Semigroup Properties for the Second Fundamental Form

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Abstract. Let M be a compact Riemannian manifold with boundary ∂M and $L=\delta+Z$ for a C^1-vector field Z on M. Several equivalent statements, including the gradient and Poincaré/log-Sobolev type inequalities of the Neumann semigroup generated by L, are presented for lower bound conditions on the curvature of L and the second fundamental form of ∂M. The main result not only generalizes the corresponding known ones on manifolds without boundary, but also clarifies the role of the second fundamental form in the analysis of the Neumann semigroup. Moreover, the Lévy-Gromov isoperimetric inequality is also studied on manifolds with boundary.

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1 Introduction

The main purpose of this paper is to find out equivalent properties of the Neumann semigroup on manifolds with boundary for lower bounds of the second fundamental form of the boundary. To explain the main idea of the study, let us briefly recall some equivalent semigroup properties for curvature lower bounds on manifolds without boundary.

Let M be a connected complete Riemannian manifold without boundary and let $L=\Delta+Z$ for some C^1-vector field Z on M. Let P_t be the diffusion semigroup generated by L, which is unique and Markovian if the curvature of L is bounded below, namely (see [3]),

$$\mathrm{Ric}-\nabla Z\geq -K \tag{1.1}$$

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holds on M for some constant $K \in \mathbb{R}$. The following is a collection of known equivalent statements for (1.1), where the first two ones on gradient estimates are classical in geometry (see e.g. [1, 5, 6, 7]), and the remainder follows from Propositions 2.1 and 2.6 in [2] (see also [9]):

(i) $|\nabla P_t f|^2 \le e^{2Kt} P_t |\nabla f|^2, \ \ t \ge 0, \ f \in C_b^1(M)$;

(ii) $|\nabla P_t f| \le e^{Kt} P_t |\nabla f|, \ \ t \ge 0, \ f \in C_b^1(M)$;

(iii) $P_t f^2 - (P_t f)^2 \le \frac{e^{2Kt}-1}{K} P_t |\nabla f|^2, \ \ t \ge 0, \ f \in C_b^1(M)$;

(iv) $P_t f^2 - (P_t f)^2 \ge \frac{1-e^{-2Kt}}{K} |\nabla P_t f|^2, \ \ t \ge 0, \ f \in C_b^1(M)$;

(v) $P_t(f^2 \log f^2) - (P_t f^2)\log(P_t f^2) \le \frac{2(e^{2Kt}-1)}{K} P_t |\nabla f|^2, \ \ t \ge 0, \ f \in C_b^1(M)$;

(vi) $(P_t f)\{P_t(f \log f) - (P_t f)\log(P_t f)\} \ge \frac{1-e^{-2Kt}}{2K} |\nabla P_t f|^2, \ \ t \ge 0, \ f \in C_b^1(M), \ f \ge 0$.

These equivalent statements for the curvature condition are crucial in the study of heat semigroups and functional inequalities on manifolds. For the case that M has a convex boundary, these equivalences are also true for P_t the Neumann semigroup (see [10] for one more equivalent statement on Harnack inequality). The question is now can we extend this result to manifolds with non-convex boundary, and furthermore describe the second fundamental using semigroup properties?

So, from now on we assume that M has a boundary ∂M. Let N be the inward unit normal vector field on ∂M. Then the second fundamental form is a two-tensor on $T\partial M$, the tangent space of ∂M, defined by

$$\mathbb{I}(X,Y) = -\langle \nabla_X N, Y\rangle, \quad X, Y \in T\partial M.$$

If $\mathbb{I} \ge 0$(i.e. $\mathbb{I}(X,X) \ge 0$ for $X \in T\partial M$), then ∂M (or M) is called convex. In general, we intend to study the lower bound condition of $\mathbb{I}$; namely, $\mathbb{I} \ge -\sigma$ on ∂M for some $\sigma \in \mathbb{R}$.

For $x \in M$, let $\mathbb{E}^x$ be the expectation taken for the reflecting L-diffusion process X_t starting from x. So, for a bounded measurable functional Φ of X,

$$\mathbb{E}\Phi: \ x \mapsto \mathbb{E}^x \Phi$$

is a function on M. Moreover, let l_t be the local time of X_t on ∂M. According to [8, Theorem 5.1], (1.1) and $\mathbb{I} \ge -\sigma$ imply

$$|\nabla P_t f| \le e^{Kt} \mathbb{E}\big[|\nabla f|(X_t) e^{\sigma l_t}\big], \quad t > 0, f \in C^1(M). \tag{1.2}$$

To see that (1.2) is indeed equivalent to (1.1) and $\mathbb{I} \geq -\sigma$, we shall make use of the following formula for the second fundamental form established recently by the author in [12]: for any $f \in C^\infty(M)$ satisfying the Neumann condition $Nf|_{\partial M} = 0$,

$$\mathbb{I}(\nabla f, \nabla f) = \frac{\sqrt{\pi}|\nabla f|^2}{2} \lim_{t\to 0} \frac{1}{\sqrt{t}} \log \frac{(P_t|\nabla f|^p)^{1/p}}{|\nabla P_t f|} \tag{1.3}$$

holds on ∂M for any $p \in [1, \infty)$. With help of this result and stochastic analysis on the reflecting diffusion process, we are able to prove the following main result of the paper.

Theorem 1.1. *Let M be a compact Riemannian manifold with boundary and let P_t be the Neumann semigroup generated by $L = \Delta + Z$. Then for any constants $K, \sigma \in \mathbb{R}$, the following statements are equivalent to each other:*

(1) $\mathrm{Ric} - \nabla Z \geq -K$ *on* M *and* $\mathbb{I} \geq -\sigma$ *on* ∂M*;*

(2) (1.2) *holds;*

(3) $|\nabla P_t f|^2 \leq \mathrm{e}^{2Kt}(P_t|\nabla f|^2)\mathbb{E}\mathrm{e}^{2\sigma l_t}, \quad t \geq 0,\ f \in C^1(M)$;

(4) $P_t(f^2 \log f^2) - (P_t f^2)\log P_t f^2 \leq 4\mathbb{E}\big[|\nabla f|^2(X_t) \int_0^t \mathrm{e}^{2\sigma(l_t - l_{t-s}) + 2Ks}\mathrm{d}s\big]$, $t \geq 0,\ f \in C^1(M)$;

(5) $P_t f^2 - (P_t f)^2 \leq 2\mathbb{E}\big[|\nabla f|^2(X_t) \int_0^t \mathrm{e}^{2\sigma(l_t - l_{t-s}) + 2Ks}\mathrm{d}s\big],\ t \geq 0,\ f \in C^1(M)$;

(6) $|\nabla P_t f|^2 \leq$
$\leq \Big(\frac{2K}{1 - \mathrm{e}^{-2Kt}}\Big)^2 \big(P_t(f\log f) - (P_t f)\log P_t f\big)\mathbb{E}\big[f(X_t)\int_0^t \mathrm{e}^{2\sigma l_s - 2Ks}\mathrm{d}s\big]$, $t > 0,\ f \geq 0,\ f \in C^1(M)$;

(7) $|\nabla P_t f|^2 \leq \dfrac{2K^2}{(1 - \mathrm{e}^{-2Kt})^2}\big(P_t f^2 - (P_t f)^2\big)\mathbb{E}\int_0^t \mathrm{e}^{2\sigma l_s - 2Ks}\mathrm{d}s, \quad t \geq 0,\ f \in C^1(M)$.

Theorem 1.1 can be extended to a class of non-compact manifolds with boundary such that the local times l_t is exponentially integrable. According to [13] the later is true provided $\mathbb{I}$ is bounded, the sectional curvature around ∂M is bounded above, the drift Z is bounded around ∂M, and the injectivity radius of the boundary is positive. To avoid technical complications, here we simply consider the compact case.

In the next section, we shall provide a result on gradient estimate and non-constant lower bounds of curvature and second fundamental form, which implies the equivalences among (1), (2) and (3) as a special case. Then we present a complete proof for the remainder of Theorem 1.1 in Section 3. As mentioned above, for manifolds without boundary or with a convex boundary an equivalent Harnack inequality for the curvature condition has been presented in [10].

Due to unboundedness of the local time which causes an essential difficulty in the study of Harnack inequality, the corresponding result for lower bound conditions of the curvature and the second fundamental form is still open. Nevertheless, log-Harnack and Harnack inequalities for the Neumann semigroup on non-convex manifolds have been provided by [13, Theorem 5.1] and [14, Theorem 4.1] respectively. Finally, as an extension to a result in [4] where manifolds without boundary is considered, the Lévy-Gromov isoperimetric inequality is derived in Section 4 for manifolds with boundary.

2 Gradient estimate

Let $K_1, K_2 \in C(M)$ be such that

$$\mathrm{Ric} - \nabla Z \geq -K_1 \text{ on } M, \quad \mathbb{I} \geq -K_2 \text{ on } \partial M. \tag{2.1}$$

According to [8, Theorem 5.1] this condition implies

$$|\nabla P_t f| \leq \mathbb{E}\big[|\nabla f|(X_t)\mathrm{e}^{\int_0^t K_1(X_s)\mathrm{d}s + \int_0^t K_2(X_s)\mathrm{d}l_s}\big], \quad t \geq 0, f \in C^1(M). \tag{2.2}$$

The main purpose of this section is to prove that these two statements are indeed equivalent to each other. To prove that (2.2) implies (2.1), we need the following results collected from [11, Proof of Lemma 2.1] and [13, Theorem 2.1, Lemma 2.2, Proposition A.2] respectively:

(I) For any $\lambda > 0$, $\mathbb{E}\mathrm{e}^{\lambda l_t} < \infty$.

(II) For $X_0 = x \in \partial M$, $\limsup_{t \to 0} \frac{1}{t}|\mathbb{E}l_t - 2\sqrt{t/\pi}| < \infty$.

(III) For $X_0 = x \in \partial M$, there exists a constant $c > 0$ such that $\mathbb{E}l_t^2 \leq ct, \quad t \in [0,1]$.

(IV) Let ρ be the Riemannian distance. For $\delta > 0$ and $X_0 = x \in M \setminus \partial M$ such that $\rho(x, \partial M) \geq \delta$, the stopping time $\tau_\delta := \inf\{t > 0 : \rho(X_t, x) \geq \delta\}$ satisfies $\mathbb{P}(\tau_\delta \leq t) \leq c\exp[-\delta^2/(16t)]$ for some constant $c > 0$ and all $t > 0$.

Theorem 2.1. (2.1), (2.2) *and the following inequality are equivalent to each other:*

$$|\nabla P_t f|^2 \leq (P_t|\nabla f|^2)\mathbb{E}\big[\mathrm{e}^{2\int_0^t K_1(X_s)\mathrm{d}s + 2\int_0^t K_2(X_s)\mathrm{d}l_s}\big], \quad t \geq 0, f \in C^1(M). \tag{2.3}$$

Proof. Since by [8] (2.1) implies (2.2) which is stronger than (2.3) due to the Schwartz inequality, it remains to deduce (2.1) from (2.3).
(a) Proof of $\mathrm{Ric} - \nabla Z \geq -K_1$. It suffices to prove at points in the interior. Let $X_0 = x \in M \setminus \partial M$. For any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\bar{B}(x,\delta) \subset M \setminus \partial M, \qquad \sup_{y\in\bar{B}(x,\delta)} |K_1(y) - K_1(x)| \le \varepsilon, \tag{2.4}$$

where $\bar{B}(x,\delta)$ is the closed geodesic ball at x with radius δ. Since $l_t = 0$ for $t \le \tau_\delta$, by (2.3), (I) and (IV) we have

$$\begin{aligned}|\nabla P_t f|^2(x) &\le (P_t|\nabla f|^2(x))\mathbb{E}\mathrm{e}^{2\int_0^t K_1(X_s)\mathrm{d}s + 2\int_0^t K_2(X_s)\mathrm{d}l_s} \\ &\le (P_t|\nabla f|^2(x))\Big\{\mathrm{e}^{2t(K_1(x)+\varepsilon)}\mathbb{P}(\tau_\delta \ge t) + \sqrt{\mathbb{P}(\tau_\delta < t)\mathbb{E}\mathrm{e}^{4t\|K_1\|_\infty + 4\|K_2\|_\infty l_t}}\Big\} \\ &\le (P_t|\nabla f|^2(x))\mathrm{e}^{2t(K_1(x)+\varepsilon)} + C\mathrm{e}^{-\lambda/t}, \quad t \in (0,1]\end{aligned}$$

for some constants $C, \lambda > 0$.
This implies

$$\limsup_{t\to 0} \frac{|\nabla P_t f|^2(x) - |\nabla f|^2(x)}{t} \le \limsup_{t\to 0} \frac{\mathrm{e}^{2t(K_1(x)+\varepsilon)} P_t|\nabla f|^2(x) - |\nabla f|^2(x)}{t}. \tag{2.5}$$

Now, let $f \in C^\infty(M)$ with $Nf|_{\partial M} = 0$, we have

$$P_t f = f + \int_0^t P_s L f \mathrm{d}s, \quad t \ge 0.$$

Then

$$\begin{aligned}&\limsup_{t\to 0} \frac{|\nabla P_t f|^2(x) - |\nabla f|^2(x)}{t} \\ &= \lim_{t\to 0} \frac{1}{t}\Big\{\Big|\int_0^t \nabla P_s L f \mathrm{d}s\Big|^2 + 2\int_0^t \langle \nabla f, \nabla P_s L f\rangle \mathrm{d}s\Big\}(x).\end{aligned} \tag{2.6}$$

Moreover, according to the last display in the proof of [8, Theorem 5.1] (the initial data $u_0 \in O_x(M)$ was missed in the right hand side therein),

$$\nabla P_t L f = u_0 \mathbb{E}\big[M_t u_t^{-1} \nabla L f(X_t)\big],$$

where u_t is the horizontal lift of X_t on the frame bundle $O(M)$, and M_t is a $d \times d$-matrices valued right continuous process satisfying $M_0 = I$ and (see [8, Corollary 3.6])

$$\|M_t\| \le \exp\big[\|K_1\|_\infty t + \|K_2\|_\infty l_t\big].$$

So, due to (I), $|\nabla P_\cdot L f|$ is bounded on $[0,1] \times M$ and $\nabla P_s L f \to \nabla L f$ as $s \to 0$. Combining this with (2.6) we obtain

$$\limsup_{t\to 0} \frac{|\nabla P_t f|^2(x) - |\nabla f|^2(x)}{t} = 2\langle \nabla f, \nabla L f\rangle(x). \tag{2.7}$$

On the other hand, applying the Itô formula to $|\nabla f|^2(X_t)$ we have

$$\begin{aligned} P_t|\nabla f|^2(x) &= |\nabla f|^2(x) + \int_0^t P_s L|\nabla f|^2(x)\mathrm{d}s + \mathbb{E}\int_0^t N|\nabla f|^2(X_s)\mathrm{d}l_s \\ &\le |\nabla f|^2(x) + \int_0^t P_s L|\nabla f|^2(x)\mathrm{d}s + \|\nabla|\nabla f|^2\|_\infty \mathbb{E} l_t. \end{aligned} \tag{2.8}$$

Since $l_t = 0$ for $t \le \tau_\delta$, by (III) and (IV) we have

$$\mathbb{E} l_t \le \sqrt{(\mathbb{E} l_t^2)\mathbb{P}(\tau_\delta \le t)} \le c_1 \mathrm{e}^{-\lambda/t}, \quad t \in (0,1]$$

for some constants $c_1, \lambda > 0$. So, it follows from (2.8) that

$$\limsup_{t\to 0} \frac{P_t|\nabla f|^2(x) - |\nabla f|^2(x)}{t} \le L|\nabla f|^2(x).$$

Combining this with (2.5) and (2.7), we arrive at

$$\frac{1}{2}L|\nabla f|^2(x) - \langle \nabla f, \nabla L f\rangle(x) \ge -(K_1(x)+\varepsilon), \quad f \in C^\infty(M), Nf|_{\partial M} = 0.$$

According to the Bochner-Weitzenböck formula, this is equivalent to $(\mathrm{Ric} - \nabla Z)(x) \ge -(K_1(x)+\varepsilon)$. Therefore, $\mathrm{Ric} - \nabla Z \ge -K_1$ holds on M by the arbitrariness of $x \in M \setminus \partial M$ and $\varepsilon > 0$.

(b) Proof of $\mathbb{I} \ge -K_2$. Let $X_0 = x \in \partial M$. For any $f \in C^\infty(M)$ with $Nf|_{\partial M} = 0$, (2.3) implies that

$$|\nabla P_t f|^2(x) \le \mathrm{e}^{C_1 t}(P_t|\nabla f|^2(x))\mathbb{E}\mathrm{e}^{2\int_0^t K_2(X_s)\mathrm{d}l_s}, \tag{2.9}$$

where $C_1 = 2\|K_1\|_\infty$. Let

$$\varepsilon_t = 2\sup_{s\in[0,t]} |K_2(X_s) - K_2(x)|.$$

By the continuity of the reflecting diffusion process we have $\varepsilon_t \downarrow 0$ as $t \downarrow 0$. Since there exists $c_0 > 0$ such that for any $r \ge 0$ one has $\mathrm{e}^r \le 1 + r + c_0 r^{3/2}\mathrm{e}^r$, we obtain

$$\log \mathbb{E}\mathrm{e}^{2\int_0^t K_2(X_s)\mathrm{d}l_s} \le \log\left\{1 + 2K_2(x)\mathbb{E}l_t + \mathbb{E}(\varepsilon_t l_t) + C_2\mathbb{E}(l_t^{3/2}\mathrm{e}^{C_2 l_t})\right\} \tag{2.10}$$

for some constant $C_2 > 0$. Moreover, by (I) and (III) we have

$$\mathbb{E}(l_t^{3/2}\mathrm{e}^{C_2 l_t}) \le (\mathbb{E}l_t^2)^{3/4}(\mathbb{E}\mathrm{e}^{4C_2 l_t})^{1/4} \le C_3 t^{3/4}, \quad t \in (0,1]$$

for some constant $C_3 > 0$. Substituting this and (2.10) into (2.9), we arrive at

$$\limsup_{t\to 0}\frac{1}{\sqrt{t}}\log\frac{|\nabla P_t f|^2(x)}{P_t|\nabla f|^2(x)}\le \limsup_{t\to 0}\frac{2K_2(x)\mathbb{E}l_t+\mathbb{E}(\varepsilon_t l_t)}{\sqrt{t}}.$$

Since $\mathbb{E}\varepsilon_t^2\to 0$ as $t\to 0$ and $\mathbb{E}l_t^2\le ct$ due to (III), this and (II) imply

$$\limsup_{t\to 0}\frac{1}{\sqrt{t}}\log\frac{|\nabla P_t f|^2(x)}{P_t|\nabla f|^2(x)}\le\frac{4K_2(x)}{\sqrt{\pi}}.$$

Combining this with (1.3) for $p=2$ we complete the proof. $\square$

3 Proof of Theorem 1.1

Applying Theorem 2.1 to $K_1=K$ and $K_2=\sigma$ we conclude that (1), (2) and (3) are equivalent to each other. Noting that the log-Sobolev inequality (4) implies the Poincaré inequality (5) (see e.g. [6]), it suffices to prove that (2) $\Rightarrow$ (4), (5) $\Rightarrow$ (1), and (2) $\Rightarrow$ (6) $\Rightarrow$ (7) $\Rightarrow$ (1), where " $\Rightarrow$" stands for "implies". We shall complete the proof step by step.

(a) (2) $\Rightarrow$ (4). By approximations we may assume that $f\in C^\infty(M)$ with $Nf|_{\partial M}=0$. In this case

$$\frac{\mathrm{d}}{\mathrm{d}t}P_t f=LP_t f=P_t Lf.$$

So, for fixed $t>0$ it follows from (2) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}P_{t-s}\{(P_s f^2)\log P_s f^2\}&=-P_{t-s}\frac{|\nabla P_s f^2|^2}{P_s f^2}\\&\ge -4\mathrm{e}^{2Ks}P_{t-s}\frac{(\mathbb{E}[f|\nabla f|(X_s)\mathrm{e}^{\sigma l_s}])^2}{P_s f^2}\\&\ge -4\mathrm{e}^{2Ks}P_{t-s}\mathbb{E}[|\nabla f|^2(X_s)\mathrm{e}^{2\sigma l_s}].\end{aligned}\tag{3.1}$$

Next, by the Markov property, for $\mathscr{F}_s=\sigma(X_r:r\le s), s\ge 0$, we have

$$\begin{aligned}P_{t-s}(\mathbb{E}[|\nabla f|^2(X_s)\mathrm{e}^{2\sigma l_s}])(x)&=\mathbb{E}^x\mathbb{E}^{X_{t-s}}[|\nabla f|^2(X_s)\mathrm{e}^{2\sigma l_s}]\\&=\mathbb{E}^x[\mathbb{E}^x(\mathrm{e}^{2\sigma(l_t-l_{t-s})}|\nabla f|^2(X_t)|\mathscr{F}_{t-s})]\\&=\mathbb{E}^x[|\nabla f|^2(X_t)\mathrm{e}^{2\sigma(l_t-l_{t-s})}].\end{aligned}$$

Combining this with (3.1) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}s}P_{t-s}\{(P_s f^2)\log P_s f^2\}\ge -4\mathbb{E}\big[|\nabla f|^2(X_t)\mathrm{e}^{2Ks+2\sigma(l_t-l_{t-s})}\big],\quad s\in(0,t).$$

This implies (4) by integrating both sides with respect to ds from 0 to t.

(b1) (5) $\Rightarrow$ $\mathrm{Ric} - \nabla Z \geq -K$. Let $X_0 = x \in M \setminus \partial M$ and $f \in C^\infty(M)$ with $Nf|_{\partial M} = 0$. By (5) we have

$$P_t f^2 - (P_t f)^2 \leq 2\mathbb{E}\bigg[|\nabla f|^2(X_t) \int_0^t \mathrm{e}^{2Ks + 2\sigma(l_t - l_{t-s})}\mathrm{d}s\bigg]. \tag{3.2}$$

Let $\delta > 0$ and τ_δ be as in the proof of Theorem 2.1(a). Then

$$\begin{aligned}
&\mathbb{E}\bigg[|\nabla f|^2(X_t) \int_0^t \mathrm{e}^{2Ks + 2\sigma(l_t - l_{t-s})}\mathrm{d}s\bigg] \\
&\leq (P_t|\nabla f|^2) \int_0^t \mathrm{e}^{2Ks}\mathrm{d}s + t\|\nabla f\|_\infty \mathrm{e}^{2Kt}\mathbb{E}[\mathrm{e}^{2\sigma l_t} 1_{\{\tau_\delta < t\}}] \\
&\leq \frac{\mathrm{e}^{2Kt} - 1}{2K} P_t|\nabla f|^2(x) + c\mathrm{e}^{-\lambda/t}, \quad t \in (0, 1]
\end{aligned}$$

holds for some constants $c, \lambda > 0$ according to (IV). Combining this with (3.2) we conclude that

$$P_t f^2(x) - (P_t f)^2(x) \leq \frac{\mathrm{e}^{2Kt} - 1}{K} P_t|\nabla f|^2(x) + 2c\mathrm{e}^{-\lambda/t}, \quad t \in (0, 1]. \tag{3.3}$$

Since $f \in C^\infty(M)$ with $Nf|_{\partial M = 0}$, we have

$$\begin{aligned}
P_t f^2 - (P_t f)^2 &= f^2 + \int_0^t P_s L f^2 \mathrm{d}s - \left(f + \int_0^t P_s L f \mathrm{d}s\right)^2 \\
&= \int_0^t (P_s L f^2 - 2f P_s L f)\mathrm{d}s - \left(\int_0^t P_s L f \mathrm{d}s\right)^2.
\end{aligned} \tag{3.4}$$

Moreover, by the continuity of $s \mapsto P_s L f$, we have

$$\left(\int_0^t P_s L f \mathrm{d}s\right)^2 = (Lf)^2 t^2 + \circ(t^2), \tag{3.5}$$

where and in what follows, for a positive function $(0, 1] \ni t \mapsto \xi_t$ the notion $\circ(\xi_t)$ stands for a variable such that $\circ(\xi_t)/\xi_t \to 0$ as $t \to 0$; while $\bigcirc(\xi_t)$ satisfies that $\bigcirc(\xi_t)/\xi_t$ is bounded for $t \in (0, 1]$. Moreover, since

$$\begin{aligned}
P_s L f^2 - 2f P_s L f =& L f^2 - 2f L f + \int_0^s (P_r L^2 f^2 - 2f P_r L^2 f)\mathrm{d}r \\
&+ \mathbb{E} \int_0^s (N L f^2 - 2f(x) N L f)(X_r)\mathrm{d}l_r,
\end{aligned}$$

and due to (IV)

$$\left|\mathbb{E}\int_0^t \{NLf^2 - 2f(x)NLf\}(X_r)\mathrm{d}l_r\right| \le c_1\mathbb{E}l_s \le c_2\mathrm{e}^{-\lambda/s}, \quad s\in(0,1]$$

holds for some constants $c_1, c_2, \lambda > 0$, it follows from the continuity of P_s in s that

$$\int_0^t (P_sLf^2 - 2fP_sLf)\mathrm{d}s = 2t|\nabla f|^2 + \frac{t^2}{2}(L^2f^2 - 2fL^2f) + \circ(t^2).$$

Combining this with (3.4) and (3.5) we obtain

$$\begin{aligned} P_tf^2(x) - (P_tf)^2(x) = & \\ &= 2t|\nabla f|^2(x) + \frac{t^2}{2}(L^2f^2 - 2fL^2f)(x) - t^2(Lf)^2(x) + \circ(t^2) \\ &= 2t|\nabla f|^2(x) + t^2(2\langle\nabla f, \nabla Lf\rangle + L|\nabla f|^2)(x) + \circ(t^2). \end{aligned} \tag{3.6}$$

Similarly,

$$\begin{aligned} P_t|\nabla f|^2(x) &= |\nabla f|^2(x) + \int_0^t P_sL|\nabla f|^2(x)\mathrm{d}s + \mathbb{E}\int_0^t N|\nabla f|^2(X_s)\mathrm{d}l_s \\ &= |\nabla f|^2(x) + tL|\nabla f|^2(x) + \circ(t). \end{aligned}$$

Combining this with (3.3) and (3.6) we arrive at

$$\begin{aligned} &\frac{1}{t^2}\{t^2(2\langle\nabla f, \nabla Lf\rangle + L|\nabla f|^2)(x) + \circ(t^2)\} \\ &\le \frac{\mathrm{e}^{2Kt}-1}{Kt}L|\nabla f|^2(x) + \circ(1) + \frac{1}{t}\Big(\frac{\mathrm{e}^{2Kt}-1}{Kt} - 2\Big)|\nabla f|^2(x). \end{aligned}$$

Letting $t\to 0$ we obtain

$$L|\nabla f|^2(x) - 2\langle\nabla f, \nabla Lf\rangle(x) \ge -2K|\nabla f|^2(x),$$

which implies $(\mathrm{Ric} - \nabla Z)(x) \ge -K$ by the Bochner-Weitzenböck formula.
(b2) (5) $\Rightarrow$ $\mathbb{I} \ge -\sigma$. Let $X_0 = x \in \partial M$ and $f \in C^\infty(M)$ with $Nf|_{\partial M} = 0$. Noting that $Lf^2 - 2fLf = 2|\nabla f|^2$, by the Itô formula we have

$$\begin{aligned} &P_tf^2(x) - (P_tf)^2(x) = f^2 + \int_0^t P_sLf^2\mathrm{d}s - \Big(f + \int_0^t P_sLf\mathrm{d}s\Big)^2 \\ &= 2\int_0^t P_s|\nabla f|^2(x)\mathrm{d}s + 2\int_0^t [P_s(fLf)(x) - f(x)P_sLf(x)]\mathrm{d}s + \bigcirc(t^2). \end{aligned} \tag{3.7}$$

Since $Nf|_{\partial M} = 0$ implies

$$0 = \langle \nabla f, \nabla \langle N, \nabla f \rangle \rangle = \mathrm{Hess}_f(N, \nabla f) - \mathbb{I}(\nabla f, \nabla f),$$

it follows that

$$\mathbb{I}(\nabla f, \nabla f) = \mathrm{Hess}_f(N, \nabla f) = \frac{1}{2} N |\nabla f|^2. \tag{3.8}$$

So, by the Itô formula, (II) and (III) yield

$$\begin{aligned} P_s|\nabla f|^2(x) &= |\nabla f|^2(x) + \int_0^s P_r L |\nabla f|^2(x) \mathrm{d}r + \mathbb{E} \int_0^s N|\nabla f|^2(X_r) \mathrm{d}l_r \\ &= |\nabla f|^2(x) + \bigcirc(s) + 2\mathbb{E} \int_0^s \mathbb{I}(\nabla f, \nabla f)(X_r) \mathrm{d}l_r \\ &= |\nabla f|^2(x) + \frac{4\sqrt{s}}{\sqrt{\pi}} \mathbb{I}(\nabla f, \nabla f)(x) + \circ(s^{1/2}). \end{aligned} \tag{3.9}$$

Moreover, since $(fNLf)(X_r) - f(x)(NLf)(X_r)$ is bounded and goes to zero as $r \to 0$, it follows from (III) that

$$2\mathbb{E} \int_0^t \mathrm{d}s \int_0^s [(fNf)(X_r) - f(x)(NLf)(X_r)] \mathrm{d}l_r = \circ(t^{3/2}).$$

So, by the Iô formula

$$\begin{aligned} &2 \int_0^t [P_s(fLf)(x) - f(x) P_s Lf(x)] \mathrm{d}s \\ &= 2 \int_0^t \mathrm{d}s \int_0^s [P_r L(fLf)(x) - f(x) P_r L^2 f(x)] \mathrm{d}r \\ &\quad + 2\mathbb{E} \int_0^t \mathrm{d}s \int_0^s [(fNLf)(X_r) - f(x)(NLf)(X_r)] \mathrm{d}l_r = \circ(t^{3/2}). \end{aligned}$$

Combining this with (3.7) and (3.9) we arrive at

$$\begin{aligned} &\lim_{t \to 0} \frac{1}{t\sqrt{t}} \big(P_t f^2(x) - (P_t f)^2(x) - 2t |\nabla f|^2(x) \big) \\ &= \frac{8}{\sqrt{\pi}} \mathbb{I}(\nabla f, \nabla f)(x) \lim_{t \to 0} \frac{1}{t\sqrt{t}} \int_0^t \sqrt{s}\, \mathrm{d}s = \frac{16}{3\sqrt{\pi}} \mathbb{I}(\nabla f, \nabla f)(x). \end{aligned} \tag{3.10}$$

On the other hand, by the Itô formula for $|\nabla f|^2(X_t)$, it follows from (3.8) and (II) that

$$
\begin{aligned}
&A_t := \\
&= \frac{1}{t\sqrt{t}}\mathbb{E}\Big\{|\nabla f|^2(X_t)\int_0^t \mathrm{e}^{2Ks+2\sigma(l_t-l_{t-s})}\mathrm{d}s - t|\nabla f|^2(x)\Big\} \\
&= \frac{1}{\sqrt{t}}\big(\mathbb{E}|\nabla f|^2(X_t) - |\nabla f|^2(x)\big) + \mathbb{E}\Big\{\frac{|\nabla f|^2(X_t)}{t\sqrt{t}}\int_0^t \big(\mathrm{e}^{2Ks+2\sigma(l_t-l_{t-s})}-1\big)\mathrm{d}s\Big\} \\
&= \frac{1}{\sqrt{t}}\Big\{\int_0^t P_sL|\nabla f|^2(x)\mathrm{d}s + \mathbb{E}\int_0^t N|\nabla f|^2(X_s)\mathrm{d}l_s\Big\} \\
&\quad + \mathbb{E}\Big\{\frac{|\nabla f|^2(X_t)}{t\sqrt{t}}\int_0^t \big(\mathrm{e}^{2Ks+2\sigma(l_t-l_{t-s})}-1\big)\mathrm{d}s\Big\} \\
&= \frac{4}{\sqrt{\pi}}\mathbb{I}(\nabla f,\nabla f)(x) + \circ(1) + \mathbb{E}\Big\{\frac{|\nabla f|^2(X_t)}{t\sqrt{t}}\int_0^t \big(\mathrm{e}^{2Ks+2\sigma(l_t-l_{t-s})}-1\big)\mathrm{d}s\Big\}.
\end{aligned}
\tag{3.11}
$$

Since by (I) and (III)

$$
\begin{aligned}
&\Big|\mathbb{E}\Big[\big(|\nabla f|^2(X_t) - |\nabla f|^2(x)\big)\int_0^t \big(\mathrm{e}^{2Ks+2\sigma(l_t-l_{t-s})}-1\big)\mathrm{d}s\Big]\Big| \\
&\le t\Big\{\mathbb{E}\big(|\nabla f|^2(X_t) - |\nabla f|^2(x)\big)^2\Big\}^{1/2}\Big\{\mathbb{E}\big(\mathrm{e}^{2Kt+2\sigma l_t}-1\big)^2\Big\}^{1/2} \\
&= \circ(t)\cdot\big(\mathbb{E}[4\sigma^2 l_t^2] + \circ(t)\big) = \circ(t^2),
\end{aligned}
$$

it follows from (I) and (II) that

$$
\begin{aligned}
&\mathbb{E}\Big[|\nabla f|^2(X_t)\int_0^t \big(\mathrm{e}^{2Ks+2\sigma(l_t-l_{t-s})}-1\big)\mathrm{d}s\Big] \\
&= \circ(t^2) + |\nabla f|^2(x)\mathbb{E}\int_0^t \big(\mathrm{e}^{2Ks+2\sigma(l_t-l_{t-s})}-1\big)\mathrm{d}s \\
&= \circ(t^{3/2}) + \frac{4\sigma|\nabla f|^2(x)}{\sqrt{\pi}}\int_0^t \big(\sqrt{t}-\sqrt{t-s}\big)\mathrm{d}s \\
&= \frac{4\sigma t\sqrt{t}}{3\sqrt{\pi}}|\nabla f|^2(x) + \circ(t^{3/2}).
\end{aligned}
$$

Combining this with (3.11) we arrive at

$$A_t \le \circ(1) + \frac{4}{\sqrt{\pi}}\mathbb{I}(\nabla f,\nabla f)(x) + \frac{4\sigma}{3\sqrt{\pi}}|\nabla f|^2(x).$$

So, (3.10) and (5) imply that

$$\frac{16}{3\sqrt{\pi}}\mathbb{I}(\nabla f,\nabla f)(x) \le \limsup_{t\to 0} 2A_t \le \frac{8}{\sqrt{\pi}}\mathbb{I}(\nabla f,\nabla f)(x) + \frac{8\sigma}{3\sqrt{\pi}}|\nabla f|^2(x).$$

Therefore, $\mathbb{I}(\nabla f, \nabla f)(x) \geq -\sigma |\nabla f|^2(x)$.
(c) (2) $\Rightarrow$ (6). Let $f \geq 0$ be smooth satisfying the Neumann boundary condition. We have

$$\frac{\mathrm{d}}{\mathrm{d}s} P_s\big\{(P_{t-s}f)\log P_{t-s}f\big\} = P_s \frac{|\nabla P_{t-s}f|^2}{P_{t-s}f}.$$

This implies

$$P_t(f\log f) - (P_tf)\log P_tf = \int_0^t P_s \frac{|\nabla P_{t-s}f|^2}{P_{t-s}f}\mathrm{d}s. \tag{3.12}$$

On the other hand, by (2) and applying the Schwartz inequality to the probability measure $\frac{2K}{1-\exp[-2Kt]}\mathrm{e}^{-2Ks}\mathrm{d}s$ on $[0,t]$, we obtain

$$\begin{aligned}
&|\nabla P_tf|^2 = \\
&= \bigg\{\frac{2K}{1-\mathrm{e}^{-2Kt}}\int_0^t |\nabla P_s(P_{t-s}f)|\mathrm{e}^{-2Ks}\mathrm{d}s\bigg\}^2\\
&\leq \bigg\{\frac{2K}{1-\mathrm{e}^{-2Kt}}\int_0^t E\big[|\nabla P_{t-s}f|(X_s)\mathrm{e}^{\sigma l_s - Ks}\big]\mathrm{d}s\bigg\}^2\\
&\leq \Big(\frac{2K}{1-\mathrm{e}^{-2Kt}}\Big)^2\bigg(\mathbb{E}\int_0^t \frac{|\nabla P_{t-s}f|^2}{P_{t-s}f}(X_s)\mathrm{d}s\bigg)\int_0^t \mathbb{E}\big[P_{t-s}f(X_s)\mathrm{e}^{2\sigma l_s - 2Ks}\big]\mathrm{d}s\\
&= \Big(\frac{2K}{1-\mathrm{e}^{-2Kt}}\Big)^2\bigg(\int_0^t P_s\frac{|\nabla P_{t-s}f|^2}{P_{t-s}f}\mathrm{d}s\bigg)\int_0^t \mathbb{E}\big[P_{t-s}f(X_s)\mathrm{e}^{2\sigma l_s - 2Ks}\big]\mathrm{d}s.
\end{aligned}$$

Combining this with (3.12) and noting that the Markov property implies

$$\begin{aligned}
\mathbb{E}[P_{t-s}f(X_s)\mathrm{e}^{2\sigma l_s}] &= \mathbb{E}[(\mathbb{E}^{X_s}f(X_{t-s}))\mathrm{e}^{2\sigma l_s}] = \mathbb{E}[\mathrm{e}^{2\sigma l_s}\mathbb{E}(f(X_t)|\mathscr{F}_s)]\\
&= \mathbb{E}[\mathbb{E}(f(X_t)\mathrm{e}^{2\sigma l_s}|\mathscr{F}_s)] = \mathbb{E}[f(X_t)\mathrm{e}^{2\sigma l_s}],
\end{aligned}$$

we obtain (6).
(d) (6) $\Rightarrow$ (7). The proof is similar to the classical one for the log-Sobolev inequality to imply the Poincaré inequality. Let $f \in C^\infty(M)$. SInce M is compact, $1+\varepsilon f > 0$ for small $\varepsilon > 0$. Applying (6) to $1+\varepsilon f$ in place of f, we obtain

$$\begin{aligned}
|\nabla P_tf|^2 \leq &\frac{2K}{\varepsilon^2(1-\mathrm{e}^{-2Kt})}\big\{P_t(1+\varepsilon f)\log(1+\varepsilon f) - (1+\varepsilon P_tf)\log(1+\varepsilon P_tf)\big\}\\
&\cdot \mathbb{E}\bigg\{(1+\varepsilon f(X_t))\int_0^t \mathrm{e}^{2\sigma l_s - 2Ks}\mathrm{d}s\bigg\}.
\end{aligned} \tag{3.13}$$

Since by Taylor's expansion

$$P_t(1+\varepsilon f)\log(1+\varepsilon f)-(1+\varepsilon P_t f)\log(1+\varepsilon P_t f)=\frac{\varepsilon^2}{2}\big(P_t f^2-(P_t f)^2\big)+\circ(\varepsilon^2),$$

letting $\varepsilon\to 0$ in (3.13) we obtain (7).
(e1) (7) $\Rightarrow$ $\mathrm{Ric}-\nabla Z\geq -K$. Let $X_0=x\in M\setminus\partial M$ and $f\in C^\infty(M)$ with $Nf|_{\partial M}=0$. by (I) and (IV) we have

$$\mathbb{E}\mathrm{e}^{2\sigma l_s}=1+\mathbb{E}[\mathrm{e}^{2\sigma l_s}1_{\{\tau_\delta\leq s\}}]=1+\circ(s).$$

So,

$$\mathbb{E}\int_0^t \mathrm{e}^{2\sigma l_s-2Ks}\mathrm{d}s=\frac{1-\exp[-2Kt]}{2K}+\circ(t).$$

Combining this with (3.6) and (7), we conclude that, at point x,

$$\begin{aligned}
&\frac{|\nabla P_t f|^2-|\nabla f|^2}{t}\leq\\
&\leq\frac{K}{1-\mathrm{e}^{-2Kt}}\big\{2|\nabla f|^2+t\big(2\langle\nabla f,\nabla Lf\rangle+L|\nabla f|^2\big)\big\}-\frac{|\nabla f|^2}{t}+\circ(1)\\
&=\frac{1}{t}\Big(\frac{2Kt}{1-\mathrm{e}^{-2Kt}}-1\Big)|\nabla f|^2+\frac{Kt}{1-\mathrm{e}^{-2Kt}}\big(2\langle\nabla f,\nabla Lf\rangle+L|\nabla f|^2\big)+\circ(1).
\end{aligned}$$

Letting $t\to 0$ and using (2.7), we obtain

$$2\langle\nabla f,\nabla Lf\rangle\leq K|\nabla f|^2+\langle\nabla f,\nabla Lf\rangle+\frac{1}{2}L|\nabla f|^2$$

at point x. This implies $\mathrm{Ric}-\nabla Z\geq -K$ at this point according to the Bochner-Weitzenböck formula.
(e2) (7) $\Rightarrow$ $\mathbb{I}\geq-\sigma$. Let $X_0=x\in\partial M$ and $f\in C^\infty(M)$ with $Nf|_{\partial M}=0$. It follows from (3.10), (7) and (II) that at point x,

$$\begin{aligned}
&|\nabla P_t f|^2\leq\\
&\leq\frac{2K^2}{(1-\mathrm{e}^{-2Kt})^2}\Big(2t|\nabla f|^2+\frac{16t^{3/2}}{3\sqrt{\pi}}\mathbb{I}(\nabla f,\nabla f)+\circ(t^{3/2})\Big)\Big(t+\frac{8\sigma t^{3/2}}{3\sqrt{\pi}}+\circ(t^{3/2})\Big)\\
&=\frac{4K^2t^2}{(1-\mathrm{e}^{-2Kt})^2}|\nabla f|^2+\frac{4K^2t^{5/2}}{(1-\mathrm{e}^{-2Kt})^2}\Big(\frac{8}{3\sqrt{\pi}}\mathbb{I}(\nabla f,\nabla f)+\frac{8\sigma}{3\sqrt{\pi}}|\nabla f|^2\Big)+\circ(t^{1/2}).
\end{aligned}$$

Combining this with (2.7) we deduce at point x that

$$\begin{aligned}
0&=\lim_{t\to 0}\frac{1}{\sqrt{t}}\Big(|\nabla P_t f|^2-\frac{4K^2t^2}{(1-\mathrm{e}^{-2Kt})^2}|\nabla f|^2\Big)\\
&\leq\lim_{t\to 0}\frac{4K^2t^2}{(1-\mathrm{e}^{-2Kt})^2}\Big(\frac{8}{3\sqrt{\pi}}\mathbb{I}(\nabla f,\nabla f)+\frac{8\sigma}{3\sqrt{\pi}}|\nabla f|^2\Big)\\
&=\frac{8}{3\sqrt{\pi}}\mathbb{I}(\nabla f,\nabla f)+\frac{8\sigma}{3\sqrt{\pi}}|\nabla f|^2.
\end{aligned}$$

Therefore, $\mathbb{I}(\nabla f, \nabla f)(x) \geq -\sigma |\nabla f|^2(x)$.

4 Lévy-Gromov isoperimetric inequality

As a dimension-free version of the classical Lévy-Gromov isoperimetric inequality, it is proved in [4] that if M does not have boundary then for $V \in C^2(M)$ such that $\mathrm{Ric} - \mathrm{Hess}_V \geq R > 0$ the following inequality

$$\mathscr{U}(\mu(f)) \leq \int_M \sqrt{\mathscr{U}^2(f) + R^{-1}|\nabla f|^2}\,\mathrm{d}\mu, \tag{4.1}$$

holds for any smooth function f with values in $[0,1]$, where $\mu(\mathrm{d}x) := C(V)^{-1}\mathrm{e}^{V(x)}\mathrm{d}x$ for $C(V) = \int_M \mathrm{e}^{V(x)}\mathrm{d}x$ is a probability measure on M, and $\mathscr{U} = \varphi \circ \Phi^{-1}$ for $\Phi(r) = (2\pi)^{-1}\int_{-\infty}^r \mathrm{e}^{-s^2/2}\mathrm{d}s$ and $\varphi = \Phi'$. Since $\mathscr{U}(0) = \mathscr{U}(1) = 0$, taking $f = 1_A$ (by approximations) in (4.1) for a smooth domain $A \subset M$, we obtain the isoperimetric inequality

$$R\mathscr{U}(A) \leq \mu_\partial(\partial A), \tag{4.2}$$

where $\mu_\partial(\partial A)$ is the area of ∂A induced by μ. This inequality is crucial in the study of Gaussian type concentration of μ (see [4, 9]). Obviously, (4.1) follows from the following semigroup inequality by letting $t \to \infty$:

$$\mathscr{U}(P_t f) \leq P_t\sqrt{\mathscr{U}^2(f) + R^{-1}(1 - \mathrm{e}^{-2Rt})|\nabla f|^2}. \tag{4.3}$$

In this section we aim to extend (4.3) to manifolds with boundary.

Now, let again M be compact with boundary ∂M, and let P_t be the Neumann semigroup generated by $L = \Delta + Z$. We shall prove an analogue of (4.3) for the curvature and second fundamental condition in Theorem 1.1(1).

Theorem 4.1. *Let* $\mathrm{Ric} - \nabla Z \geq -K$ *and* $\mathbb{I} \geq -\sigma$ *for some constants* $K \in \mathbb{R}$ *and* $\sigma \geq 0$. *Then for any smooth function* f *with values in* $[0,1]$,

$$\mathscr{U}(P_t f) \leq \mathbb{E}\sqrt{\mathscr{U}^2(f)(X_t) + |\nabla f|^2(X_t)\frac{(\mathrm{e}^{2Kt} - 1)\mathrm{e}^{2\sigma l_t}}{K}}, \quad t \geq 0. \tag{4.4}$$

If in particular ∂M *is convex (i.e.* $\sigma = 0$*), then*

$$\mathscr{U}(P_t f) \leq P_t\sqrt{\mathscr{U}^2(f) + |\nabla f|^2(X_t)\frac{\mathrm{e}^{2Kt} - 1}{K}}, \quad t \geq 0.$$

If moreover $K < 0$, *then* (4.1) *and* (4.2) *hold for* $R = -K > 0$.

Proof. It suffices to prove the first assertion. To this end, we shall use the following equivalent condition for $\mathrm{Ric} - \nabla Z \geq -K$ (see e.g. the proof of [9, (1.14)]):

$$\Gamma_2(f,f) := \frac{1}{2}L|\nabla f|^2 - \langle \nabla f, \nabla L f\rangle \geq -K|\nabla f|^2 + \frac{|\nabla|\nabla f|^2|^2}{4|\nabla f|^2}. \tag{4.5}$$

To prove (4.4), we consider the process

$$\eta_s = \mathscr{U}^2(P_{t-s}f)(X_s) + |\nabla P_{t-s}f|^2(X_s)\frac{(\mathrm{e}^{2Ks}-1)\mathrm{e}^{2\sigma l_s}}{K}, \quad s \in [0,t].$$

To apply the Itô formula for η_s, recall that X_s solves the equation

$$\mathrm{d}X_s = \sqrt{2}\, u_s \circ \mathrm{d}B_s + N(X_s)\mathrm{d}l_s,$$

where u_s is the horizontal lift of X_s and B_s is the Brownian motion on $\mathbb{R}^d$ provided M is d-dimensional. So,

$$\begin{aligned}
\mathrm{d}\eta_s = {} & \sqrt{2}\Big\langle 2(\mathscr{U}\mathscr{U}')(P_{t-s}f)(X_s) + \frac{(\mathrm{e}^{2Ks}-1)\mathrm{e}^{2\sigma l_s}}{K}\nabla|\nabla P_{t-s}f|^2(X_s), u_s\mathrm{d}B_s\Big\rangle \\
& + \Big\{2(\mathscr{U}'^2 + \mathscr{U}\mathscr{U}'')(P_{t-s}f)|\nabla P_{t-s}f|^2 + 2\Gamma_2(P_{t-s}f, P_{t-s}f)\frac{(\mathrm{e}^{2Ks}-1)\mathrm{e}^{2\sigma l_s}}{K} \\
& + 2|\nabla P_{t-s}f|^2\mathrm{e}^{2Ks+2\sigma l_s}\Big\}(X_s)\mathrm{d}s \\
& + \frac{(\mathrm{e}^{2Ks}-1)\mathrm{e}^{2\sigma l_s}}{K}\big(N|\nabla P_{t-s}f|^2 + 2\sigma|\nabla P_{t-s}f|^2\big)(X_s)\mathrm{d}l_s.
\end{aligned}$$

Noting that $\mathscr{U}\mathscr{U}'' = -1$ and $\sigma \geq 0$ so that $\mathrm{e}^{2\sigma l_s} \geq 1$, combining this with (3.8), $\mathbb{I} \geq -\sigma$ and (4.5), we obtain

$$\begin{aligned}
\mathrm{d}\eta_s \geq {} & \sqrt{2}\Big\langle 2(\mathscr{U}\mathscr{U}')(P_{t-s}f)(X_s) + \frac{(\mathrm{e}^{2Ks}-1)\mathrm{e}^{2\sigma l_s}}{K}\nabla|\nabla P_{t-s}f|^2(X_s), u_s\mathrm{d}B_s\Big\rangle \\
& + \Big\{2\mathscr{U}'^2(P_{t-s}f)|\nabla P_{t-s}f|^2 + \frac{(\mathrm{e}^{2Ks}-1)\mathrm{e}^{2\sigma l_s}|\nabla|\nabla P_{t-s}f|^2|^2}{2K|\nabla P_{t-s}f|^2}\Big\}(X_s)\mathrm{d}s.
\end{aligned}$$

Therefore, there exists a martingale M_s for $s \in [0,t]$ such that

$$\begin{aligned}
\mathrm{d}\eta_s^{1/2} = {} & \mathrm{d}M_s + \frac{\mathrm{d}\eta_s}{2\eta_s^{1/2}} - \\
& - \frac{\big|2(\mathscr{U}\mathscr{U}')(P_{t-s}f)\nabla P_{t-s}f + \frac{(\mathrm{e}^{2Ks}-1)\mathrm{e}^{2\sigma l_s}}{K}\nabla|\nabla P_{t-s}f|^2\big|^2(X_s)}{4\eta_s^{3/2}} \\
= {} & \mathrm{d}M_s + \frac{1}{4\eta_s^{3/2}}B_s\mathrm{d}s,
\end{aligned}$$

where

$$
\begin{aligned}
B_s :=&2\eta_s\Big(2\mathscr{U}'^2(P_{t-s}f)|\nabla P_{t-s}f|^2 + \frac{(\mathrm{e}^{2Ks}-1)\mathrm{e}^{2\sigma l_s}|\nabla|\nabla P_{t-s}f|^2|^2}{2K|\nabla P_{t-s}f|^2}\Big)(X_s)\\
&-\Big|2(\mathscr{U}\mathscr{U}')(P_{t-s}f)\nabla P_{t-s}f + \frac{\mathrm{e}^{2Ks}-1}{K}\mathrm{e}^{2\sigma l_s}\nabla|\nabla P_{t-s}f|^2\Big|^2(X_s)\\
\ge&\frac{(\mathrm{e}^{2Ks}-1)\mathrm{e}^{2\sigma l_s}}{K}\Big\{\frac{\mathscr{U}^2(P_{t-s}f)|\nabla|\nabla P_{t-s}f|^2|^2}{2|\nabla P_{t-s}f|^2} + 4|\nabla P_{t-s}f|^4\mathscr{U}'^2(P_{t-s}f)\\
&\qquad - 4(\mathscr{U}\mathscr{U}')(P_{t-s}f)\langle\nabla P_{t-s}f, \nabla|\nabla P_{t-s}f|^2\rangle\Big\}(X_s)\\
\ge&0.
\end{aligned}
$$

So, $\eta_s^{1/2}$ is a sub-martingale on $[0,t]$. Therefore, $\mathbb{E}\eta_0^{1/2} \le \mathbb{E}\eta_t^{1/2}$, which is nothing but (4.4). $\square$

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