f_v}=q_v^{1/2}\alpha_v(\varpi_v)} = D_{cris}(\rho_{x_0}|_{D_v})^{\varphi^{f_v}=x_0(\phi_v)}$$

is also a $\overline{\mathbf{Q}}_p \otimes_{\mathbf{Q}_p} F_{v,0}$-module of rank at least one.

While the existence of $\mathcal{R}$ and ϕ is essentially proved in the work of Buzzard, there is no convenient reference in [Bu1]. So we conclude by explaining how their existence follows from this work. Let D be the quaternion algebra over F that is split at all finite places and compact modulo the center at all archimedean places. Fix a maximal order $\mathcal{O}_D$ of D, and for each finite place v of F fix an isomorphism $\mathcal{O}_{D,v} \cong M_2(\mathcal{O}_v)$. This identifies $\mathrm{GL}_2(\mathbf{A}_{F,f})$ with $(D \otimes_F \mathbf{A}_{F,f})^\times$. Let $\mathfrak{n}$ be the conductor of π and let $U_0 \subseteq \mathrm{GL}_2(\mathcal{O} \otimes \widehat{\mathbf{Z}})$ be the subgroup of matrices with lower left entries in $\mathfrak{n} \otimes \widehat{\mathbf{Z}}$, and let $U = U_0 \cap I_p$. Let $J := \{v|p\}$. For $\mathbf{a} \in \mathbf{Z}_{>0}^J$ let $U_{\mathbf{a}} := \prod_{v \in J} U_v^{a_v}$. For $v \in J$ let $\sigma_v := \mathrm{ord}_v(\alpha_v(\varpi_v)q_v^{1/2})$, and let $\sigma_{\mathbf{a}} := \sum_{v \in J} a_v\sigma_v$.

For $r \in |K^\times|$ with $r \le 1$ we define a homomorphism $\kappa : \mathcal{O}_p^\times \times \mathcal{O}_p^\times \to A_r^\times$ by

$$\kappa((x_v), (y_v)) = \prod_{v \in S_p} \prod_{j \in \mathrm{Hom}_{\mathbf{Q}_p}(F_v, \overline{\mathbf{Q}}_p)} j(y_v)(1+T)^{\log_p \mathrm{Nm}_{F_v/\mathbf{Q}_p}(x_v)}.$$

[9] Proposition (5.14) and Corollary (5.15) of [Ki2] are only stated for representations of $G_{\mathbf{Q}_p} = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. But it is easily checked that the arguments extend to the case of the representations of $D_v = \mathrm{Gal}(\overline{F}_v/F_v)$ under consideration here; the necessary results with φ replaced by φ^{f_v} (e.g., Corollary (3.7)) are easily deduced from those for φ. A key point is that our hypotheses on the weights in the family $\mathcal{X}$ ensure that the polynomial $P(X) \in (\mathcal{O}(\mathcal{X}) \otimes_{\mathbf{Q}_p} F_v)[X]$ provided by Sen's theory as in [Ki2, (2.2)] is of the form $P(X) = XQ(X)$ with the constant coefficient of Q not a zero-divisor.

Let $\mathcal{W}$ be the rigid analytic weight space over K defined in §8 of [Bu1]. Then B_r is identified with a reduced affinoid subspace of $\mathcal{W}$ such that κ is the induced weight in the sense of *loc. cit.* Let $m \in |K^\times|$ be so small that the A_r-Banach module $\mathbf{S}_\kappa^D(U;m)$ of overconvergent automorphic forms is defined (notation as in [Bu1, §9]). This is equipped with an A_r-linear action of $\mathbf{T}^S$ such that each U_v is a completely continuous operator. For $b \in B_r(\overline{\mathbf{Q}}_p)$ such that the induced map $e_b : A_r \to \overline{\mathbf{Q}}_p$ sends $1+T$ to $(1+q)^{n_b}$ with $n_b \in p(p-1)\mathbf{Z}_{\geq 0}$ we have a $\mathbf{T}^S$-equivariant inclusion of the classical forms of weight $(\mathbf{k}_b, \mathbf{w}_b)$: $S^D_{\mathbf{k}_b,\mathbf{w}_b}(U) \subseteq \mathbf{S}_\kappa^D(U;m) \otimes_{\mathcal{O}(B_r),e_b} \overline{\mathbf{Q}}_p$, $\mathbf{w}_b := (n_b+2)_{i\in I} \in \mathbf{Z}^I$ (see [Bu1, §11]). By the Jacquet-Langlands correspondence, there exists $f_0 \in S_{2,2}(U)$ having the same $\mathbf{T}^S$-eigenvalues as f_π. Recall that by the theory of Fredholm series and orthonormalizable Banach modules as developed by Coleman, Ash and Stevens, and Buzzard, if r is small enough then there is a finite A_r-direct summand $\mathcal{N} \subset \mathbf{S}_\kappa^D(U;m)$ that is stable under $\mathbf{T}^S$ and such that for each $\mathbf{a} \in \mathbf{Z}^J_{>0}$ the Fredholm series for $U_\mathbf{a}$ on $\mathcal{N}$ is a factor of the slope $\sigma_\mathbf{a}$ part of the Fredholm series $P_\mathbf{a}(X) \in A_r\{X\}$ associated to the completely continuous operator $U_\mathbf{a}$ on $\mathbf{S}_\kappa^D(U;m)$ (the latter is well-defined for r small enough), and furthermore is such that $f_0 \in \mathcal{N} \otimes_{A_r,e_0} K$. If r is sufficiently small then for any $b \in A_r$ with $n_b \in p(p-1)\mathbf{Z}_{>0}$ it follows from the arguments in [Bu2, §7] (see also the comment at the end of §11 of [Bu1]) that $N_b := \mathcal{N} \otimes_{A_r,e_b} \overline{\mathbf{Q}}_p$ is comprised of classical forms in $S^D_{n_b+2,n_b+2}(U)$ (n_b is divisible by a high power of p; the smaller r is, the larger the power of p). By the definition of $\mathcal{N}$, any $\mathbf{T}^S$-eigenform in N_b is such that the eigenvalue of U_v has slope σ_v, and so if r is small enough relative to σ_v then it is easily seen that the v-constituent of the irreducible representation of $\mathrm{GL}_2(\mathbf{A}_{F,f})$ generated by f is not special and therefore must be an unramified principal series.

Let R be the A_r-algebra generated by the image of $\mathbf{T}^S$ in $\mathrm{End}_{A_r}(\mathcal{N})$; this is a finite torsion-free A_r-algebra and so an affinoid K-algebra. Note that there exists a K-homomorphism $\phi_0 : R \to K$ giving the eigenvalues of the $\mathbf{T}^S$-action on f_0. Let A be the normalization of the quotient of R by a minimal prime containing the kernel of ϕ_0. This is a reduced finite torsion-free A_r-algebra and so also an affinoid K-algebra. Let $\phi : \mathbf{T}^S \to A$ be the canonical homomorphism. It follows from the definitions that (i), (ii), and (iii) hold with $\mathcal{R}$ replaced by A. For each $x \in \mathrm{Hom}_K(A, \overline{\mathbf{Q}}_p)$ as in (i), let $T_x : G_F \to \overline{\mathbf{Q}}_p$ be the continuous pseudo-representation associated with ρ_{π_x} (so $T_x = \mathrm{trace}\,\rho_{\pi_x}$). Since for a place $w \nmid \mathfrak{n}p$, $T_x(\mathrm{frob}_w) = x \circ \phi(\mathrm{char}(\mathrm{GL}_2(\mathcal{O}_w)\mathrm{diag}(\varpi_w, 1)\mathrm{GL}_2(\mathcal{O}_w))$, $\varpi_w \in \mathcal{O}_w$ a uniformizer, it follows easily from the Cebotarev density theorem and the Zariski density of the set Σ_A of $x \in \mathrm{Hom}_K(A, \overline{\mathbf{Q}}_p)$ as in (i) that there is a continuous pseudo-representation $T : G_F \to A$ such that $T_x = x \circ T$. From the general theory of pseudo-representations (cf. [Tay3]) there is a semisimple Galois representation $\rho_A : G_F \to \mathrm{GL}_2(F_A)$, F_A the field of fractions of A, such that $T = \mathrm{trace}\,\rho_A$. It is easy to see that there is a finite A-module $M \subset F_A^2$ on which G_F acts continuously and such that $V_x := M_x \otimes_{A_x,x} \overline{\mathbf{Q}}_p$ is isomorphic to the representation ρ_{π_x}, $x \in \Sigma_A$ or x any extension of ϕ_0 to A (here the subscript x on M and A denotes the localization at the kernel of x). Such a module M is given explicitly as follows. Fix a basis of ρ_A such that for some $i \in I$ the corresponding complex conjugation in G_F is diagonalized (with eigenvalues 1 and -1). Writing $\rho_A(\sigma) = \left(\begin{smallmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{smallmatrix}\right)$, we have $a_\sigma, d_\sigma, b_\sigma c_{\sigma'} \in A$ for all $\sigma, \sigma' \in G_F$ and that these define continuous functions of σ and σ'. It follows that the A-submodules $\mathcal{B}$ and $\mathcal{C}$ of F_A generated by $\{b_\sigma \ : \sigma \in G_F\}$ and $\{c_\sigma \ : \sigma \in G_F\}$, respectively, are fractional ideals of A satisfying $\mathcal{C}\mathcal{B} \subseteq A$ (note that by the semisimplicity of ρ_A and the diagonalization of the chosen complex conjugation, $\mathcal{B} = 0$ if and only of $\mathcal{C} = 0$). We can then take $M = A \oplus A$ if $\mathcal{C} = 0$ and $M = A \oplus \mathcal{C}$ otherwise. Being a finite A-module, M is a Banach A-module and the continuity of the action of G_F on M is clear from the continuity of the functions a_σ, d_σ, and $b_\sigma c_{\sigma'}$. As A is normal, for any $x \in \mathrm{Hom}_K(A, \overline{\mathbf{Q}}_p)$ the localization A_x is a DVR, and so M_x is a free A_x-module of rank two. The representation V_x is then two-dimensional and its associated pseudo-representation is $x \circ T$. Therefore if $x \in \Sigma_A$ or x any extension of ϕ_0 to A, the pseudo-representation associated with V_x equals that associated with ρ_{π_x}. As the latter representation is irreducible (this irreducibility is well-known, but see also the remark below) it follows that $V_x \cong \rho_{\pi_x}$. As A is normal and finite over A_r, there is an $f \notin TA_r$ (in fact one can pick f not to be zero on any given finite set of points of B_r) such that M_f is free over A_f. Let $r_0 \leq r$ be so small that $f \in A_{r_0}^\times$. Then (i)-(iv) hold with $\mathcal{R}$ the quotient of $A \otimes_{A_r} A_{r_0}$ by any minimal prime (a finite A_{r_0}-algebra and so an affinoid K-algebra) and with $\rho_{\mathcal{R}}$ the representation of G_F on the free $\mathcal{R}$-module $M \otimes_A \mathcal{R}$.

Remark. We recall that there is a quick proof of the irreducibility of ρ_π using that it is potentially semistable (really only that it is Hodge-Tate), which was established in 2.4.1. If $\rho_\pi \cong \chi_1 \oplus \chi_2$, then each χ_i is potentially semistable and hence is the Galois representation associated to an algebraic Hecke character ψ_i of F (cf. [Se], esp. III,2.3-2.4). It then follows that $L(\pi \otimes \psi_2^{-1}, s - 1/2) = L(\psi_1/\psi_2, s)\zeta_F(s)$. As $\psi_i = \psi_i' |\cdot|_F^{a_i}$ with $a_i \in \mathbf{Z}$ and ψ_i' finite and since we may assume $a_1 \geq a_2$, $L(\psi_1/\psi_2, 1) = L(\psi_1'/\psi_2', a_1 - a_2 + 1) \neq 0$. But this implies that $L(\pi \otimes \psi_2^{-1}, s)$ has a pole at $s = 1$, contradicting the cuspidality of π.

References

[B1] D. Blasius, *Automorphic forms and Galois representations: some examples*, Automorphic forms, Shimura varieties, and L-functions, Vol. II (Ann Arbor, MI, 1988), Perspect. Math., vol 11, Academic Press, Boston, MA, 1990, pp. 1–13.

[B2] D. Blasius, *Hilbert modular forms and the Ramanujan conjecture*, Noncommutative geometry and number theory, Aspects Math., E37, Vieweg, Wiesbaden, 2006, pp. 35–56.

[BR] D. Blasius and J. Rogawski, *Motives for Hilbert modular forms*, Invent. Math. 114 (1993), no. 1, 55–87.

[Br] C. Breuil, *Une remarque sur les représentations locales p-adiques et les congruences entre formes modulaires de Hilbert*, Bull. Soc. Math. France 127 (1999), no. 3, 459–472.

[Bu1] K. Buzzard, *Eigenvarieties*, L-functions and Galois representations, London Math. Soc. Lecture Note Ser., vol. 320, Cambridge Univ. Press, Cambridge, 2007, pp. 59–120.

[Bu2] K. Buzzard,*On p-adic families of automorphic forms*, Modular curves and abelian varieties, Progr. Math., vol. 224, Birkhuser, Basel, 2004, pp. 23–44.

[Ca1] H. Carayol, *Sur la mauvaise rduction des courbes de Shimura*, Compositio Math. 59 (1986), no. 2, 151–230.

[Ca2] H. Carayol, *Sur les représentations l-adiques associées aux formes modulaires de Hilbert*, Ann. Sci. Ecole Norm. Sup. (4) 19 (1986), no. 3, 409–468.

[Fo] J.-M. Fontaine, *Arithmtique des représentations galoisiennes p-adiques*, Cohomologies p-adiques et applications arithmétiques. III. AstÉrisque No. 295 (2004), xi, 1–115.

[GJ] S. Gelbart and H. Jacquet, *A relation between automorphic representations of* GL(2) *and* GL(3), Ann. Sci. cole Norm. Sup. (4) 11 (1978), no. 4, 471–542.

[Ki1] M. Kisin, *Potentially semi-stable deformation rings*, J. Amer. Math. Soc. 21 (2008), no. 2, 513–546.

[Ki2] M. Kisin, *Overconvergent modular forms and the Fontaine-Mazur conjecture*, Invent. Math. 153 (2003), no. 2, 373–454.

[Ko] R. Kottwitz, *Points on some Shimura varieties over finite fields*, J. Amer. Math. Soc. 5 (1992) no. 2, 373–444.

[L] T. Liu, *Lattices in filtered* (φ, N)*-modules*, preprint 2008.

[Mo] *The zeta functions of Picard modular surfaces*, edited by Robert P. Langlands and Dinakar Ramakrishnan, Universit de Montrèal, Centre de Recherches Mathèmatiques, Montreal, QC, 1992.

[Sa1] T. Saito, *Modular forms and p-adic Hodge theory*, Invent. Math. 129 (1997), no. 3, 607–620.

[Sa2] T. Saito, *Hilbert modular forms and p-adic Hodge theory*, preprint (2006).

[Se] J.-P. Serre, *Abelian l-adic representations and elliptic curves*, W. A. Benjamin, Inc., New York-Amsterdam 1968 xvi+177

[Ta] J. Tate, *Number theoretic background*, Automorphic forms, representations and L-functions, Part 2, Proc. Sympos. Pure Math., vol. XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 3–26.

[Tay1] R. Taylor, *On Galois representations associated to Hilbert modular forms*, Invent. Math. 98 (1989), no. 2, 265–280.

[Tay2] R. Taylor, *On Galois representations associated to Hilbert modular forms. II*, Elliptic curves, modular forms, and Fermat's last theorem (Hong Kong, 1993), Ser. Number Theory, I, Int. Press, Cambridge, MA, 1995, pp. 185–191.

[Tay3] R Taylor, *Galois representations associated to siegel modular forms of low weight*, Duke Math. J. 63 (1991), no. 2, 281–332.

[W] A. Wiles, *On ordinary λ-adic representations associated to modular forms*, Invent. Math. 94 (1988), no. 3, 529–573.

[Win1] J.-P. Wintenberger, *Relëvement selon une isogènie de systëmes l-adiques de reprsentations galoisiennes associès aux motifs*, Invent. Math. 120 (1995), no. 2, 215–240.

[Win2] J.-P. Wintenberger, *Propriètès du groupe tannakien des structures de Hodge p-adiques et torseur entre cohomologies cristalline et tale*, Ann. Inst. Fourier (Grenoble) 47 (1997), no. 5, 1289–1334.

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